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On Conditions of Solvability of the Goursat Problem for Generalized Aller Equation

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Abstract—In terms of coefficients of an equation under consideration, we derive 16 variants of collections of conditions of resolvability of this equation in quadratures. Every collection consists of three identities that interconnect five coefficients appearing in left-hand side of an equation.

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We speak about equation

$$u_{xxy} + mu_{xx} + nu_{xy} + pu_x + qu_y + ru = f, \tag{1}$$

and its particular case, Aller equation $u_t = (mu_x + bu_{xt})_x$ which appear in modeling of soil moisture transfer process in aeration zones [1]. Eq. (1) was investigated in papers by D. Colton, W. Rundell and M. Stecher, M. Kh. Shkhanukov, A. P. Soldatov, A. I. Kozhanov, V. A. Vodakhova, O. M. Jokhadze, A. Maher and E. A. Utkina (e.g., [2–11]) from different points of view.

Current paper can be considered a continuation of paper [12] (also [13], pp. 132–137), where we use the Riemann method to obtain the Goursat problem solution formula and specify some particular cases where the Riemann function included in this formula can be written explicitly which means that the problem can be solved in quadratures. Here we seek for other cases of the Goursat problem solution in quadratures. We also use the method of factorization of the left-hand side of Eq. (1) with differential operators containing the derivatives of unknown function with respect to only one of the independent variable (x, y) instead of the Riemann method.

Let us recall the Goursat problem setting in the domain $D = \{x_0 < x < x_1,\ y_0 < y < y_1\}$ for function $u(x, y)$ which is a solution to (1) in D with boundary condition

$$u(x_0, y) = \varphi(y), u_x(x_0, y) = \varphi_1(y),\ \ y \in [y_0, y_1],\ \ \varphi, \varphi_1 \in C^1, \tag{2}$$

$$u(x, y_0) = \psi(x),\ \ x \in [x_0, x_1],\ \ \psi \in C^2,$$

where the given functions also satisfy matching conditions

$$\varphi(y_0) = \psi(x_0),\ \ \psi'(x_0) = \varphi_1(y_0).$$

First let us find functions α, β, λ with which the left-hand side of Eq. (1) becomes either

$$\left(\frac{\partial^2}{\partial x^2} + \alpha\frac{\partial}{\partial x} + \beta\right)\left(\frac{\partial u}{\partial y} + \lambda u\right) \tag{3}$$

or

$$\left(\frac{\partial}{\partial y} + \lambda\right)\left(\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u\right). \tag{4}$$

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