Splitting and nonsplitting in the difference hierarchy

Marat M. Arslanov

Dedicated to the fond memory of Barry Cooper

Abstract

In this paper we investigate splitting and non-splitting properties in the Ershov difference hierarchy, in which area major contributions have been made by Barry Cooper with his students and colleagues. In the first part of the paper we give a brief survey of his research in this area and discuss a number of related open questions. In the second part of the paper we consider a splitting of $\mathbf{0}'$ with some additional properties.

The notion of a *computably enumerable (c.e.) set*, i. e. a set of integers whose members can be effectively listed, is a fundamental one. Another way of approaching this definition is via an approximating function $\{A_s\}_{s\in\omega}$ to the set A in the following sense: we begin by guessing $x \notin A$ at stage 0 (i.e. $A_0(x) = 0$); when later x enters A at a stage $s+1$, we change our approximation from $A_s(x) = 0$ to $A_{s+1}(x) = 1$. Note that this approximation (for fixed) x may change at most once as s increases, namely when x enters A. An obvious variation of this definition is to allow more than one change: a set A is *2-c.e.* (or *d-c.e.*) if, for each x, $A_s(x)$ changes at most twice as s increases. This is equivalent to requiring the set A to be the difference of two c.e. sets $A_1 - A_2$. Similarly, one can define n-c.e. sets by allowing n changes at most for each x. A direct generalization of this reasoning leads to sets which are computably approximable in the following sense: for a set A there is a set of uniformly computable sequences $\{f(0,x), f(1,x), ...f(s,x), ...|x \in \omega\}$ consisting of 0 and 1 such that for any x the limit of the sequence $f(0,x), f(1,x), ...$ exists and is equal to the value of the characteristic function $A(x)$ of A. The well-known Shoenfield Lemma states that the class of such sets coincides with the class

[1]The author was partially supported by RFBR Grants 15-01-08252 and 15-41-02507. The author would like to thank the anonymous referees for many suggestions and improvements throughout the paper.

of all Δ^0_2-sets. Thus, for a set A, $A \leqslant_T \emptyset'$ if and only if there is a computable function $f(s,x)$ such that $A(x) = lim_s f(s,x)$.

The notion of d-c.e. and n-c.e. sets goes back to Putnam [1965] and Gold [1965] and was first investigated and generalized by Ershov [1968a,b, 1970]. The arising hierarchy of sets is now known as *the Ershov difference hierarchy.* The position of a set A in this hierarchy is determined by the number of changes in the approximation of A described above, i.e. by the number of different pairs of neighboring elements of the sequence.

The Turing degrees of the sets from the finite levels of the Ershov hierarchy have been intensively studied since the 1970's. It turned out that they (partially ordered by Turing reducibility) have a quite rich inner structure, in many respects reflecting its paramount representative, the class of the c.e. degrees.

Barry Cooper took the first step toward this analysis in his PhD dissertation [1971] where a 2-c.e. (d-c.e.) set was constructed whose Turing degree does not contain any c.e. set. His construction easily generalized to all finite levels of the difference hierarchy: for any $n < \omega$ there is a Turing degree which contains an $(n+1)$-c.e. but no n-c.e. sets. This result for many years remained the only result on the Turing degrees of the n-c.e. sets until Arslanov [1985,1988] and Downey [1989] showed that some pathological properties of c.e. degrees disappear in the difference hierarchy.

After that, Cooper was actively involved in these investigations. First of all he, jointly with Harrington, Lachlan, Lempp and Soare [1991], established a nondensity result for the n-c.e. degrees, $n > 1$, thus giving another difference between these structures: for every $n > 1$, there exists a maximal n-c.e. (in fact a d-c.e.) degree in the n-c.e. degrees. It is natural to ask: how far can this degree $\mathbf{d}$ be from $\mathbf{0}'$? Does it have to be high degree? Is it possible to choose it low? Considering these questions in a joint work we (Arslanov, Cooper and Li [2000,2004]) proved that at least $\mathbf{d}$ cannot be low. Moreover, for any low d-c.e. degree $\mathbf{d}$ any c.e. degree $\mathbf{a}$ above $\mathbf{d}$ is splittable into two incomparable low d-c.e. degrees above $\mathbf{d}$. This raises the natural question: for which $n > 1$ is

- $\mathbf{0}'$ splittable over any low$_n$ d-c.e. degree?
- any c.e. degree splittable over any low$_n$ d-c.e. degree below it?

