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Research Article

The algebra of thin measurable operators is directly finite

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau)=S_0(\mathcal{M},\tau)+\mathbb{C}I$ be the *-algebra of all operators $X=A+\lambda I$ with $A\in S_0(\mathcal{M},\tau)$ and $\lambda\in\mathbb{C}$. It is proved that every operator of $T(\mathcal{M},\tau)$ that is left-invertible in $T(\mathcal{M},\tau)$ is in fact invertible in $T(\mathcal{M},\tau)$. It is a generalization of Sterling Berberian theorem (1982) on the subalgebra of thin operators in $\mathcal{B}(\mathcal{H})$. For the singular value function $\mu(t;Q)$ of $Q=Q^2\in S(\mathcal{M},\tau)$, the inclusion $\mu(t;Q)\in\{0\}\bigcup[1,+\infty)$ holds for all $t>0$. It gives the positive answer to the question posed by Daniyar Mushtari in 2010.

Keywords: Hilbert space, von Neumann algebra, semifinite trace, τ-measurable operator, τ-compact operator, singular value function, idempotent.

2020 Mathematics Subject Classification: 16E50, 46L51.

1. INTRODUCTION

In this paper, we extend the Sterling Berberian's result [2] (see also [12]) on direct finiteness of the algebra of thin operators on an infinite-dimensional Hilbert space to the Irving Segal's non-commutative integration setting [16]. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau)=S_0(\mathcal{M},\tau)+\mathbb{C}I$ be the *-algebra of all operators $X=A+\lambda I$ with $A\in S_0(\mathcal{M},\tau)$ and a complex number λ. We prove that every operator of $T(\mathcal{M},\tau)$ left-invertible in $T(\mathcal{M},\tau)$ is actually invertible in $T(\mathcal{M},\tau)$ (Theorem 3.1). Assume that $A\in S(\mathcal{M},\tau)$ and $B\in T(\mathcal{M},\tau)$. We have $AB\in T(\mathcal{M},\tau)$ if and only if $BA\in T(\mathcal{M},\tau)$ (Theorem 3.2). For the singular value function $\mu(t;Q)$ of $Q=Q^2\in S(\mathcal{M},\tau)$, we have $\mu(t;Q)\in\{0\}\bigcup[1,+\infty)$ for all $t>0$ (Theorem 3.3). It is the positive answer to the question by Daniyar Mushtari of year 2010.

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2. PRELIMINARIES

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$, I be the unit of $\mathcal{M}$. Also $\mathcal{M}^+$ denotes the cone of positive elements in $\mathcal{M}$. A mapping $\varphi:\mathcal{M}^+\to[0,+\infty]$ is called *a trace*, if $\varphi(X+Y)=\varphi(X)+\varphi(Y)$, $\varphi(\lambda X)=\lambda\varphi(X)$ for all $X,Y\in\mathcal{M}^+$, $\lambda\geq 0$ (moreover, $0\cdot(+\infty)\equiv 0$); $\varphi(Z^*Z)=\varphi(ZZ^*)$ for all $Z\in\mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X)>0$ for all $X\in\mathcal{M}^+$, $X\neq 0$; *normal*, if $X_i\uparrow X$ $(X_i,X\in\mathcal{M}^+)\Rightarrow$

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$\varphi(X) = \sup \varphi(X_i)$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{P}(\mathcal{M})$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(I-P) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [16], [14]. Let $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)^{\mathrm{h}}$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$, then $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(s; X) = \inf\{\|XP\| : \ P \in \mathcal{P}(\mathcal{M}) \text{ and } \tau(I-P) \leq s\}.$$

Lemma 2.1. (see [10]) *We have $\mu(s+t; XY) \leq \mu(s; X)\mu(t; Y)$ for all $X, Y \in S(\mathcal{M}, \tau)$ and $s, t > 0$.*

The sets $U(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \ (\|XP\| \leq \varepsilon \text{ and } \tau(I-P) \leq \delta \text{ for some } P \in \mathcal{P}(\mathcal{M}))\}$, where $\varepsilon > 0$, $\delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $S(\mathcal{M}, \tau)$, called *the measure topology* [14]. Equipped with this topology, $S(\mathcal{M}, \tau)$ is a complete metrizable topological *-algebra in which $\mathcal{M}$ is dense. We will write $X_n \xrightarrow{\tau} X$ if a sequence $\{X_n\}_{n=1}^{\infty}$ converges to $X \in S(\mathcal{M}, \tau)$ in the measure topology on $S(\mathcal{M}, \tau)$.

