

THE TOPOLOGIES OF LOCAL CONVERGENCE IN MEASURE ON THE ALGEBRAS OF MEASURABLE OPERATORS

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Abstract: Given a von Neumann algebra \mathcal{M} of operators on a Hilbert space \mathcal{H} and a faithful normal semifinite trace τ on \mathcal{M} , denote by $S(\mathcal{M}, \tau)$ the $*$ -algebra of τ -measurable operators. We obtain a sufficient condition for the positivity of an hermitian operator in $S(\mathcal{M}, \tau)$ in terms of the topology $t_{\tau l}$ of τ -local convergence in measure. We prove that the $*$ -ideal $\mathcal{F}(\mathcal{M}, \tau)$ of elementary operators is $t_{\tau l}$ -dense in $S(\mathcal{M}, \tau)$. If t_{τ} is locally convex then so is $t_{\tau l}$; if $t_{\tau l}$ is locally convex then so is the topology $t_{w\tau l}$ of weakly τ -local convergence in measure. We propose some method for constructing F -normed ideal spaces, henceforth F -NIPs, on (\mathcal{M}, τ) starting from a prescribed F -NIP and preserving completeness, local convexity, local boundedness, or normability whenever present in the original. Given two F -NIPs \mathcal{X} and \mathcal{Y} on (\mathcal{M}, τ) , suppose that $A\mathcal{X} \subseteq \mathcal{Y}$ for some operator $A \in S(\mathcal{M}, \tau)$. Then the multiplier $\mathbf{M}_A X = AX$ acting as $\mathbf{M}_A : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous. In particular, for $\mathcal{X} \subseteq \mathcal{Y}$ the natural embedding of \mathcal{X} into \mathcal{Y} is continuous. We inspect the properties of decreasing sequences of F -NIPs on (\mathcal{M}, τ) .

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Introduction

Suppose that a von Neumann operator algebra \mathcal{M} acts on a Hilbert space \mathcal{H} and τ is a faithful normal semifinite trace on \mathcal{M} with $\tau(I) = +\infty$. This article continues the study of [1–10], of the properties of the topologies $t_{\tau l}$ of τ -local convergence in measure and $t_{w\tau l}$ of weakly τ -local convergence in measure on the $*$ -algebra $S(\mathcal{M}, \tau)$ of τ -measurable operators. The topologies $t_{\tau l}$ and $t_{w\tau l}$ match the order structure of the hermitian part of $S(\mathcal{M}, \tau)$ better than the classical topology t_{τ} of convergence in measure [2, 6, 7, 9].

In this article, Theorem 1 establishes a sufficient condition for the positivity of a hermitian operator in $S(\mathcal{M}, \tau)$ in terms of $t_{\tau l}$. Finding such conditions is an important problem attracting many researchers; see, for instance, [11–13] and the bibliographies therein. It is well known that each τ -measurable operator is a linear combination of four positive τ -measurable operators. In view of the operator inequality

$$|X + Y| \leq U|X|U^* + V|Y|V^*, \quad X, Y \in S(\mathcal{M}, \tau), \quad |X| = \sqrt{X^*X},$$

where $U, V \in \mathcal{M}$ are suitable partial isometries (see [14]), to prove the triangle inequality for an F -normed ideal space, henceforth spelled F -NIP, on (\mathcal{M}, τ) , it suffices to verify the triangle inequality for the F -norm just for pairs of positive τ -measurable operators.

In Theorem 2 we show that the $*$ -ideal $\mathcal{F}(\mathcal{M}, \tau)$ of elementary operators is $t_{\tau l}$ -dense in $S(\mathcal{M}, \tau)$. Therefore, the space $L_p(\mathcal{M}, \tau)$ with $0 < p < \infty$ is $t_{\tau l}$ -dense in $S(\mathcal{M}, \tau)$; see Corollary 1. Recall that the t_{τ} -closures of the $*$ -ideal $\mathcal{F}(\mathcal{M}, \tau)$ and the spaces $L_p(\mathcal{M}, \tau)$ with $0 < p < \infty$ coincide with the $*$ -ideal $S_0(\mathcal{M}, \tau)$ of all τ -compact operators.

