ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2018, Vol. 301, pp. 114–133. © Pleiades Publishing, Ltd., 2018.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2018, Vol. 301, pp. 124–143.

# Chern–Simons Action and Disclinations

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Received July 25, 2017

**Abstract**—We review the main properties of the Chern–Simons and Hilbert–Einstein actions on a three-dimensional manifold with Riemannian metric and torsion. We show a connection between these actions that is based on the gauge model for the inhomogeneous rotation group. The exact solution of the Euler–Lagrange equations is found for the Chern–Simons action with the linear source. This solution is proved to describe one straight linear disclination in the geometric theory of defects.

**DOI:** 10.1134/S0081543818040107

## 1. INTRODUCTION

Most physical properties of crystals, such as plasticity, melting, growth, etc., are described by defects in the crystalline structure. The space–time defects (cosmic strings, domain walls) play a significant role in cosmology as well. In spite of hundreds of papers and dozens of monographs devoted to this subject, a fundamental theory of defects is missing at present.

One of the most promising approaches to describe defects is based on the Riemann–Cartan geometry with nontrivial curvature and torsion. In this approach, a crystal is considered as a manifold (continuous medium) with given unit vector field (spin structure). If dislocations are absent, then there exists a continuous displacement vector field which corresponds to diffeomorphisms of Euclidean space. When the displacement vector field has discontinuities, we say that there are defects, called dislocations. This leads to nontrivial geometry. Namely, dislocations correspond to nontrivial torsion, which has the physical meaning of the surface density of the Burgers vector. Defects (discontinuities) of the unit vector field are called disclinations. They correspond to a nontrivial curvature tensor, which has the physical meaning of the surface density of the Frank vector.

The idea of relating dislocations to torsion appeared in the 1950s [2, 18–20]. A review and references to earlier papers can be found in the monograph [17].

In the geometric approach to describing defects [9, 12, 15, 16], there is a model that differs substantially from the other approaches in two respects. First, the displacement and rotation fields are absent in our approach as independent variables, because they are not continuous functions in the presence of defects. Instead, a vielbein and an $\mathbb{SO}(3)$ connection are introduced, which are the only variables. In those domains of the medium where the defects are absent, the vielbein and $\mathbb{SO}(3)$ connection are equal to partial derivatives of the displacement and rotation fields, respectively, and can be restored. Second, the system of equations is significantly different. We propose a purely geometric approach that coincides with Euclidean three-dimensional gravity with torsion. The nonlinear equations of elasticity theory and the principal chiral $\mathbb{SO}(3)$ model for the unit vector field appear in the model by imposing the elastic and Lorentz gauges [7–9]. These gauge conditions

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