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Paranormal Measurable Operators Affiliated with a Semifinite von Neumann Algebra

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Abstract—Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, τ be a faithful normal semifinite trace on $\mathcal{M}$. We define two (closed in the topology of convergence in measure τ) classes $\mathcal{P}_1$ and $\mathcal{P}_2$ of τ-measurable operators and investigate their properties. The class $\mathcal{P}_2$ contains $\mathcal{P}_1$. If a τ-measurable operator T is hyponormal, then T lies in $\mathcal{P}_1$; if an operator T lies in $\mathcal{P}_k$, then UTU^* belongs to $\mathcal{P}_k$ for all isometries U from $\mathcal{M}$ and $k = 1, 2$; if an operator T from $\mathcal{P}_1$ admits the bounded inverse T^{-1} then T^{-1} lies in $\mathcal{P}_1$. If a bounded operator T lies in $\mathcal{P}_1$ then T is normaloid, T^n belongs to $\mathcal{P}_1$ and a rearrangement $\mu_t(T^n) \geq \mu_t(T)^n$ for all $t > 0$ and natural n. If a τ-measurable operator T is hyponormal and T^n is τ-compact operator for some natural number n then T is both normal and τ-compact. If an operator T lies in $\mathcal{P}_1$ then T^2 belongs to $\mathcal{P}_1$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, then the class $\mathcal{P}_1$ coincides with the set of all paranormal operators on $\mathcal{H}$. If a τ-measurable operator A is q-hyponormal ($1 \geq q > 0$) and $|A^*| \geq \mu_\infty(A)I$ then A is normal. In particular, every τ-compact q-hyponormal (or q-cohyponormal) operator is normal. Consider a τ-measurable nilpotent operator $Z \neq 0$ and numbers $a, b \in \mathbb{R}$. Then an operator $Z^*Z - ZZ^* + a\Re Z + b\Im Z$ cannot be nonpositive or nonnegative. Hence a τ-measurable hyponormal operator $Z \neq 0$ cannot be nilpotent.

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1. INTRODUCTION

Let $\mathcal{M}$ be a von Neumann operator algebra on a Hilbert space $\mathcal{H}$, τ be a faithful normal semifinite trace on $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be the $*$-algebra of all τ-measurable operators, a number $0 < p < \infty$ and $L_p(\mathcal{M}, \tau)$ be the space of integrable (with respect to τ) in p-th degree operators. Let $\mathcal{M}_1 = \{X \in \mathcal{M} : \ ||X|| = 1\}$, $\mu_t(X)$ be a rearrangement of operator $X \in \widetilde{\mathcal{M}}$ and $\mu_\infty(X) = \lim\limits_{t\to\infty} \mu_t(X)$. In this paper we introduce two classes

$$\mathcal{P}_1 = \{T \in \widetilde{\mathcal{M}} : \ ||T^2A|| \geq ||TA||^2 \ \text{ for all } \ A \in \mathcal{M}_1 \ \text{ with } \ TA \in \mathcal{M}\},$$

$$\mathcal{P}_2 = \{T \in \widetilde{\mathcal{M}} : \ \mu_t(T^2) \geq \mu_t(T)^2 \ \text{ for all } \ t > 0\}$$

of τ-measurable operators and investigate their properties. The classes $\mathcal{P}_1$ and $\mathcal{P}_2$ are closed in the topology of convergence in measure τ and $\mathcal{P}_1 \subset \mathcal{P}_2$ (Propositions 3.5 and 3.30). In Theorem 3.1 we obtain an equivalent definition of the class $\mathcal{P}_1$, that allows us to call $\mathcal{P}_1$ a class of all paranormal τ-measurable operators. If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal then $T \in \mathcal{P}_1$; if an operator $T \in \mathcal{P}_1$ has the inverse $T^{-1} \in \mathcal{M}$ then $T^{-1} \in \mathcal{P}_1$ (Theorem 3.6). If an operator $T \in \mathcal{P}_k$ then $UTU^* \in \mathcal{P}_k$ for all

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