I think these are open questions.

In general, the investigation of splitting properties occupied a special place in Cooper's research. The reason is that splitting and non-splitting techniques have had a number of consequences for definability and elementary equivalence in degree structures below $\mathbf{0}'$, major research areas in local degree theory.

First of all, Cooper proved (Cooper [1992]) that any n-c.e. degree $> \mathbf{0}$ is splittable in the n-c.e. degrees for any $n > 0$. The proof of this theorem is nonuniform. The methods for dealing with the c.e. case (the Sacks [1963] splitting theorem) and those for the properly n-c.e. case are different.

Later Cooper repeatedly returned to this issue. First, in joint work with Li (Cooper and Li [2002a]) he showed that there is no uniform proof of the splitting theorem for the d-c.e. degrees. Later Cooper and Li [2002b] showed that this non-uniformity leads to the following non-splitting theorem: given $n > 1$, there exist an n-c.e. degree $\mathbf{a}$, and a c.e. degree $\mathbf{b}$ with $\mathbf{0} < \mathbf{b} < \mathbf{a}$ such that any splitting of $\mathbf{a}$ cannot avoid the upper cone of $\mathbf{b}$ (as it can be done in the c.e. case by the Sacks Splitting Theorem): for all n-c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{x} \cup \mathbf{y} = \mathbf{a}$, then either $\mathbf{b} \leq \mathbf{x}$ or $\mathbf{b} \leq \mathbf{y}$. In this work it was also noted that the degree of $\mathbf{a}$ can be made low_3, thus not every low_3 n-c.e. degree $\mathbf{a}$ is splittable in the n-c.e. degrees avoiding upper cones of c.e. degrees below $\mathbf{a}$. Since in the c.e. degrees such a splitting of low_3 c.e. degrees is possible by the Sacks splitting theorem, this result provided a nice elementary difference between the low_3 c.e. and the low_3 d-c.e. degrees. Earlier, Cooper [1991] had shown that in the low_2 n-c.e. degrees density and splitting properties can be combined: for all low_2 n-c.e. degree $\mathbf{a} < \mathbf{b}$, there exist n-c.e. degrees $\mathbf{x}_0, \mathbf{x}_1$ such that $\mathbf{a} < \mathbf{x_0}, \mathbf{x_1} < \mathbf{b}$, and $\mathbf{x}_0 \cup \mathbf{x}_1 = \mathbf{b}$. Since this result also holds for the low_2 c.e. degrees (Harrington, unpubl., see Shore and Slaman [1990] for the proof) the question of the elementary equivalence of the low_2 c.e. and low_2 d-c.e. degrees arises. Faizrahmanov [2010] proved that the low c.e. and the low d-c.e. degrees are not elementarily equivalent: it is known (Welch [1980]) that in the c.e. degrees the following sentence holds: there are low c.e. degrees $\mathbf{x}_0, \mathbf{x}_1$ such that for any c.e. degree $\mathbf{y}$, there are c.e. degrees $\mathbf{y}_0 \leq \mathbf{x}_0$ and $\mathbf{y}_1 \leq \mathbf{x}_1$ with $\mathbf{y} = \mathbf{y}_0 \cup \mathbf{y}_1$. Faizrachmanov proved that for all low d-c.e. degrees $\mathbf{x}_0$ and $\mathbf{x}_1$ there is a low d-c.e. degree $\mathbf{y}$ such that $\mathbf{x}_0 \cup \mathbf{x}_1 \neq \mathbf{y}$. These results leave open the question of the elementary equivalence of the semilattices of low_n c.e. and low_n d-c.e. degrees for $n = 2$ and $n > 3$. So far we have no examples which would distinguish these structures.