Theorem 3 shows that if t_{τ} is locally convex then so is $t_{\tau l}$; if $t_{\tau l}$ is locally convex then so is $t_{w\tau l}$. We propose some method for constructing “weighted” F -NIPs on (\mathcal{M}, τ) starting from a prescribed

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F -NIP and preserving completeness, local convexity, local boundedness, or normability whenever present in the original; see Theorem 4 and Corollaries 4, 5. Weighted F -NIPs on an abelian von Neumann algebra arise naturally in studying integral operators. In this case $\mathcal{M} \simeq L^\infty(\Omega, \Sigma, \mu)$ and $\tau(f) = \int_\Omega f \, d\mu$, where (Ω, Σ, μ) is a localizable measure space.

Given two F -NIPs \mathcal{X} and \mathcal{Y} on (\mathcal{M}, τ) , suppose that $A\mathcal{X} \subseteq \mathcal{Y}$ for some operator $A \in S(\mathcal{M}, \tau)$. Then the multiplier $\mathbf{M}_A X = AX$ acting as $\mathbf{M}_A : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous. In particular, for $\mathcal{X} \subseteq \mathcal{Y}$ the natural embedding of \mathcal{X} into \mathcal{Y} is continuous (see Theorem 5) which yields a new proof of the key Lemma 4.3 of [15]. Theorem 6 studies the properties of decreasing sequences of F -NIPs on (\mathcal{M}, τ) .

Most of the results are new even for the $*$ -algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators on \mathcal{H} which is equipped with the canonical trace $\tau = \text{tr}$.

1. Notation and Definitions

Consider a von Neumann algebra \mathcal{M} of operators on a Hilbert space \mathcal{H} . Denote by \mathcal{M}^{pr} the projection lattice ($P = P^2 = P^*$) in \mathcal{M} ; and by I , the identity of \mathcal{M} . So $P^\perp = I - P$ for $P \in \mathcal{M}^{\text{pr}}$. Denote by \mathcal{M}^+ the cone of positive elements of \mathcal{M} . A projection $P \in \mathcal{M}$ is *minimal* or an *atom*, if $Q \in \mathcal{M}^{\text{pr}}$ and $Q \leq P$ implies that either $Q = 0$ or $Q = P$. A von Neumann algebra \mathcal{M} is *atomic* whenever each nonzero projection in \mathcal{M} majorizes a nonzero minimal projection.

A mapping $\varphi : \mathcal{M}^+ \rightarrow [0, +\infty]$ is a *trace* whenever $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$ and $\lambda \geq 0$, with $0 \cdot (+\infty) \equiv 0$, and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is *faithful* whenever $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$ with $X \neq 0$, *semifinite* whenever $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$, and *normal* whenever $X_i \nearrow X$ with $X_i, X \in \mathcal{M}^+$ implies that $\varphi(X) = \sup \varphi(X_i)$; see [16, Chapter V, §2].

An operator in \mathcal{H} , not necessarily bounded or densely defined, is *affiliated to the von Neumann algebra* \mathcal{M} whenever it commutes with every unitary operator in the commutator \mathcal{M}' of \mathcal{M} . Henceforth, τ stands for a faithful normal semifinite trace on \mathcal{M} and $\mathcal{M}_\tau^{\text{pr}} = \{P \in \mathcal{M}^{\text{pr}} : \tau(P) < \infty\}$.

A closed operator X affiliated to \mathcal{M} whose domain $\mathcal{D}(X)$ is dense in \mathcal{H} is τ -*measurable* whenever, given $\varepsilon > 0$, there exists $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ -measurable operators forms a $*$ -algebra with the operations of conjugate transpose, multiplication by scalars, and strong addition and multiplication obtained by closing the ordinary operations [17, Chapter IX]. Given a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, denote by \mathcal{L}^+ and \mathcal{L}^{h} the positive and hermitian parts of \mathcal{L} . Denote by \leq the partial order on $S(\mathcal{M}, \tau)^{\text{h}}$ generated by the proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

Endow the $*$ -algebra $S(\mathcal{M}, \tau)$ with the topology t_τ of convergence in measure [17, Chapter IX, §2] whose base of neighborhoods of zero consists of the sets

$$\mathcal{U}(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \exists Q \in \mathcal{M}^{\text{pr}} (\|XQ\| \leq \varepsilon \text{ and } \tau(Q^\perp) \leq \delta)\}, \quad \varepsilon > 0, \delta > 0.$$

It is known that $\langle S(\mathcal{M}, \tau), t_\tau \rangle$ is a complete metrizable topological $*$ -algebra; furthermore, \mathcal{M} is dense in $\langle S(\mathcal{M}, \tau), t_\tau \rangle$. To indicate the convergence of a net $\{X_j\}_{j \in J} \subset S(\mathcal{M}, \tau)$ to $X \in S(\mathcal{M}, \tau)$ in the topology t_τ , we write $X_j \xrightarrow{\tau} X$ and say that $\{X_j\}_{j \in J}$ *converges to X in measure τ* .

