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ITERATIVE METHODS FOR THE GRID APPROXIMATIONS OF VARIATIONAL INEQUALITIES WITH CONSTRAINTS ON THE GRADIENT OF SOLUTION [1)]

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ИТЕРАЦИОННЫЕ МЕТОДЫ ДЛЯ СЕТОЧНЫХ АППРОКСИМАЦИЙ ВАРИАЦИОННЫХ НЕРАВЕНСТВ С ОГРАНИЧЕНИЯМИ НА ГРАДИЕНТ РЕШЕНИЯ

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Summary

The iterative solution methods are investigated for the finite difference approximations of variational inequalities with diffusion-convection operator and constraints on the gradient of solution. The methods are easily implementable because on every iterative step one has to solve only a system of linear equations (even in the case of nonlinear diffusion-convection operator) and a set of the minimization problems of small dimensions. The results on the convergence of the proposed methods are given. Computational experiments confirm the high rate of their convergence.

Key words: Variational inequality, finite difference method, iterative method, constrained saddle point problem.

Аннотация

Исследованы итерационные методы для конечно-разностных аппроксимаций вариационных неравенств с оператором диффузии-конвекции и ограничениями на градиент решения. Методы легко реализуемы, потому что на каждом итерационном шаге требуется решить лишь систему линейных уравнений (даже в случае нелинейного оператора) и множество задач минимизации малой размерности. Приведены результаты о сходимости предложенных методов. Вычислительные эксперименты подтверждают их высокую скорость сходимости.

Ключевые слова: Вариационное неравенство, конечно-разностный метод, итерационный метод, седловая задача с ограничениями.

Introduction

Variational inequalities with constraints on the gradient of the solution are mathematical models of a number of physical problems. These include the problem of elastic-plastic deformation, flow of Bingham viscoplastic materials, problem of nonlinear fluid flow in porous media. Practically all known methods for solving these problems are based on the introduction of an auxiliary function that is equal to the gradient of the solution, and finding the saddle point of the corresponding Lagrange function. Widely used algorithms based on the construction of so-called augmented Lagrangian technique [1], [2]. Slightly different approach to solving this class of problems have been proposed in the articles [3], [4]. Namely, preconditioned Uzawa-type

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iterative methods for the equivalently transformed constrained saddle point problems have been proposed and investigated. All of the above methods can be effectively implemented when solving the variational inequalities with potential operators: on every iterative step one has to solve only a system of linear equations and a set of two-dimensional minimization problems. But this is not so in the case of variational inequalities with nonlinear operators, depending on both solution and its gradient. For these problems the algorithms from [1] - [4] includes the solution of a system of nonlinear equations instead of linear ones, and this is the most time consuming part of the algorithms.

In [5] and [6] a new method was proposed and investigated for a finite dimensional constrained saddle point problem with linear and non-linear operators. This method being applied to a grid approximation of variational inequality with nonlinear diffusion-convection operator keeps the same efficiency of the implementation as in the case of potential operator. In this article we report the results on the iterative solution methods for finite difference approximation of a variational inequality with linear and nonlinear diffusion-convection operator and constraints on the gradient of solution. Results on the convergence of the iterative methods are cited and numerical results for several model problems are given.

1. Formulation of the problem.

Let Ω is a bounded domain with piecewise smooth boundary. Define the functions $g_1(\bar t) : \mathbb{R}^2 \to \mathbb{R}^2$ and $g_2(s, \bar t) : \mathbb{R}^3 \to \mathbb{R}$ which satisfy for all $\bar t \in \mathbb{R}^2$ and $s \in \mathbb{R}$ the following assumptions:

$$\begin{cases} g_1(\bar t) \text{ and } g_2(s,\bar t) \text{ are continuous}, \\ |g_1(\bar t)| \leqslant c|\bar t|, \quad (g_1(\bar t_1) - g_1(\bar t_2), \bar t_1 - \bar t_2) \geqslant \alpha |\bar t_1 - \bar t_2|^2, \quad \alpha > 0, \\ |g_2(s_1, \bar t_1) - g_2(s_2, \bar t_2)| \leqslant \beta_1 |s_1 - s_2| + \beta_2 |\bar t_1 - \bar t_2|, \; \beta_i \geqslant 0. \end{cases}$$

