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DIFFERENCES OF IDEMPOTENTS IN C^*-ALGEBRAS AND THE QUANTUM HALL EFFECT

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Let φ be a trace on the unital C^*-algebra $\mathcal{A}$ and $\mathfrak{M}_\varphi$ be the ideal of the definition of the trace φ. We obtain a C^* analogue of the quantum Hall effect: if $P, Q \in \mathcal{A}$ are idempotents and $P - Q \in \mathfrak{M}_\varphi$, then $\varphi((P-Q)^{2n+1}) = \varphi(P-Q) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let the isometries $U \in \mathcal{A}$ and $A = A^* \in \mathcal{A}$ be such that $I + A$ is invertible and $U - A \in \mathfrak{M}_\varphi$ with $\varphi(U - A) \in \mathbb{R}$. Then $I - A$, $I - U \in \mathfrak{M}_\varphi$ and $\varphi(I - U) \in \mathbb{R}$. Let $n \in \mathbb{N}$, $\dim \mathcal{H} = 2n + 1$, the symmetry operators $U, V \in \mathcal{B}(\mathcal{H})$, and $W = U - V$. Then the operator W is not a symmetry, and if $V = V^*$, then the operator W is nonunitary.

Keywords: Hilbert space, linear operator, idempotent, symmetry, projection, unitary operator, trace-class operator, C^*-algebra, trace, quantum Hall effect

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1. Introduction

Let P and Q be idempotents in the Hilbert space $\mathcal{H}$. Various properties (invertibility, Fredholm properties, trace-class, positivity, etc.) of the difference $X = P - Q$ were studied in [1]–[6]. Each tripotent ($A = A^3$) is the difference $P - Q$ of some idempotents P and Q with $PQ = QP = 0$ (see Proposition 1 in [7]). Therefore, tripotents inherit some idempotent properties [8]. If X is a trace-class operator, then the traces of all odd powers of X coincide:

$$\operatorname{tr}(P-Q) = \operatorname{tr}((P-Q)^{2n+1}) = \dim\ker(X-I) - \dim\ker(X+I) \in \mathbb{Z}, \tag{1}$$

where I is the identity operator in $\mathcal{H}$. If X is a compact operator, then the right-hand side of (1) yields a natural "regularization" for the trace and shows that it is always an integer [5], [6].

Pairs of idempotents play an important role in the quantum Hall effect [9]. For idempotents P, Q, and R with the trace-class operators $P - Q$ and $Q - R$, from the equality $\operatorname{tr}(P-Q) = \operatorname{tr}(P-R) + \operatorname{tr}(R-Q)$ and (1), we obtain

$$\operatorname{tr}((P-Q)^3) = \operatorname{tr}((P-R)^3) + \operatorname{tr}((R-Q)^3). \tag{2}$$

The physical meaning of the additivity in Eq. (2) comes from the interpretation of $\operatorname{tr}((P-Q)^3)$ as *Hall conductivity*. The additivity of (cubic) Eq. (2) can be considered a variant of Ohm's law for the additivity of conductivity [10].

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