One of the most important open problems that Cooper was interested in is the problem of the definability of the c.e. degrees in the broader classes of n-c.e. degrees. In particular, Cooper and Li [2002c] proved that in the class of d-c.e. degrees the c.e. degrees have the following property: if $\mathbf{a}$ is c.e. then for every d-c.e. degree $\mathbf{d} > \mathbf{a}$, $\mathbf{d}$ is splittable above $\mathbf{a}$ into two incomparable d-c.e. degrees. Since there is a maximal d-c.e. degree, not all d-c.e. degrees $\mathbf{a}$ have this property. Thus, we have the following non-trivial d-c.e. *Turing approximation* to the class of the c.e. degrees: for every c.e. degree $\mathbf{a}$ each d-c.e. degree $\mathbf{b} > \mathbf{a}$ is splittable in the d-c.e. cone above $\mathbf{a}$. (A Turing approximation to a class S of degrees is a Turing definable class A for which $A \subset S$ or $S \subset A$.) If any properly d-c.e. degree $\mathbf{a}$ had above it a d-c.e. degree which is not so splittable, we would get the following natural definition of the c.e. degrees in the class of the d-c.e. degrees: $\mathbf{a} < \mathbf{0}'$ is c.e. if and only if $\forall \mathbf{b} > \mathbf{a} \exists \mathbf{x_0}, \mathbf{x_1} (\mathbf{a} < \mathbf{x_0}, \mathbf{x_1} < \mathbf{b} \ \& \ \mathbf{b} = \mathbf{x_0} \cup \mathbf{x_1})$. Although in the literature the (positive or negative) answer to this question (as far as I know) has not been published, it is not difficult to see that, using the technique of the Robinson Splitting Theorem for the c.e. case (see Soare [1987, p.224]), it can be proved that there is a low properly d-c.e. degree such that any d-c.e. degree above it so splittable.

Cooper's idea of using d-c.e. sets and 2-CEA operators induced by these sets to obtain a natural definition of $\mathbf{0}'$ and, relativising, of the jump operator, was a culmination of these investigations. Cooper [1990] claimed the existence of a d-c.e. set and a 2-CEA operator J induced by the construction of this set such that for every set C there are degrees $\mathbf{a}$ and $\mathbf{b}$ with the following property: the Turing degree $\deg(J(C))$ is not splittable over $\mathbf{a}$ avoiding $\mathbf{b}$. Such a result would have provided a very pleasant natural definition of $\mathbf{0}'$ as the largest degree $\mathbf{x}$ satisfying $\neg(\exists \mathbf{a}, \mathbf{b})(\mathbf{x} \cup \mathbf{a}$ is unsplittable over $\mathbf{a}$ avoiding $\mathbf{b})$, and, by relativization, a natural definition of the jump operator, as well as of the relation "$\mathbf{a}$ is c.e. in $\mathbf{b}$". Although this specific claim, as it turned out, was erroneous (Shore and Slaman [2001]), it seems likely that his basic idea may be resurrected in some modified form, and we would get a natural definition of the jump operator along Cooper's line of thought. (The definability of the jump operator has been proved by Shore and Slaman [1999], but, as the authors themselves write (see, for instance Shore [2000, p.263]), their definition cannot be considered natural.)

It was mentioned already that any d-c.e. degree $\mathbf{a} > \mathbf{0}$ is splittable into two d-c.e. degrees $\mathbf{c}_0$ and $\mathbf{c}_1$ which are above a given c.e. degree $\mathbf{b} < \mathbf{a}$: $\mathbf{c}_0 \cup \mathbf{c}_1 = \mathbf{a}$ and $\mathbf{b} < \mathbf{c}_0, \mathbf{c}_1 < \mathbf{a}$, and that any c.e. degree is splittable in the d-c.e. degrees over any *low* d-c.e. degree. Below we prove that $\mathbf{0}'$ is splittable into c.e. degrees $\mathbf{v}_0$ and $\mathbf{v}_1$ such that for every d-c.e. degree $\mathbf{d}$ and each $i \leq 1$, if $\mathbf{v}_i \leq \mathbf{d}$, then $\mathbf{d}$ is c.e. in $\mathbf{v}_{1-i}$.

We adopt the usual notational conventions, found, for instance, in Soare [1987]. In particular, for an oracle X and c.e. functional Φ, $\Phi(X; y, s)$ means that s steps are made in the computation from oracle X. For a partial computable (p.c.) functional Φ say, the use function is denoted by the corresponding lower case letter φ. When using a c.e. oracle, we adopt the common practice of taking the use function to be nondecreasing in the stage.