Denote by $\mu(X, t)$ a *rearrangement* of $X \in S(\mathcal{M}, \tau)$, meaning a nonincreasing right-continuous function $\mu(X, \cdot) : (0, \infty) \rightarrow [0, \infty)$ defined as

$$\mu(X, t) = \inf\{\|XP\| : P \in \mathcal{M}^{\text{pr}}, \tau(P^\perp) \leq t\}, \quad t > 0.$$

The set of τ -compact operators $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \lim_{t \rightarrow \infty} \mu(X, t) = 0\}$ is an ideal of $S(\mathcal{M}, \tau)$ and the set of elementary operators $\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : (\exists s > 0) \mu(X, t) = 0 \forall t > s\}$ is an ideal of \mathcal{M} . The topology t_τ is also determined by the F -norm $\rho_\tau(X) = \inf_{t > 0} \max\{t, \mu(X, t)\}$, $X \in S(\mathcal{M}, \tau)$.

Lemma 1 [18]. Take $X, Y, X_j \in S(\mathcal{M}, \tau)$ for $j \in J$. Then

- (i) $\mu(X, t) = \mu(|X|, t) = \mu(X^*, t)$ for all $t > 0$;
- (ii) $\mu(X^*X, t) = \mu(XX^*, t)$ for all $t > 0$;
- (iii) if $|X| \leq |Y|$ then $\mu(X, t) \leq \mu(Y, t)$ for all $t > 0$;
- (iv) if $X \in \mathcal{M}$ then $\lim_{t \rightarrow +0} \mu(X, t) = \sup_{t > 0} \mu(X, t) = \|X\|$;
- (v) $\mu(XY, t+s) \leq \mu(X, t)\mu(Y, s)$ for all $t, s > 0$;
- (vi) $\mu(|X|^\alpha, t) = \mu(X, t)^\alpha$ for all $\alpha > 0$ and $t > 0$;
- (vii) $X_j \xrightarrow{\tau} X \iff \mu(X_j - X, t) \rightarrow 0$ for each $t > 0$.

Denote the linear Lebesgue measure on \mathbb{R} by m . Define the noncommutative Lebesgue space L_p , with $0 < p < \infty$, associated with (\mathcal{M}, τ) as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \mu(X, \cdot) \in L_p(\mathbb{R}^+, m)\}$$

with the F -norm (a norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu(X, \cdot)\|_p$, $X \in L_p(\mathcal{M}, \tau)$.

2. The Topologies of Local Convergence in Measure on $S(\mathcal{M}, \tau)$

The topology t_τ of convergence in measure can be localized as follows. Given $\varepsilon, \delta > 0$ and $P \in \mathcal{M}_\tau^{\text{pr}}$, define the sets

$$\mathcal{V}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \exists Q \in \mathcal{M}^{\text{pr}} (Q \leq P, \|XQ\| \leq \varepsilon \text{ and } \tau(P - Q) \leq \delta)\},$$

$$\mathcal{W}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : (\exists Q \in \mathcal{M}^{\text{pr}}) (Q \leq P, \|QXQ\| \leq \varepsilon \text{ and } \tau(P - Q) \leq \delta)\}.$$