The set of τ-compact operators $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \lim\limits_{t \to \infty} \mu(t; X) = 0\}$ is an ideal in $S(\mathcal{M}, \tau)$. For any closed and densely defined linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, the *null projection* $\mathrm{n}(X) = \mathrm{n}(|X|)$ is the projection onto its kernel $\mathrm{Ker}(X)$, the *range projection* $\mathrm{r}(X)$ is the projection onto the closure of its range $\mathrm{Ran}(X)$ and the *support projection* $\mathrm{supp}(X)$ of X is defined by $\mathrm{supp}(X) = I - \mathrm{n}(X)$.

The two-sided ideal $\mathcal{F}(\mathcal{M}, \tau)$ in $\mathcal{M}$ consisting of all elements of τ-finite range is defined by

$$\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : \ \tau(\mathrm{r}(X)) < \infty\} = \{X \in \mathcal{M} : \ \tau(\mathrm{supp}(X)) < \infty\}.$$

Equivalently, $\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : \ \mu(t; X) = 0 \text{ for some } t > 0\}$. Clearly, $S_0(\mathcal{M}, \tau)$ is the closure of $\mathcal{F}(\mathcal{M}, \tau)$ with respect to the measure topology [9].

3. Main results

Throughout the sequel, let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra, with some distinguished faithful normal semifinite trace τ.

Lemma 3.2. *We have $|X| \in T(\mathcal{M}, \tau)$ for every $X \in T(\mathcal{M}, \tau)$.*

Proof. The ideal $\mathcal{F}(\mathcal{M}, \tau)$ is a C^*-subalgebra in $\mathcal{M}$. Hence $F(\mathcal{M}, \tau) = \mathcal{F}(\mathcal{M}, \tau) + \mathbb{C}I$ is an unital C^*-subalgebra in $\mathcal{M}$ and if $X \in F(\mathcal{M}, \tau)$, then $|X| \in F(\mathcal{M}, \tau)$. Assume that $X \in T(\mathcal{M}, \tau)$, i.e., $X = A + \lambda I$ with $A \in S_0(\mathcal{M}, \tau)$ and $\lambda \in \mathbb{C}$. Since $\mathcal{F}(\mathcal{M}, \tau)$ is t_τ-dense in $S_0(\mathcal{M}, \tau)$, there exists a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}(\mathcal{M}, \tau)$ such that $A_n \xrightarrow{\tau} A$ as $n \to \infty$. Then the sequence $X_n = A_n + \lambda I$, $n \in \mathbb{N}$, lies in $F(\mathcal{M}, \tau)$ and t_τ-converges to the operator X as $n \to \infty$. According to the results given above, $|X_n| = B_n + |\lambda| I$ with some $B_n \in F(\mathcal{M}, \tau)^{\mathrm{h}}$, $n \in \mathbb{N}$. Since $X_n \xrightarrow{\tau} X$ as $n \to \infty$, we have $X_n^* \xrightarrow{\tau} X^*$ as $n \to \infty$ by t_τ-continuity of the involution in $S(\mathcal{M}, \tau)$. Then via joint t_τ-continuity of the multiplication in $S(\mathcal{M}, \tau)$, we have $X_n^* X_n \xrightarrow{\tau} X^*X$ as $n \to \infty$. Therefore, we obtain $|X_n| \xrightarrow{\tau} |X|$ as $n \to \infty$ by t_τ-continuity of the real function $f(t) = \sqrt{t}$, $t \geq 0$ [18]. Thus the sequence $\{B_n\}_{n=1}^{\infty}$ t_τ-converges to a some operator $B \in S_0(\mathcal{M}, \tau)^{\mathrm{h}}$ and $|X| = B + |\lambda| I$. $\square$

Lemma 3.3. (see [4, Corollary 2.4]) *If $X \in T(\mathcal{M}, \tau)$ and $XX^* \le X^*X$, then $XX^* = X^*X$.*

Lemma 3.4. *The idempotents of $T(\mathcal{M}, \tau)$ are the operators P, $I - P$, where P runs over the idempotent operators of $S_0(\mathcal{M}, \tau)$.*