Under formulated assumptions on the functions g_1 and g_2 semilinear form

$$a(u,v) = \int_\Omega g_1(\nabla u) \cdot \nabla v \, dx + \int_\Omega g_2(u, \nabla u) v \, dx$$

is bounded from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $\mathbb{R}$. We suppose that it is also strongly monotone:

$$a(u, u-v) - a(v, u-v) \geqslant \sigma \|u\|^2_{H_0^1(\Omega)} = \sigma \int_\Omega |\nabla u|^2 \, dx, \quad \sigma > 0. \tag{1}$$

This property is ensured, for example, if $\alpha - |k|\beta_1 c_f^2 - b\beta_2 c_f \equiv \sigma > 0$, where c_f is the constant in Friedrichs inequality: $\|u\|_{L^2(\Omega)} \leqslant c_f \|u\|_{H_0^1(\Omega)}$. In particular case $g_2(u, \nabla u) = \bar v \cdot \nabla u$ property (1) is true for any constant vector $\bar v$.

Let further $\psi(z) : L_2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $f \in L_2(\Omega)$. Two classical examples of $\psi(z)$ are: $I(z) = \int_\Omega |z(x)| dx$ and the indicator function $I_K(z)$ of the convex and closed set $K = \{z \in L_2(\Omega) : |z(x)| \leqslant 1 \; \forall x \in \Omega\}$.

We consider the variational inequality: find $u \in H_0^1(\Omega)$ such that

$$a(u, v-u) + \psi(|\nabla v|) - \psi(|\nabla u|) \geqslant \int_\Omega f(v-u) \, dx \quad \forall v \in H_0^1(\Omega). \tag{2}$$

Because of the aforementioned properties of the form $a(.,.)$ and functional ψ variational inequality (2) has a unique solution (see e.g. [7], Chapter 2, Theorem 3.1).

2. Finite difference approximation.

Let Ω be unit square $(0,1) \times (0,1)$. Approximate problem (2) by a finite difference problem on the uniform grid $\bar\omega = \{x = (ih, jh) : 0 \leqslant i, j \leqslant m+1, \; (m+1)h = 1\}$. By V_h we denote the space of grid functions, which are defined in the nodes of $\bar{\omega}$ and equal to 0 in the nodes of $\partial\omega$. The finite difference approximation of variational inequality (2) reads as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h - u_h) + \psi_h(|\overline{\nabla}_h v_h|) - \psi_h(|\overline{\nabla}_h u_h|) \geqslant f_h(v_h - u_h) \; \forall v_h \in V_h, \tag{3}$$

where a_h, ψ_h and f_h are the approximations of a, ψ and f, respectively, while $\overline{\nabla}_h u_h = (\overline{\partial}_1 u_h, \overline{\partial}_2 u_h)$ is the grid gradient. Let $y \in \mathbb{R}^{m^2}$ be the vector of nodal values of a grid function u_h, matrix L corresponds to $\overline{\nabla}_h$, and convex, proper and lower semicontinuous function $\varphi : \mathbb{R}^{m^2} \to \mathbb{R} \cup \{+\infty\}$ corresponds to ψ_h. Then grid problem (3) is equivalent to the following discrete variational inequality: find $y \in \mathbb{R}^{m^2}$ such that

$$(L^T g_1(Ly) + g_2(y, Ly), v - y) + \varphi(Lv) - \varphi(Ly) \geqslant (f, v - y) \;\; \forall v \in \mathbb{R}^{m^2}. \tag{4}$$

Suppose the Slater condition be satisfied:

$$\exists\, u_0 \text{ such that the vector } Lu_0 \in \operatorname{int}\operatorname{dom}\varphi.$$

Then $\partial[\varphi(Lu)] = L^T \partial\varphi(Lu)$, (cf. [7], Chapter 1, proposition 5.7) and (4) is equivalent to inclusion

$$L^T g_1(Lu) + g_2(u, Lu) + L^T \partial\varphi(Lu) \ni f. \tag{5}$$

Note that for two mentioned above particular cases of the functional ψ, $\psi = I$ and $\psi = I_K$, Slater condition is satisfied with $u_0 = 0$.