The space $S(\mathcal{M}, \tau)$ becomes a topological vector space if endowed with the topology $t_{\tau l}$ of τ -local, respectively $t_{w\tau l}$ of weakly τ -local, convergence in measure whose base of neighborhoods of zero consists of the family $\Theta = \{\mathcal{V}(\varepsilon, \delta, P)\}_{\varepsilon, \delta > 0; P \in \mathcal{M}_\tau^{\text{pr}}}$, respectively $\Theta = \{\mathcal{W}(\varepsilon, \delta, P)\}_{\varepsilon, \delta > 0; P \in \mathcal{M}_\tau^{\text{pr}}}$. We write $X_i \xrightarrow{\tau l} X$ and $X_i \xrightarrow{w\tau l} X$ to indicate $t_{\tau l}$ -convergence and $t_{w\tau l}$ -convergence. Using the standard techniques for von Neumann algebra reduction, we can show (see also [3, 6]) that $X_i \xrightarrow{\tau l} X$, respectively $X_i \xrightarrow{w\tau l} X$, if and only if $X_i P \xrightarrow{\tau} XP$, cf. [1, p. 114], respectively $PX_i P \xrightarrow{\tau} PXP$, cf. [1, p. 114; 2, p. 746], for all $P \in \mathcal{M}_\tau^{\text{pr}}$. It is clear that $t_{w\tau l} \leq t_{\tau l} \leq t_\tau$ and $t_{w\tau l}$ -convergence coincides with convergence in measure in $\langle S(PMP) = PS(\mathcal{M}, \tau)P, t_{\tau(P.P)} \rangle$ for all $P \in \mathcal{M}_\tau^{\text{pr}}$. We can also define the topologies $t_{\tau l}$ and $t_{w\tau l}$ in terms of nonincreasing rearrangements. The family $\tilde{\Theta} = \{\tilde{\mathcal{V}}(\varepsilon, \delta, P)\}_{\varepsilon, \delta > 0; P \in \mathcal{M}_\tau^{\text{pr}}}$, where $\tilde{\mathcal{V}}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \mu(XP, \delta) < \varepsilon\}$, also determines a base of neighborhoods of zero for $t_{\tau l}$. If $\tau(I) < \infty$ then $t_\tau = t_{\tau l} = t_{w\tau l}$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$ -algebra of all bounded linear operators on \mathcal{H} and $\tau = \text{tr}$ is the canonical trace then $S(\mathcal{M}, \tau)$, $S_0(\mathcal{M}, \tau)$, $L_p(\mathcal{M}, \tau)$, and $\mathcal{F}(\mathcal{M}, \tau)$ coincide with $\mathcal{B}(\mathcal{H})$, the ideal $\mathfrak{S}_\infty(\mathcal{H})$ of compact operators, the Schatten-von Neumann ideal $\mathfrak{S}_p(\mathcal{H})$, and the ideal $\mathcal{F}(\mathcal{H})$ of finite-rank operators on \mathcal{H} respectively. The topology t_τ coincides with the norm topology $\|\cdot\|$, while $t_{\tau l}$ and $t_{w\tau l}$, coincide with the topologies of strong and weak operator convergence, respectively. We have

$$\mu(X, t) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1, n)}(t)$$

for $t > 0$, where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s -numbers of the compact operator X and χ_A is the indicator of some set $A \subset \mathbb{R}$.

If \mathcal{M} is abelian, i.e., commutative; then $\mathcal{M} \simeq L^\infty(\Omega, \Sigma, \mu)$ and $\tau(f) = \int_\Omega f d\mu$, where (Ω, Σ, μ) is a localizable measure space, $S(\mathcal{M}, \tau)$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, μ) bounded outside a set of finite measure. The topology t_τ is the usual topology of convergence in measure and $t_{\tau l}$ coincides with $t_{w\tau l}$ and available topology of convergence in measure on the sets of finite measure.

Theorem 1. Suppose that $X, Y \in S(\mathcal{M}, \tau)$ with $Y = Y^*$ and $X^n \xrightarrow{\tau^l} 0$ as $n \rightarrow \infty$. If $X^*YX \leq Y$ then $Y \geq 0$.

PROOF. The assumption $X^*YX \leq Y$ implies the chain of inequalities

$$Y \geq X^*YX \geq X^{*2}YX^2 \geq \dots \geq X^{*n}YX^n \geq \dots$$

Take the Jordan decomposition $Y = Y_+ - Y_-$ into the positive and negative parts with $Y_+Y_- = 0$ and put $|Y| = Y_+ + Y_-$. Given an arbitrary projection $P \in \mathcal{M}_\tau^{\text{pf}}$, write down the equalities

