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ISOMORPHISMS OF FORMAL MATRIX RINGS WITH ZERO TRACE IDEALS

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Abstract: We obtain explicit criteria for the isomorphism of formal matrix rings with zero trace ideals. In particular, we consider the case of formal upper-triangular matrix rings with semicentral reduced rings on the principal diagonal.

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1. Introduction

All rings are assumed associative with unity, while modules and bimodules are assumed unitary. The Jacobson radical, the center, and the group of invertible elements of a ring R will be denoted by $J(R)$, $C(R)$, and $U(R)$ respectively.

Formal matrix rings play a substantial role in the theory of rings and modules. An important class of formal matrices is constituted by Morita context rings (see, for instance, [1]). Formal triangular matrix rings often appear in the representation theory of Artin algebras and serve as a source for examples of rings with asymmetric properties (for example, right but not left Artin, etc.).

Each ring with nontrivial idempotents is isomorphic to some formal matrix ring. The endomorphism ring of an indecomposable module is also a formal matrix ring. This justifies the relevance of the study of formal matrix rings.

Let R_1, R_2, … , R_n be rings and let M_{ij} be (R_i, R_j)-bimodules, where $M_{ii} = R_i$, for all $1 \le i, j \le n$. Suppose also that $\varphi_{ijk} : M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ are (R_i, R_k)-bimodule homomorphisms with the reservation that φ_{iij} and φ_{ijj} are canonical isomorphisms for all $1 \le i, j \le n$. Put $a \circ b = \varphi_{ijk}(a \otimes b)$ for $a \in M_{ij}$ and $b \in M_{jk}$. Denote by K the set of all $(n \times n)$-matrices (m_{ij}) with entries $m_{ij} \in M_{ij}$ for all $1 \le i, j \le n$. A simple check shows that K with the usual addition and multiplication is a ring if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a \in M_{ik}$, $b \in M_{kl}$, $c \in M_{lj}$, $1 \le i, k, l, j \le n$. The so-obtained ring K is called a *formal matrix ring* of order n and denoted by $K(\{M_{ij}\} : \{\varphi_{ikj}\})$. In the particular case of $n = 2$, these rings are called *Morita context rings.*

An idempotent e of a ring K is called *left semicentral* [2] if $(1 - e)Ke = 0$. The set of all left semicentral idempotents of K is denoted by $\mathbf{S}_l(K)$. A zero idempotent e of K is *left semicentral reduced* if $\mathbf{S}_l(eKe) = \{0, e\}$. Right semicentral reduced idempotents can be introduced similarly. But since an idempotent e is left semicentral reduced if and only if it is right semicentral reduced, the prefix left/right is usually omitted. A ring K is called *semicentral reduced* if 1 is a semicentral reduced idempotent. This condition is equivalent to the fact that K is strictly indecomposable (see [3]), i.e., if, for every idempotent $e \in K$, the relation $eK(1 - e) = 0$ implies that e is either 0 or 1. If K is strictly indecomposable then it is not representable as a nontrivial Morita context $(M \ne 0)$ of the form $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S

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