Proof. Assume that $X = A + \lambda I \in T(\mathcal{M}, \tau)$ and $X^2 = X$. Then $A^2 + 2\lambda A + \lambda^2 I = A + \lambda I$, i.e., $\lambda \in \{0, 1\}$. If $\lambda = 0$, then $A^2 = A$ and $A \in S_0(\mathcal{M}, \tau)$ is an idempotent operator. Then $I - A \in T(\mathcal{M}, \tau)$ and is also an idempotent. If $\lambda = 1$, then $A^2 = -A = (-A)^2$ and $-A \in S_0(\mathcal{M}, \tau)$ is an idempotent operator. Then $I - (-A) \in T(\mathcal{M}, \tau)$ and is also an idempotent. □

Consider $F_0(\mathcal{M}, \tau) = \{A \in S_0(\mathcal{M}, \tau) : \tau(\mathrm{r}(A)) < +\infty\}$ and $\mathcal{A}(\mathcal{M}, \tau) = F_0(\mathcal{M}, \tau) + \mathbb{C}I$. Then $\mathcal{A}(\mathcal{M}, \tau)$ is a *-subalgebra of $T(\mathcal{M}, \tau)$.

Lemma 3.5. *$\mathcal{A}(\mathcal{M}, \tau)$ contains every idempotent of $T(\mathcal{M}, \tau)$.*

Proof. Let Q be an idempotent operator of $S(\mathcal{M}, \tau)$. Then

$$(Q + Q^* - I)^2 = I + (Q - Q^*)(Q - Q^*)^*$$

and by [6, Theorem 2.21] there exists a unique "range" projection $Q^\sharp \in \mathcal{P}(\mathcal{M})$, defined by the formula $Q^\sharp = Q(Q + Q^* - I)^{-1}$ with $(Q + Q^* - I)^{-1} \in \mathcal{M}$ and subject to the condition $Q^\sharp \cdot S(\mathcal{M}, \tau) = Q \cdot S(\mathcal{M}, \tau)$. By [6, Theorem 2.23], there exists a unique decomposition $Q = P + Z$, where $P = Q^\sharp \in \mathcal{P}(\mathcal{M})$ and $Z \in S(\mathcal{M}, \tau)$ is a nilpotent so that $Z^2 = 0$ and $ZP = 0$, $PZ = Z$. Thus $QP = P$ and $PQ = Q$. Assume that $Q \in S_0(\mathcal{M}, \tau)$. Since $QP = P$, we have $P \in S_0(\mathcal{M}, \tau)$. Since the singular function $\mu(t; P) = \chi_{(0, \tau(P)]}(t)$ for all $t > 0$, we conclude that $P \in \mathcal{F}(\mathcal{M}, \tau)$. Then by equality $PQ = Q$, we have $Q \in F_0(\mathcal{M}, \tau)$ and apply Lemma 3.4. □

Lemma 3.6. *$F_0(\mathcal{M}, \tau)$ is a regular ring.*

Proof. We show that for every operator $A \in F_0(\mathcal{M}, \tau)$ the equation $AXA = A$ possesses a solution in $F_0(\mathcal{M}, \tau)$. For $A \in F_0(\mathcal{M}, \tau)$, the range projection $\mathrm{r}(A)$ and the support projection $\mathrm{supp}(A)$ lie in $\mathcal{F}(\mathcal{M}, \tau)$. Consider the projection $P = \mathrm{r}(A) \bigvee \mathrm{supp}(A)$ in $\mathcal{F}(\mathcal{M}, \tau)$ and the reduced von Neumann algebra $\mathcal{M}_P = P\mathcal{M}P$, the reduced faithful normal finite trace τ_P with $\tau_P(X) = \tau(PXP)$, $X \in \mathcal{M}_P^+$. The algebra $\mathcal{M}_P$ is finite, therefore $S(\mathcal{M}_P, \tau_P)$ is a regular ring by [15, Theorem 4.3]. Since $A \in S(\mathcal{M}_P, \tau_P)$, the equation $AXA = A$ admits a solution in $S(\mathcal{M}_P, \tau_P) \subset F_0(\mathcal{M}, \tau)$. □

Idempotents P, Q of a ring $\mathcal{R}$ are said to be *equivalent* (in $\mathcal{R}$), written $P \sim Q$, if there exist elements $X, Y \in \mathcal{R}$ such that $XY = P$ and $YX = Q$ (replacing X, Y by PXQ, QYP, one can suppose that $X \in P\mathcal{R}Q$, $Y \in Q\mathcal{R}P$ [13, p. 22]). Projections (=self-adjoint idempotents) P, Q of a ring with involutions are said to be **-equivalent* if there exists an element X such that $XX^* = P$ and $X^*X = Q$.