$$PX^{*n}YX^nP = PX^{*n}Y_+X^nP - PX^{*n}Y_-X^nP, \quad n \in \mathbb{N}.$$

For each $t > 0$ claims (ii), (vi), and (v) of Lemma 1 yield

$$\mu(PX^{*n}Y_+X^nP, t) = \mu(\sqrt{Y_+}X^nP, t)^2 \leq \mu(\sqrt{Y_+}, t/2)^2 \mu(X^nP, t/2)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $X^{*n}Y_+X^n \xrightarrow{w\tau^l} 0$ as $n \rightarrow \infty$, see claim (vii) of Lemma 1. Similarly we obtain $X^{*n}Y_-X^n \xrightarrow{w\tau^l} 0$ as $n \rightarrow \infty$. By the $t_{w\tau^l}$ -continuity of the addition $(A, B) \mapsto A + B$ from $S(\mathcal{M}, \tau) \times S(\mathcal{M}, \tau)$ to $S(\mathcal{M}, \tau)$, we have $X^{*n}YX^n \xrightarrow{w\tau^l} 0$ as $n \rightarrow \infty$. Thus, $Y \geq 0$. \square

Theorem 2. The $*$ -ideal $\mathcal{F}(\mathcal{M}, \tau)$ is t_{τ^l} -dense in $S(\mathcal{M}, \tau)$.

PROOF. STEP 1. Take $X \in S(\mathcal{M}, \tau)$ and its polar decomposition $X = U|X|$. If $(X_j)_{j \in J} \subset \mathcal{F}(\mathcal{M}, \tau)$ and $X_j \xrightarrow{\tau^l} |X|$ then by the separate t_{τ^l} -continuity of multiplication [5, Theorem 1] we have $UX_j \xrightarrow{\tau^l} U|X| = X$. Therefore, it suffices to show that

$$\forall X \in S(\mathcal{M}, \tau)^+ \exists (Y_j)_{j \in J} \subset \mathcal{F}(\mathcal{M}, \tau) \quad (Y_j \xrightarrow{\tau^l} X).$$

STEP 2. Take $X \in S(\mathcal{M}, \tau)^+$ such that $X = Y + Z$ with $Y \in \mathcal{M}^+$ and $Z \in S_0(\mathcal{M}, \tau)^+$ [19]. If $Z = \int_0^\infty \lambda dE_\lambda$ is the spectral representation then the operator $Z_n = \int_{1/n}^n \lambda dE_\lambda$ lies in the cone $\mathcal{F}(\mathcal{M}, \tau)^+$ for each $n \in \mathbb{N}$. We have

$$Z - Z_n = \int_{[0, 1/n)} \lambda dE_\lambda + \int_{(n, \infty)} \lambda dE_\lambda \equiv Z_{n,1} + Z_{n,2}, \quad n \in \mathbb{N}.$$

It is obvious that $\|Z_{n,1}\| \leq \frac{1}{n}$; hence, $Z_{n,1} \xrightarrow{\tau} 0$ as $n \rightarrow \infty$. Take the projection $s_r(A)$ onto the support of $A \in S(\mathcal{M}, \tau)^h$. We have $s_r(Z_{n,2}) = E_n^\perp$; since by assumption Z is τ -measurable and the trace τ is normal, we infer that $\tau(E_n^\perp) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $Z_{n,2} \xrightarrow{\tau} 0$ as $n \rightarrow \infty$. Thus, $Z_n \xrightarrow{\tau} 0$ as $n \rightarrow \infty$.

Suppose that $\mathcal{M}_\tau^{\text{pf}} = (P_j)_{j \in J}$. Then $I = \bigvee_{j \in J} P_j$ because the trace τ is semifinite. We have $Y_j = YP_j \in \mathcal{F}(\mathcal{M}, \tau)$ for all $j \in J$. Since $P_j \xrightarrow{\tau^l} I$, for every projection $P \in \mathcal{M}_\tau^{\text{pf}}$ we obtain $Y_jP \xrightarrow{\tau} YP$. Consequently, $Y_j \xrightarrow{\tau^l} Y$ and the assertion follows from the t_{τ^l} -continuity of addition in $S(\mathcal{M}, \tau)$. \square

Since $\mathcal{F}(\mathcal{M}, \tau) \subset L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$, we have

Corollary 1. Every space $L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$ is t_{τ^l} -dense in $S(\mathcal{M}, \tau)$.

Theorem 3. (i) If t_τ is locally convex then so is t_{τ^l} .

(ii) If t_{τ^l} is locally convex then so is $t_{w\tau^l}$.