Note 1. *1) Discrete variational inequality of the form* (4) *(respectively, inclusion* (5)*) can be obtained for the nodal parameters of the mesh function when solving the problem by finite element method with linear or bilinear elements.*

2) Inclusion (5) *can be obtained by approximation directly differential inclusion (formal writing of variational inequality* (2)*):*

$$-\mathrm{div} g_1(\nabla u) + k\, g_2(u, \nabla u) - \mathrm{div}\partial\psi(|\nabla v|) \ni f.$$

1. Construction of saddle point problem.

Define the auxiliary vectors p and λ: $p = Lu$ and $\lambda \in g_1(p) + \partial\varphi(p)$. Then the triple (u, p, λ) satisfies the system

$$g_2(u, p) + L^T\lambda = f, \; g_1(p) + \partial\varphi(p) - \lambda \ni 0, \; Lu - p = 0.$$

We make an equivalent transformation of this system by multiplying the equation $p - Lu = 0$ by rL^T, $r = \mathrm{const}$, and adding to the first inclusion. This results in the following saddle point problem:

$$rL^T Lu - rL^T p + g_2(u, p) + L^T\lambda = f, \; g_1(p) + \partial\varphi(p) - \lambda \ni 0, \; Lu - p = 0. \tag{6}$$

Lemma 1. *Let $x = (u, p)^T$ and*

$$A(x) = \begin{pmatrix} rL^T Lu - rL^T p + g_2(u, p) \\ g_1(p) \end{pmatrix}.$$

Then it is strongly monotone if the parameter r is such that

$$0 < 2\alpha - k\beta_2 c_f - 2\sqrt{\alpha\sigma} < r < 2\alpha - k\beta_2 c_f + 2\sqrt{\alpha\sigma},$$

and problem (6) *has a solution (u, p, λ) with the unique components (u, p) (see [6]).*

Further we take for the definiteness $r = 2\alpha - k\beta_2 c_f$.

1. Iterative solution method.

Consider the following method for problem (6):

$$\begin{array}{c} rL^T Lu^{k+1} - rL^T p^{k+1} + g_2(u^k, p^{k+1}) + L^T \lambda^k = f, \\ g_1(p^{k+1}) + \partial\varphi(p^{k+1}) - \lambda^k \ni 0, \\ \lambda^{k+1} = \lambda^k + \tau(p^{k+1} - Lu^{k+1}) \end{array} \qquad (7)$$

with an initial guess (u^0, λ^0).

Theorem 1. *Let $r = 2\alpha - k\beta_2 c_f$. Then iterative method (7) converges for $0 < \tau < r$.*

The proof follows from the properties of the operator A and general results on the convergence of Uzawa-type iterative method from [6].

The implementation of method (7) consists of solving the inclusion

$$g_1(p^{k+1}) + \partial\varphi(p^{k+1}) \ni \lambda^k$$

and the system of linear equations

$$rL^T Lu^{k+1} = f + rL^T p^{k+1} - g_2(u^k, p^{k+1}) - L^T \lambda^k$$

with symmetric and positive definite matrix $rL^T L$. Owing to the block diagonal form of the operators g_1 and $\partial\varphi$ the inclusion is split up into non-coupled two-dimensional problems corresponding to current node of the grid. Thus, method (7) is very easy to implement.

5. Numerical results.

We applied iterative method (7) to the finite difference schemes approximating variational inequality (2) with several variants of differential operator and functional, namely:

Problem 1 $g_1(\nabla u) = \{\nabla u \text{ if } |\nabla u| < 1/2;\ \dfrac{\nabla u}{\sqrt{2|\nabla u|}} \text{ if } |\nabla u| \geqslant 1/2\}$, $g_2(u, \nabla u) = k \sin u \dfrac{\partial u}{\partial x_1}$, $\psi(z) = I_K(z)$ and $f(x) = 5$.

Problem 2 Non-linear differential operator with $g_1(\nabla u) = \{\nabla u \text{ if } |\nabla u| < 1/2;\ \dfrac{\nabla u}{\sqrt{2|\nabla u|}} \text{ if } |\nabla u| \geqslant 1/2\}$, $g_2(u, \nabla u) = k\, u \dfrac{\partial u}{\partial x_1}$, $\psi(z) = I_K(z)$ and $f(x) = 5$.

Problem 3 Linear diffusion-convection operator $P(u) = -\Delta u + k\dfrac{\partial u}{\partial x_1}$, $\psi(z) = I_K(z)$ and $f(x) = 10$.

Problem 4 Linear diffusion-convection operator $P(u) = -\Delta u + k\dfrac{\partial u}{\partial x_1}$, $\psi(z) = I(z)$ and $f(x) = C(\delta_{i_1 j_1}(x) - \delta_{i_2 j_2}(x))$, where $\delta_{i_1 j_1}(x)$ is grid δ-function, which equals h^{-2} in $x_{i_1 j_1}$, $x_{i_1 j_1} = (0.1, 0.1)$ and $x_{i_2 j_2} = (0.9, 0.9)$, $C = \text{const}$.