Theorem 3.1. *If $X, Y \in T(\mathcal{M}, \tau)$ such that $XY = I$, then $YX = I$.*

Proof. In the terms of ring theory, we assert that the ring $T(\mathcal{M}, \tau)$ is "directly finite" [11, p. 49]. Since $F_0(\mathcal{M}, \tau)$ (by Lemma 3.6) and $\mathcal{A}(\mathcal{M}, \tau)/F_0(\mathcal{M}, \tau) \cong \mathbb{C}$ are both regular rings, $\mathcal{A}(\mathcal{M}, \tau)$ is a regular ring [11, p. 2, Lemma 1.3]; since, moreover, the involution of $\mathcal{A}(\mathcal{M}, \tau)$ is proper ($AA^* = 0$ implies $A = 0$), the algebra $\mathcal{A}(\mathcal{M}, \tau)$ is *-regular in the sense of von Neumann [1, p. 229].

If X, Y are elements of $T(\mathcal{M}, \tau)$ such that $XY = I$, then $P = YX$ is an idempotent of $T(\mathcal{M}, \tau)$ such that $P \sim I$ in $T(\mathcal{M}, \tau)$. By Lemma 3.5, we have $P \in \mathcal{A}(\mathcal{M}, \tau)$; since $\mathcal{A}(\mathcal{M}, \tau)$ is *-regular, there exists a projection $Q \in \mathcal{A}(\mathcal{M}, \tau)$ such that $Q \cdot \mathcal{A}(\mathcal{M}, \tau) = P \cdot \mathcal{A}(\mathcal{M}, \tau)$ [1, p. 229, Proposition 3]. Then $P \sim Q$ in $\mathcal{A}(\mathcal{M}, \tau)$ [13, p. 21, Theorem 14], a fortiori $P \sim Q$ in $T(\mathcal{M}, \tau)$; already $P \sim I$ in $T(\mathcal{M}, \tau)$, so $Q \sim I$ in $T(\mathcal{M}, \tau)$ by transitivity. Since $T(\mathcal{M}, \tau)$

satisfies the "square root" axiom (SR) and contains square roots of its positive elements (see Lemma 3.2 and [13, p. 90]), it follows that the projections P, I are *-equivalent in $T(\mathcal{M}, \tau)$ [13, p. 35, Theorem 27], say $X \in T(\mathcal{M}, \tau)$ with $XX^* = P$, $X^*X = I$. By Lemma 3.3, $P = I$; then $Q \cdot \mathcal{A}(\mathcal{M}, \tau) = P \cdot \mathcal{A}(\mathcal{M}, \tau) = \mathcal{A}(\mathcal{M}, \tau)$ shows that $P = I$, that is, $YX = I$. $\square$

Theorem 3.1 can obviously be reformulated as follows: if $A, B \in S_0(\mathcal{M}, \tau)$ and $A+B+AB = 0$, then $AB = BA$. On invertibility in $S(\mathcal{M}, \tau)$, see [17], [7] and [8].

Theorem 3.2. *Assume that $A \in S(\mathcal{M}, \tau)$ and $B \in T(\mathcal{M}, \tau)$. Then $AB \in T(\mathcal{M}, \tau)$ if and only if $BA \in T(\mathcal{M}, \tau)$.*

Proof. "$\Rightarrow$". If $B \in S_0(\mathcal{M}, \tau)$, then $BA \in S_0(\mathcal{M}, \tau) \subset T(\mathcal{M}, \tau)$. Assume that $B \notin S_0(\mathcal{M}, \tau)$. Then $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M}, \tau)$. Hence,

$$AB = \lambda A + AK = \mu I + K_1 \tag{3.1}$$

for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M}, \tau)$.