PROOF. (i): If t_τ is locally convex then t_τ is determined by a family $(p_j)_{j \in J}$ of seminorms on $S(\mathcal{M}, \tau)$ [20, 1.10.1]. Then the family $(p_{j,P})_{j \in J, P \in \mathcal{M}_\tau^{\text{pf}}}$ of seminorms on $S(\mathcal{M}, \tau)$, where $p_{j,P}(X) = p_j(XP)$ for all $j \in J$, with $P \in \mathcal{M}_\tau^{\text{pf}}$ and $X \in S(\mathcal{M}, \tau)$, is a defining family for t_{τ^l} .

(ii): If $t_{\tau l}$ is locally convex then $t_{\tau l}$ is determined by a family $(q_j)_{j \in J}$ of seminorms on $S(\mathcal{M}, \tau)$ [20, 1.10.1]. Then the family $(q_{j,P})_{j \in J; P \in \mathcal{M}^{\text{pr}}}$ of seminorms on $S(\mathcal{M}, \tau)$, where $q_{j,P}(X) = q_j(PXP)$ for all $j \in J$, with $P \in \mathcal{M}^{\text{pr}}$ and $X \in S(\mathcal{M}, \tau)$, is a defining family for $t_{w\tau l}$. \square

Recall [19] that the following are equivalent:

- (i) $\mathcal{M} = S(\mathcal{M}, \tau)$;
- (ii) $\inf\{\tau(P) : P \in \mathcal{M}^{\text{pr}}, \tau(P) \neq 0\} > 0$;
- (iii) t_{τ} coincides with the $\|\cdot\|$ -topology on \mathcal{M} .

It is easy to verify that if these conditions hold then \mathcal{M} is atomic.

Corollary 2. *For an atomic von Neumann algebra \mathcal{M} with $\inf\{\tau(P) : P \in \mathcal{M}^{\text{pr}}, \tau(P) \neq 0\} = 0$, if*

$$\exists K > 0 \quad \sum_{\tau(P) < K, P \text{ is an atom}} \tau(P) < \infty \quad (1)$$

then $t_{\tau l}$ and $t_{w\tau l}$ are locally convex.

PROOF. The claim follows from [21, Theorem 3.2] and Theorem 3. \square

Corollary 3. *Suppose that \mathcal{M} is an abelian atomic von Neumann algebra and one of the following holds: (a) $\inf\{\tau(P) : P \in \mathcal{M}^{\text{pr}}, \tau(P) \neq 0\} > 0$ or (b) $\inf\{\tau(P) : P \in \mathcal{M}^{\text{pr}}, \tau(P) \neq 0\} = 0$ and condition (1). Then $t_{\tau l} = t_{w\tau l}$ is locally convex.*

PROOF. The claim follows from [21, Corollary 3.5] and Theorem 3. \square

3. Sequences of F -NIPs on (\mathcal{M}, τ)

Consider a faithful normal semifinite trace τ on a von Neumann algebra \mathcal{M} . A $*$ -linear space $\mathcal{X} \subset S(\mathcal{M}, \tau)$ endowed with the F -norm $\|\cdot\|_{\mathcal{X}}$ is an F -normed ideal space (F -NIP) on (\mathcal{M}, τ) whenever

- (i) $\|A\|_{\mathcal{X}} = \|A^*\|_{\mathcal{X}}$ for all $A \in \mathcal{X}$;
- (ii) $A \in \mathcal{X}$ and $B \in S(\mathcal{M}, \tau)$ with $|B| \leq |A|$ imply that $B \in \mathcal{X}$ and $\|B\|_{\mathcal{X}} \leq \|A\|_{\mathcal{X}}$.

The natural embedding $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle \subset S(\mathcal{M}, \tau)$ is $t_{w\tau l}$ -continuous (see [22, Theorem 1] as well as [15]). Regarding the ideal spaces of τ -measurable operators, see [23, 24] and the bibliographies therein.