We controlled L_2-norm of the residual $r^k = p^k - Lu^k$ in the iterative method (7) and used the stopping criterion $\|r^k\|_{L_2} = \big(h^2 \sum_{i,j} (r_{ij}^k)^2\big)^{1/2} \leqslant \varepsilon_{tol}$ (see [4] on the stopping criterion for Uzawa-type iterative methods).

Calculations were made for the grids with $n \simeq h^{-1} = 100, 300, 500$. The number of iterations N to achieve fixed accuracy $\|r^N\|_{L_2} < 0.01$ is reported in table 1. It was found that N doesn't depend on the grid size n.

The typical pictures of the free boundaries for several problems are plotted in figures.

Problem 1	Convective coefficient	$k=0$	$k=1$	$k=2$
	Number of iterations	13	16	24
Problem 2	Convective coefficient	$k=0$	$k=1$	$k=2$
	Number of iterations	13	16	28
Problem 3	Convective coefficient	$k=0$	$k=1$	$k=3$
	Number of iterations	26	37	41
Problem 4	Convective coefficient	$k=0$	$k=1$	$k=5$
	Number of iterations	9	11	13

Table. 1: Number N of iterations to achieve the accuracy $\|r^N\|_{L_2} < 0.01$ depending on the convective coefficient k.

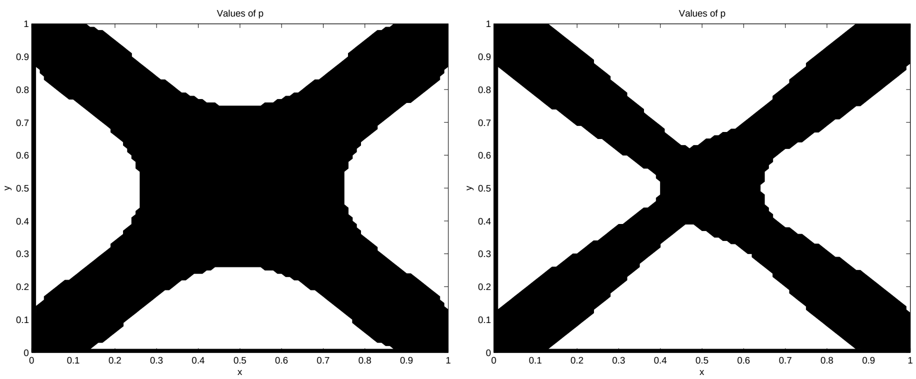

Fig. 1: Free boundary profile for problem 1 with right-hand side $f(x) = 10$; “black subdomain” corresponding to $|\nabla u_h| < 1$ and “white subdomain” corresponding to $|\nabla u_h| = 1$; left picture in the case of no convection $k = 0$ and right picture for $k = 2$.

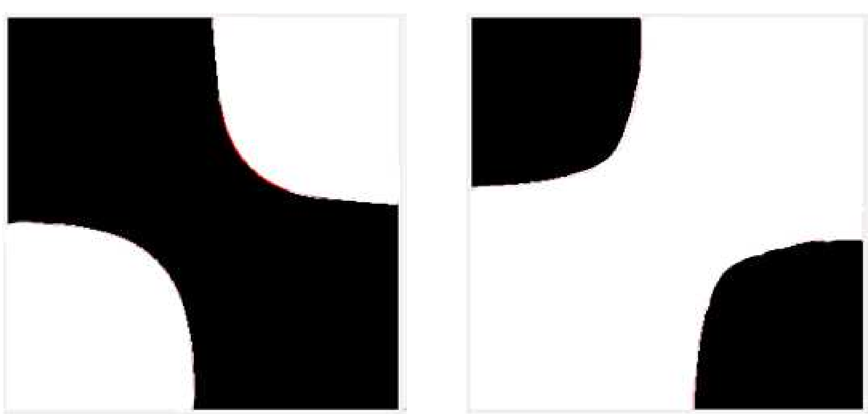

Fig. 2: Free boundary profile for problem 4 in the case of no convection ($k = 0$) with “black subdomain” corresponding to $|\nabla u_h| = 0$; left picture in the case $C = 5$ and right picture for $C = 10$.

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