Theorem 1 *There is a splitting of* $\mathbf{0}'$ *into two incomparable c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *with an infimum* $\mathbf{h}$ *of high degree such that for every d-c.e. degree* $\mathbf{d}$, *if* $\mathbf{a} \leq \mathbf{d}$, *then* $\mathbf{d}$ *is c.e. in* $\mathbf{b}$, *and if* $\mathbf{b} \leq \mathbf{d}$, *then* $\mathbf{d}$ *is c.e. in* $\mathbf{a}$.

Proof. We will construct c.e. sets V_0 and V_1 such that $\emptyset' \leq_T V_0 \oplus V_1$ and the degrees $\mathbf{v}_i = \deg V_i$ have some of the desired properties: for every d-c.e.

degree $\mathbf{d}$ and each $i \leq 1$, if $\mathbf{v}_i \leq \mathbf{d}$, then $\mathbf{d}$ is c.e. in $\mathbf{v}_{1-i}$.

Let $\mathbf{h}_0$ be a noncuppable high c.e. degree (Harrington, unpubl.; see Miller [1981]), and let $\mathbf{u}_0 = \mathbf{h}_0 \cup \mathbf{v}_0$, $\mathbf{u}_1 = \mathbf{h}_0 \cup \mathbf{v}_1$. Then $\mathbf{u}_0$ and $\mathbf{u}_1$ are both $< \mathbf{0}'$ and $\mathbf{u}_0 \cup \mathbf{u}_1 = \mathbf{0}'$. Now for the "infimum" part of the theorem we use a result from Downey, LaForte and Shore [2003, Theorem 2.1][2] which states that for the degrees $\mathbf{u}_0$ and $\mathbf{u}_1$ there exists a c.e. degree $\mathbf{h}$ such that $(\mathbf{u}_0 \cup \mathbf{h}) \cap (\mathbf{u}_1 \cup \mathbf{h}) = \mathbf{h}$ and $\mathbf{u}_0 \cup \mathbf{h} \mid \mathbf{u}_1 \cup \mathbf{h}$.

Let $\mathbf{a} = \mathbf{u}_0 \cup \mathbf{h}$, $\mathbf{b} = \mathbf{u}_1 \cup \mathbf{h}$. It is clear that $\mathbf{h}_0 \leq \mathbf{h}$. We have $\mathbf{a}$ and $\mathbf{b}$ are both $< \mathbf{0}'$, $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$, and the high degree $\mathbf{h} = \mathbf{a} \cap \mathbf{b}$. It is also clear that for every d-c.e. degree $\mathbf{d}$, if $\mathbf{a} \leq \mathbf{d}$, then $\mathbf{d}$ is c.e. in $\mathbf{b}$, and if $\mathbf{b} \leq \mathbf{d}$, then $\mathbf{d}$ is c.e. in $\mathbf{a}$.

Let $\{D_e\}_{e \in \omega}$ be a fixed effective enumeration of all d-c.e. sets. Simultaneously with the c.e. sets V_0 and V_1 we will also construct auxiliary c.e. sets U_0, U_1 to satisfy the following two types of requirements.

To ensure that $\emptyset' \not\leq_T V_i$, we satisfy requirements

- P^i_e: $U_i \neq \Theta^{V_i}_{i,e}$ (for each partial computable functional $\Theta_{i,e}$).

To ensure that for all d-c.e. sets D if $V_i \leq_T D$ then D is of degree c.e. in V_{1-i}, we satisfy requirements

- R^i_e: $D_e = \Delta^{V_i \oplus Q^i_e}_{i,e}$ & $Q^i_e \leq_T V_i \oplus D_e$ (for each d-c.e. set D_e we build a partial computable functional $\Delta_{i,e}$ and an associated d-c.e. set Q^i_e such that Q^i_e is c.e. in V_{1-i}).

  The condition $Q^i_e \leq_T V_i \oplus D_e$ will be met by a permitting argument.

To ensure that Q^i_e is c.e. in V_{1-i} we use a common method which works as follows. When an integer x is enumerated into Q^i_e at stage s we appoint a certain marker $\alpha(x)$. Then we allow x to be removed from Q^i_e at a later stage t only if $V_{1-i} \upharpoonright \alpha(x)[s] \neq V_{1-i} \upharpoonright \alpha(x)[t]$.