EXAMPLE (a “weighted” F -NIP). Consider an F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ on (\mathcal{M}, τ) . If $X \in \mathcal{X}$ and $T \in \mathcal{M}$ with $\|T\| \leq 1$ then

$$\|TX\|_{\mathcal{X}} \leq \|X\|_{\mathcal{X}}, \quad \|XT\|_{\mathcal{X}} \leq \|X\|_{\mathcal{X}}; \quad (2)$$

for $Y \notin \mathcal{X}$ we write $\|Y\|_{\mathcal{X}} = +\infty$. If $A \in S(\mathcal{M}, \tau)^{\text{h}}$ then for

$$\|X\|_{\mathcal{X}(A)} \equiv \left\| \frac{AX + XA}{2} \right\|_{\mathcal{X}}, \quad X \in \mathcal{X}(A) = \{Y \in S(\mathcal{M}, \tau) : AY + YA \in \mathcal{X}\},$$

we have

$$\|X^*\|_{\mathcal{X}(A)} = \left\| \frac{AX^* + X^*A}{2} \right\|_{\mathcal{X}} = \left\| \frac{(AX^* + X^*A)^*}{2} \right\|_{\mathcal{X}} = \|X\|_{\mathcal{X}(A)}.$$

If also $AZ = ZA$ for all $Z \in S(\mathcal{M}, \tau)$ and $X, Y \in S(\mathcal{M}, \tau)$ with $|X| \leq |Y|$ then $|X| = T|Y|T^*$ for some $T \in \mathcal{M}$ with $\|T\| \leq 1$ [25], and (2) yields

$$\begin{aligned} \| |X| \|_{\mathcal{X}(A)} &= \left\| \frac{|X| + |X|}{2} \right\|_{\mathcal{X}} = \| |X| \|_{\mathcal{X}} = \| AT|Y|T^* \|_{\mathcal{X}} \\ &\leq \| AT|Y| \|_{\mathcal{X}} = \| T|Y|A \|_{\mathcal{X}} \leq \| |Y|A \|_{\mathcal{X}} = \| |Y| \|_{\mathcal{X}(A)}. \end{aligned}$$

If, moreover, A is invertible in $S(\mathcal{M}, \tau)$ (regarding invertibility in $S(\mathcal{M}, \tau)$, see [26, 27]); then $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$ is an F -NIP on (\mathcal{M}, τ) ; if $\|\cdot\|_{\mathcal{X}}$ is a norm on \mathcal{X} then $\|\cdot\|_{\mathcal{X}(A)}$ is a norm on $\mathcal{X}(A)$. Given $\mathcal{U} \subset \mathcal{X}$, put $A^{-1} \cdot \mathcal{U} \equiv \{A^{-1}U : U \in \mathcal{U}\}$. It is obvious that \mathcal{U} is convex if and only if so is $A^{-1} \cdot \mathcal{U}$.

Theorem 4. Consider an invertible operator $A \in S(\mathcal{M}, \tau)^h$ in $S(\mathcal{M}, \tau)$ with $AZ = ZA$ for all $Z \in S(\mathcal{M}, \tau)$.

(i) An F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ is complete if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.

(ii) $(X_n)_{n=1}^{\infty}$ is a Schauder basis in an F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ if and only if $(A^{-1}X_n)_{n=1}^{\infty}$ is a Schauder basis in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.

(iii) $\mathcal{U} \subset \mathcal{X}$ is a neighborhood of zero in an F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ if and only if $A^{-1} \cdot \mathcal{U}$ is a neighborhood of zero in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.

(iv) $\mathcal{U} \subset \mathcal{X}$ is bounded in an F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ if and only if $A^{-1} \cdot \mathcal{U}$ is bounded in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.

PROOF. Recall that $\mathcal{X}(A) = \{Y \in S(\mathcal{M}, \tau) : \mathbf{M}_A Y = AY \in \mathcal{X}\}$, while $A, A^{-1} \in S(\mathcal{M}, \tau)^h$ with $AZ = ZA$ and $A^{-1}Z = ZA^{-1}$ for all $Z \in S(\mathcal{M}, \tau)$. Note that $(\mathcal{X}(A))(A^{-1}) = \mathcal{X}$ and $\|\cdot\|_{(\mathcal{X}(A))(A^{-1})} = \|\cdot\|_{\mathcal{X}}$. Therefore, we just need to establish the sufficiency of each of conditions (i)–(iv).

(i): If $(X_n)_{n=1}^{\infty} \subset \mathcal{X}(A)$ is $\|\cdot\|_{\mathcal{X}(A)}$ -fundamental then

$$\|X_n - X_m\|_{\mathcal{X}(A)} = \|A(X_n - X_m)\|_{\mathcal{X}} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

i.e., $(AX_n)_{n=1}^{\infty}$ is $\|\cdot\|_{\mathcal{X}}$ -fundamental. Since the F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ is complete, there is $X \in \mathcal{X}$ such that $\|AX_n - X\|_{\mathcal{X}} \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\|AX_n - A \cdot A^{-1}X\|_{\mathcal{X}} = \|X_n - A^{-1}X\|_{\mathcal{X}(A)} \rightarrow 0$ as $n \rightarrow \infty$.

