



Elementary theories and hereditary undecidability for semilattices of numberings

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Abstract

A major theme in the study of degree structures of all types has been the question of the decidability or undecidability of their first order theories. This is a natural and fundamental question that is an important goal in the analysis of these structures. In this paper, we study decidability for theories of upper semilattices that arise from the theory of numberings. We use the following approach: given a level of complexity, say Σ_α^0 , we consider the upper semilattice $R_{\Sigma_\alpha^0}$ of all Σ_α^0 -computable numberings of all Σ_α^0 -computable families of subsets of \mathbb{N} . We prove that the theory of the semilattice of all computable numberings is computably isomorphic to first order arithmetic. We show that the theory of the semilattice of all numberings is computably isomorphic to second order arithmetic. We also obtain a lower bound for the 1-degree of the theory of the semilattice of all Σ_α^0 -computable numberings, where $\alpha \geq 2$ is a computable successor ordinal. Furthermore, it is shown that for any of the theories T mentioned above, the Π_5 -fragment of T is hereditarily undecidable. Similar results are obtained for the structure of all computably enumerable equivalence relations on \mathbb{N} , equipped with composition.

Keywords Computability theory · Numbering theory · Computably enumerable equivalence relation · Hereditary undecidability · Elementary definability · First order arithmetic · Second order arithmetic · Upper semilattice · Rogers semilattice

Mathematics Subject Classification 03D45

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