The condition $\emptyset' \leq_T V_0 \oplus V_1$ will be deduced from other properties of the sets constructed.

The basic strategy for P-requirements in isolation is the one developed by Friedberg and Muchnik:

(1) Pick an unused witness x from the column associated with this requirement ($\langle i, \omega \rangle$ for $i \leq 1$) which is larger than all higher priority restraints and keep it out of U_i.

[2] The author thanks Guohua Wu, who drew the author's attention to this work which allowed us to simplify this part of the proof.

(2) Wait for $\Theta_e^V(x)\downarrow= 0$.

(3) Put x into U_i and protect $V_i \upharpoonright (\theta(x)+1)$.

The basic strategy for R-requirements in isolation is to build $\Delta^{V\oplus Q}$, ensuring that it is total and computes D correctly. Since we build the set V during the construction, we may easily meet this requirement by changing V, if necessary.

Before giving the explicit construction we first explain the intuition for the R-requirements. We note that only P-requirements participate in the priority ordering.

Basic module for the R_e^i-strategy.

We use an ω-sequence of "cycles", where each cycle k proceeds as follows:

(1) At a stage s set $\Delta_{i,e}^{V_i\oplus Q_e^i}(k) = D_{e,s}(k)$ with *use* $\delta_{i,e,s}(k) >$ all P-restraints, and $\delta_{i,e,s}(k) > \delta_{i,e,s}(k-1)$ and start cycle $k+1$ to run simultaneously.

(2) Wait for $D_e(k)$ to change (at a stage t, say).

(If between stages s and t there is a $V_i \oplus Q_e^i\lceil(\delta_{i,e,s}(k)+1)$-change we correct $\Delta_{i,e}^{V_i\oplus Q_e^i}(k) = D_{e,s}(k)$ with the same use.)

(3) (i) enumerate $\delta_{i,e,s}(k)$ into Q_e^i,

(ii) set $\Delta_{i,e}^{V_i\oplus Q_e^i}(k)[t] = D_{e,t}(k)$ with the same *use* $\delta_{i,e,t}(k) = \delta_{i,e,s}(k)$, and

(iii) appoint the marker $\alpha_{i,e}(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,t}(k)$ and $y = \langle 2, l\rangle$ for some l.

(4) Wait for $D_e(k)$ to change back (at a stage u, say).

(5) We need

–to keep Q_e^i "below" $V_i \oplus D_e$ (at stage t, k enters D_e and we put $\delta_{i,e,s}(k)$ into Q_e^i. Now k leaves D_e).

–to correct the axiom $\Delta_{i,e}^{V_i\oplus Q_e^i}(k) = D_e(k)$

We have two possibilities to achieve this:

- either by enumerating $\delta_{i,e,s}(k)$ into V_i,
- or by removing $\delta_{i,e,s}(k)$ from Q_e^i (in this case we need to enumerate $\alpha_{i,e}(\delta_{i,e,s}(k))$ into V_{1-i}).

The crucial point here is that our choice between these two possibilities depends upon the priority ordering of requirements P^i and P^{1-i} that may be injured:

a) If the highest priority strategy which would be injured by either of these corrections is a P^i-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_{i,e}(\delta_{i,e,s}(k))$ into V_{1-i} and remove $\delta_{i,e,s}(k)$ from Q^i_e.

b) Otherwise, put $\delta_{i,e,s}(k)$ into V_i.

Set $\Delta^{V_i \oplus Q^i_e}_{i,e}(k)[u] = D_{e,u}(k)$ with the same *use* $\delta_{i,e,u}(k) = \delta_{i,e,t}(k)$.

In both cases start cycle $k+1$ to run simultaneously.

We now give the construction. We say that the axiom $\Delta^{V_i \oplus Q^i_e}_{i,e}(k) = D_e(k)$ *requires correction* at stage s if at a stage $t < s$ we set $\Delta^{V_i \oplus Q^i_e}_{i,e}(k)[t] = D_{e,t}(k)$ with a *use* $\delta_{i,e,t}(k)$, $D_{e,s}(k) \neq D_{e,t}(k)$, and $(V_i \oplus Q^i_e)[t] \upharpoonright \delta_{i,e,t}(k) = (V_i \oplus Q^i_e)[s-1] \upharpoonright \delta_{i,e,t}(k)$.

