

Normal Coordinates in Affine Geometry

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Abstract—Manifolds endowed with an affine geometry of general type with nontrivial metric, torsion, and nonmetricity tensor are considered. Such manifolds have recently attracted much attention due to the construction of generalized gravity models. Under the assumption that all geometric objects are real analytic functions, normal coordinates in a neighborhood of an arbitrary point are constructed by expanding the connection and the metric in Taylor series. It is shown that the normal coordinates are a generalization of a Cartesian coordinate system in Euclidean space to the case of manifolds with any affine geometry. Moreover, the components of any real analytic tensor field in a neighborhood of any given point are represented in the form of a power series whose coefficients are constructed from the covariant derivatives and the curvature and torsion tensors evaluated at this point. For constant curvature spaces, these series are explicitly summed, and an expression for the metric in normal coordinates is found. It is shown that normal coordinates determine a smooth surjective mapping of Euclidean space to a constant curvature manifold. The equations for extremals are explicitly integrated in normal coordinates for constant curvature spaces. A relationship between normal coordinates and the exponential mapping is analyzed.

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INTRODUCTION

Normal coordinates, also referred to as geodesic or Riemann coordinates, play an important role in mathematical physics. They were first introduced by Riemann [1] for Riemannian manifolds, for which torsion and the nonmetricity tensor identically vanish. For details on Riemannian coordinates, see, e.g., [2, 3]. This construction is easy to transfer to pseudo-Riemannian manifolds with nonpositive definite metric [4, 5].

In recent years, in mathematical physics, models based on an affine geometry of general form with nontrivial torsion and nonmetricity tensor have been extensively studied (see, e.g., [6, 7]). In this paper, we construct normal coordinates for manifolds with an affine geometry of general form, given a metric and torsion and nonmetricity tensors. First, we introduce this coordinate system by using series. We assume that all geometric objects are real analytic, that is, their components expand in convergent power series in a neighborhood of each point of the manifold. Then, we give a definition in terms of the exponential mapping.

Normal coordinates in affine connection spaces of general form were considered, e.g., in [8, 9]. In particular, these monographs contain Theorem 1 and Lemma 1, which we cite in this paper for completeness. To the author's knowledge, Theorem 3 has not been published previously. Apparently, the explicit expression for the second-order expansion coefficients in (19) has not been known either. In Sections 3 and 4, the obtained formulas are compared with known expressions for (pseudo)Riemannian manifolds.

In what follows, we shall see that Cartesian coordinates are normal in Euclidean space \mathbb{R}^n . In this respect, normal coordinates are a generalization of a Cartesian coordinate system to the general case of a manifold with an affine geometry.

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