Case 1: $\mu = 0$. Then we have $A \in S_0(\mathcal{M}, \tau)$ by (3.1); hence $BA \in S_0(\mathcal{M}, \tau) \subset T(\mathcal{M}, \tau)$.

Case 2: $\mu \neq 0$. Then by (3.1), we have $\lambda A = \mu I + K_2$ with $K_2 = K_1 - AK \in S_0(\mathcal{M}, \tau)$. Therefore, $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2$ and

$$BA = (\lambda I + K)\Big(\frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2\Big) = I + K_3$$

with $K_3 = K_1 - AK + \frac{\mu}{\lambda} K + \frac{1}{\lambda} KK_1 - \frac{1}{\lambda} KAK \in S_0(\mathcal{M}, \tau)$. Thus $BA \in T(\mathcal{M}, \tau)$.

"$\Leftarrow$". We know that $X \in T(\mathcal{M}, \tau)$ if and only if $X^* \in T(\mathcal{M}, \tau)$, and apply the proof given above to the pair $\{A^*, B^*\}$. $\square$

Corollary 3.1. *If $A \in S(\mathcal{M}, \tau)$ and $B \in T(\mathcal{M}, \tau) \setminus S_0(\mathcal{M}, \tau)$ then the following conditions are equivalent:*

(i) *$AB \in T(\mathcal{M}, \tau)$;*
(ii) *$BA \in T(\mathcal{M}, \tau)$;*
(iii) *$A \in T(\mathcal{M}, \tau)$.*

Proof. "(i)$\Rightarrow$(iii)". Let $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M}, \tau)$. Then $AB = \lambda A + AK = \mu I + K_1$ for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M}, \tau)$. Thus $\lambda A = \mu I + K_1 - AK$ and $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_1 - \frac{1}{\lambda} AK \in T(\mathcal{M}, \tau)$. $\square$

Theorem 3.3. *If $Q \in S(\mathcal{M}, \tau)$ is such that $Q^2 = Q$, then $\mu(t; Q) \in \{0\} \bigcup [1, +\infty)$ for all $t > 0$. For the symmetry $U = 2Q - I$, we have $\mu(t; U) \geq 1$ for all $t > 0$.*

Proof. For $Q = Q^2 \notin S_0(\mathcal{M}, \tau)$, we have $\mu(t; Q) \geq 1$ for all $t > 0$, see [5, Lemma 3.8]. Let $Q = Q^2 \in S_0(\mathcal{M}, \tau)$ and P be "the range" projection of the idempotent Q, see the proof of Lemma 3.5. Since $QP = P$ and $P \in \mathcal{P}(\mathcal{M}) \bigcap \mathcal{F}(\mathcal{M}, \tau)$, by Lemma 2.1 we have

$$1 = \mu(s + t; P) = \chi_{(0, \tau(P)]}(s + t) = \mu(s + t; QP) \leq \mu(s; P)\mu(t; Q) = \mu(t; Q)$$

for all $s, t > 0$ with $s + t \leq \tau(P)$. By tending s to $0+$, we obtain $\mu(t; Q) \geq 1$ for all $0 < t < \tau(P)$. By the right continuity of the function $\mu(t; \cdot)$, we have $\mu(\tau(P); Q) \geq 1$. If $t > \tau(P)$ then $\mu(t; P) = 0$; by the equality $PQ = Q$ and by Lemma 2.1, we obtain

$$0 \leq \mu(t; Q) = \mu(t; PQ) \leq \mu(t - \varepsilon; P)\mu(\varepsilon; Q) = 0$$

for all $\varepsilon > 0$ with $t - \varepsilon > \tau(P)$.

Let $Q \in S(\mathcal{M}, \tau)$ be such that $Q^2 = Q$. For the symmetry $U = 2Q - I$, we have $U^2 = I$ and by Lemma 2.1 obtain

$$1 = \mu(2t; I) = \mu(2t; U^2) \leq \mu(t; U)\mu(t; U) = \mu(t; U)^2$$

for all $t > 0$. $\square$

Note that for $Q \in \mathcal{M}$ such that $Q^2 = Q$ the relation $\mu(t; Q) \in \{0\} \bigcup [1, \|Q\|]$ for all $t > 0$ was obtained by another way in [3, item 1) of Lemma 3.8]. Theorem 3.3 gives the positive answer to the question by Daniyar Mushtari of year 2010.

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