Stage $s = 0$. Set $U_0 = V_0 = V_1 = \emptyset[0]$. $x^0_{e,0} =< 0, e >$, $x^1_{e,0} =< 1, e >$.

Stage $s > 0$. Fix e such that $s = \langle e, m \rangle$ for some m.

Substage 1 (P^0_e-requirement).

a) If $\Theta^{V_0}_{0,e}(x^0_e)[s] \downarrow= 0$ and $x^0_{e,s-1} \notin U_{0,s-1}$, then enumerate $x^0_{e,s-1}$ into $U_{0,s}$, and protect $V_0 \upharpoonright \theta_{0,e,s}(x^0_{e,s-1})$ with priority P^0_e.

b) If $\Theta^{V_0}_{0,e}(x^0_e)[s] \downarrow= U_{0,s}(x^0_{e,s-1}) = 1$, then define

$$x^0_{e,s} = (\mu x)[(\exists y)(\forall j)(\forall i \leq 1)(x = \langle 0, y \rangle \wedge x > \text{all current } P\text{-uses}].$$

Otherwise set $x^0_{e,s} = x^0_{e,s-1}$.

Substage 2 (P^1_e-requirement). Similar to the previous case with the necessary changes (replacing $\Theta^0_e, V_0, U_0, x^0_e, \theta_{0,e}$ by $\Theta^1_e, V_1, U_1, x^1_e, \theta_{1,e}$ accordingly).

Substage 3 ($R_{e,i}$-requirement). Let z be the greatest integer such that for every $k < z$ there exists a stage $s' < s$ such that at stage s' the axiom $\Delta^{V_i \oplus Q^i_e}_{i,e}(k)[s'] = D_e(k)[s']$ was set. Let $k < z$ be the smallest integer (if any) such that the axiom $\Delta^{(V_i \oplus Q^i_e)}_{i,e}(k) = D_e(k)$ requires correction at stage s. Let t be a stage at which the axiom $\Delta^{(V_i \oplus Q^i_e)}_{i,e}(k) = D_e(k)$ was set.

We consider two cases.

Case 1) $D_{e,s}(k) = 1$. In this case we proceed as in step (3) of the Basic Module:

(i) enumerate $\delta_{i,e,t}(k)$ into Q^i_e,

(ii) set $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_{e,s}(k)$ with the same *use* $\delta_{i,e,s}(k) = \delta_{i,e,t}(k)$, and

(iii) appoint the marker $\alpha_{i,e}(\delta_{i,e,s}(k))$ as the first integer y such that $y \geq \delta_{i,e,s}(k)$ and $y =< 2, l >$ for some l.

Case 2) $D_{e,s}(k) = 0$. Therefore, there is a stage u with $t \leq u < s$ such that $D_{e,u}(k) = 1$, and at stage u we (re)set the axiom $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_e(k)$.

$2a$) If at stage u we set the axiom $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_e(k) = 1$, then enumerate $\delta_{i,e,u}(k)$ into Q_e^i and set $\Delta_{i,e}^{V_i \oplus Q_e^i}(k)[s] = D_{e,s}(k)$ with $\delta_{i,e,s}(k) = \delta_{i,e,u}(k)$.

$2b$) If at stage u we reset the axiom $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_e(k) = 1$, then it follows that at stage u we enumerated $\delta_{i,e,t}(k)$ into Q_e^i. In this case we proceed as in step (5) of the Basic Module:

$2b_1$) If the highest priority strategy which would be injured by the $Q_e^i(\delta_{i,e,t}(k))-$ or $V_i(\delta_{i,e,t}(k))-$ correction is a P^i-strategy (or there is no strategy at all that would be injured), then enumerate $\alpha_{i,e}(\delta_{i,e,t}(k))$ into V_{1-i}, and remove $\delta_{i,e,t}(k)$ from Q_e^i.

$2b_2$) Otherwise, put $\delta_{i,e,t}(k)$ into V_i.

If $\Delta_{i,e}^{V_i \oplus Q_e^i}(k)[s] \uparrow$ then set $\Delta_{i,e}^{V_i \oplus Q_e^i}(k)[s] = D_{e,s}(k)$.

