

## On Inductive Limits for Systems of $C^*$ -Algebras

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**Abstract**—We consider a covariant functor from the category of an arbitrary partially ordered set into the category of  $C^*$ -algebras and their  $*$ -homomorphisms. In this case one has inductive systems of algebras over maximal directed subsets. The article deals with properties of inductive limits for those systems. In particular, for a functor whose values are Toeplitz algebras, we show that each such an inductive limit is isomorphic to a reduced semigroup  $C^*$ -algebra defined by a semigroup of rationals. We endow an index set for a family of maximal directed subsets with a topology and study its properties. We establish a connection between this topology and properties of inductive limits.

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**Introduction.** As is well-known, the notion of the inductive limit, or the direct limit, is a fundamental categorical construction. For a given family of objects and their morphisms, it allows to construct a new object.

The present paper deals with limits for inductive systems consisting of  $C^*$ -algebras and their  $*$ -homomorphisms. In other words, we consider a covariant functor from a category associated with an arbitrary partially ordered set  $K$  to the category of  $C^*$ -algebras and their morphisms. Such a functor can be defined as a triple  $(K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ , where  $\{\mathfrak{A}_a\}$  is a family of  $C^*$ -algebras indexed by elements of the set  $K$ , and  $\{\sigma_{ba} : \mathfrak{A}_a \rightarrow \mathfrak{A}_b\}$  is a family of  $*$ -homomorphisms between  $C^*$ -algebras defined for every pair of elements  $a, b \in K$  satisfying the condition  $a \leq b$ . Moreover, the equations  $\sigma_{ca} = \sigma_{cb} \circ \sigma_{ba}$  hold for all elements  $a \leq b \leq c$ . Furthermore, for each element  $a \in K$ , the morphism  $\sigma_{aa}$  is the identity mapping. We also suppose that all morphisms  $\sigma_{ba}$  are embeddings of  $C^*$ -algebras, i.e., injective  $*$ -homomorphisms.

The motivation for our studying these covariant functors comes from several sources. On one hand, these are papers which are closely related to applications in algebraic quantum field theory. In [1–3] the authors consider the families  $(K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$  containing  $C^*$ -algebras of quantum observables for the case of curved space-times. In the above mentioned papers, the triple  $(K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$  is called a net over the set  $K$ . On the other hand, a part of motivation comes from papers concerning with semigroup  $C^*$ -algebras, in particular, Toeplitz algebras [4–15]. In [14], the authors study the net which is constructed by means of semigroup  $C^*$ -algebra generated by the path semigroup for an arbitrary partially ordered set.

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