Substage 4. If none of the axioms $\Delta_{i,e}^{V_i \oplus Q_e^i}(k) = D_e(k)$ for $k < z$ requires correction at stage s, then set a new axiom $\Delta_{i,e}^{V_i \oplus Q_e^i}(z) = D_{e,s}(z)$ with *use* $\delta_{i,e,s}(z) >$ all P-restraints.

Substage 5. Go to stage $s + 1$.

End of the construction.

Verification.

Let $\mathbf{v}_i$=deg(V_i), $i \leq 1$.

Lemma 1. $Q_e^i \leq_T V_i \oplus D_e$.

Proof. To compute relative to $V_i \oplus D_e$ whether $x \in Q_e^i$, first find a stage u at which a new axiom $D_e(y) = \Delta_{i,e}^{V_i \oplus Q_e^i}(y)$ with a *use* $\delta_{i,e,u}(y) \geq x$ was set. Obviously, such a stage u exists.

Suppose now that $x = \delta_{i,e,s}(k)$ was chosen as a *use* for some $\Delta_{i,e}^{V_i \oplus Q_e^i}(k)$ at a stage $s \leq u$ (otherwise $x \notin Q_e^i$). Find a stage $v \geq u$ at which $V_{i,v} \upharpoonright (x+1) = V_i \upharpoonright (x+1)$ and $D_{e,v}(k) = D_e(k)$. Now $x \in Q_e^i$ if and only if $x \in Q_{e,v}^i$.

To show this, consider the following two possibilities.

1) $D_{e,s}(k) = 1$. In this case by the construction $x \in Q^i_e$ if and only if $k \notin D_{e,v}$.

2) $D_{e,s}(k) = 0$. In this case by the construction either x never enters Q^i_e, or at a stage $w > s$ it enters Q^i_e, but then it remains there only if $x \in V_i$ (case $2b_2$ of the construction). In any case after $V_{i,v} \upharpoonright (x+1) = V_i \upharpoonright (x+1)$ there is no $Q^i_e(x)$-change. □

Lemma 2. Q^i_e is c.e. in V_{1-i}.

Proof. It follows immediately from the construction. □

Lemma 3. Let $\mathbf{d}_e$=deg(D_e). If $V_i \leq_T D_e$, then $\mathbf{d}_e$ c.e. in $\mathbf{v}_{1-i}$.

Proof. Let $V_i \leq_T D_e$. We have $Q^i_e \leq_T V_i \oplus D_e$ and, by construction, $D_e \leq_T V_i \oplus Q^i_e$. It follows that $D_e \equiv_T V_i \oplus D_e \equiv_T V_i \oplus Q^i_e$. Since Q^i_e is c.e. in V_{1-i}, we have $\mathbf{d}_e$ c.e. in $\mathbf{v}_{1-i}$. □

Lemma 4. For each $i \leq 1$ and $e \in \omega$, the requirements P^i_e are eventually satisfied.

Proof. Fix e and assume by induction that the Lemma holds for all $j < e$. Choose s minimal so that no P^i_j-restraints may be injured by some R requirement. By construction we may injure P^i_e by finitely many times contributing some integers into V_i to protect the V_{1-i}-restraint of higher priority. But beginning at some stage s we take witnesses for Δ-*uses* greater than the V_{1-i}-restraints, after that we meet the requirement P^i_e. □

Lemma 5. $\mathbf{0}' = \mathbf{v_0} \cup \mathbf{v_1}$.

Proof. Suppose for a contradiction that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{0}'$. Then by Arslanov, Lempp and Shore [1996, Theorem 4.1] there exists a d-c.e. degree $\mathbf{d}$ such that $\mathbf{v_0} \cup \mathbf{v_1} < \mathbf{d}$ and $\mathbf{d}$ is not c.e. in $\mathbf{v_0} \cup \mathbf{v_1}$ and, therefore, it is not c.e. in $\mathbf{v}_1$. We have $\mathbf{v}_0 < \mathbf{d}$ and $\mathbf{d}$ is not c.e. in $\mathbf{v_1}$, a contradiction. □

References

Arslanov M. M. [1985] Structural properties of the degrees below $\mathbf{0}'$, *Sov. Math. Dokl. N.S.*, **283**, 270-273.

Arslanov M. M. [1988] On the upper semilattice of Turing degrees below $\mathbf{0}'$, *Russian Mathematics (Iz. VUZ)*, **7**, 27-33.

Arslanov M. M., Cooper S.B. and Li A. [2000] There is no low maximal d.c.e. degree, *Math. Logic Quart.*, **46**, 409-416.

Arslanov M. M., Cooper S.B. and Li A. [2004] There is no low maximal d.c.e. degree - Corrigendum, *Math. Logic Quart.*, **50**, 628-636.

Arslanov M.M., Kalimullin I.Sh., Lempp S. [2010] On Downey's conjecture, *J. Symb. Logic*, **75**, 401-441.

Arslanov, M. M., Lempp, L. and Shore, R. A. [1996] On isolating r.e. and isolated d-r.e. degrees, *London Math. Soc. Lect. Note Series* **224**, Cambridge University Press, 61-80.

Cooper, S.B. [1971] *Degrees of Unsolvability*, Ph. D. Thesis, Leicester University, Leicester, England.

Cooper S.B. [1990] The jump is definable in the structure of the degrees of unsolvability, *Bull. Amer. Math. Soc.*, **23**, 151-158.

Cooper S.B. [1991] The density of the Low2 n-r.e. degrees, *Archive for Math. Logic*, **31**, 19-24

Cooper S.B. [1992] A splitting theorem for the n-r.e. degrees, *Proc. Amer. Math. Soc.*, **115**, 461-471

Cooper, S.B., Harrington, L., Lachlan, A.H., Lempp, S. and Soare, R.I. [1991] The d-r.e. degrees are not dense, *Ann. Pure and Applied Logic* , **55**, 125-151.

Cooper S.B. and Li A. [2000a] Non-uniformity and generalised Sacks splitting, *Acta Mathematica Sinica, English Series*, **18**, 327-334.

Cooper S.B. and Li A. [2002b] Splitting and cone avoidance in the d.c.e. degrees, *Science in China (Series A)*, **45**, 1135-1146.

Cooper S.B. and Li A. [2002c] Turing definability in the Ershov hierarchy, *J. London Math. Soc.* (2), **66**, 513-528.

Downey R.G. [1989] D.r.e. degrees and the Nondiamond Theorem, *Bull. London Math. Soc.*, **21**, 43-50.

Downey R.A., Laforte G.A. and Shore R.I. [2003] Decomposition and infima in the computably enumerable degrees, *J. Symb. Logic*, **68**, 551-579.

Ershov Y.L. [1968a] On a hierarchy of sets I, *Algebra i Logika*, **7**, no. 1, 47-73.

Ershov Y.L. [1968b] On a hierarchy of sets II, *Algebra i Logika*, **7** , no. 4, 15-47.

Ershov, Y.L. [1970] On a hierarchy of sets III, *Algebra i Logika*, **9** , no. 1, 34-51.

Faizrakhmanov M.Kh. [2010] Decomposability of low 2-computably enumerable degrees and Turing jumps in the Ershov hierarchy, *Russian Mathematics (Iz. VUZ)*, **12**, 51-58

Gold, E. M. [1965] Limiting recursion, *J. Symb. Logic*, **30**, 28-48.

Miller D. [1981] High recursively enumerable degrees and the anticupping property, in: Logic Year 1979-80: University of Connecticut, 230-245.

Putnam H. [1965] Trial and error predicates and the solution to a problem of Mostowski, J. Symb. Logic, **30**, 49-57.

Shore R.A. [2000] Natural definability in degree structures, in: Computability Theory and Its Applications: Current Trends and Open Problems (P. Cholak, S. Lempp, M. Lerman and R. A. Shore eds.), *Contemporary Mathematics*, AMS, Providence RI, 255-272.

Shore R.A. and Slaman T.A. [1999] Defining the Turing jump, *Math. Research Letters*, **6**, 711-722.

Shore R.A. and Slaman T.A. [2001] Working below a low2 recursively enumerable degrees, *Archive for Math. Logic*, **29**, 201-211.

Soare R. I. [1997] *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, 1997.

Welch L. [1981] A hierarchy of families of recursively enumerable degrees and a theorem on bounding minimal pairs, Ph. D. Thesis (University of Illinois, Urbana, 1980).