(ii): Suppose that $(X_n)_{n=1}^{\infty} \subset \mathcal{X}$ is a Schauder basis for \mathcal{X} [28, Chapter II, §5], i.e., for each $X \in \mathcal{X}$ there exists a unique expansion $X = \sum_{n=1}^{\infty} \lambda_n X_n$ into a $\|\cdot\|_{\mathcal{X}}$ -converging series with $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$. If $Y \in \mathcal{X}(A)$ then $AY \in \mathcal{X}$, and $AY = \sum_{n=1}^{\infty} \lambda_n X_n$ by assumption. Since this series $\|\cdot\|_{\mathcal{X}}$ -converges, [22, Theorem 1] shows that

$$\sum_{n=1}^k \lambda_n X_n \xrightarrow{w\tau l} AY \quad \text{as } k \rightarrow \infty.$$

Hence, the separate $t_{w\tau l}$ -continuity of multiplication [5, Theorem 1] implies that

$$A^{-1} \sum_{n=1}^k \lambda_n X_n = \sum_{n=1}^k \lambda_n A^{-1} X_n \xrightarrow{w\tau l} Y \quad \text{as } k \rightarrow \infty; \quad (3)$$

i.e., $Y = \sum_{n=1}^{\infty} \lambda_n A^{-1} X_n$.

UNIQUENESS. Suppose that $Y \in \mathcal{X}(A)$ admits two distinct representations

$$Y = \sum_{n=1}^{\infty} \lambda_n A^{-1} X_n = \sum_{n=1}^{\infty} \alpha_n A^{-1} X_n.$$

Then by Theorem 1 of [22] we have

$$\sum_{n=1}^k \lambda_n A^{-1} X_n \xrightarrow{w\tau l} Y, \quad \sum_{n=1}^k \alpha_n A^{-1} X_n \xrightarrow{w\tau l} Y \quad \text{as } k \rightarrow \infty.$$

Hence, the separate $t_{w\tau l}$ -continuity of multiplication [5, Theorem 1] implies that, see (3),

$$AY = \sum_{n=1}^{\infty} \lambda_n X_n = \sum_{n=1}^{\infty} \alpha_n X_n;$$

this is a contradiction.

(iii): Take an open set \mathcal{U} in \mathcal{X} ; i.e.,

$$\forall X \in \mathcal{U} \exists \varepsilon = \varepsilon(X) > 0 \forall Y \in \mathcal{X} \quad (\|X - Y\|_{\mathcal{X}} < \varepsilon \Rightarrow Y \in \mathcal{U}). \quad (4)$$

We have to show that

$$\forall \tilde{X} \in A^{-1} \cdot \mathcal{U} \exists \tilde{\varepsilon} = \tilde{\varepsilon}(\tilde{X}) > 0 \forall \tilde{Y} \in \mathcal{X}(A) \quad (\|\tilde{X} - \tilde{Y}\|_{\mathcal{X}(A)} < \tilde{\varepsilon} \Rightarrow \tilde{Y} \in A^{-1} \cdot \mathcal{U}).$$

For $\tilde{X}, \tilde{Y} \in A^{-1} \cdot \mathcal{U}$ there are $X, Y \in \mathcal{U}$ with $\tilde{X} = A^{-1}X$ and $\tilde{Y} = A^{-1}Y$; appreciating (4), we have

$$\|\tilde{X} - \tilde{Y}\|_{\mathcal{X}(A)} = \|A(A^{-1}X - A^{-1}Y)\|_{\mathcal{X}} = \|X - Y\|_{\mathcal{X}} < \varepsilon.$$

We can take $\tilde{\varepsilon} = \varepsilon$.

(iv) A set $\mathcal{U} \subset \mathcal{X}$ is bounded in $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ if and only if for each neighborhood \mathcal{V} of zero there is a real $\lambda > 0$ such that $\mathcal{U} \subset \lambda\mathcal{V}$. In other words, $\|\alpha_n X_n\|_{\mathcal{X}} \rightarrow 0$ as $n \rightarrow \infty$ for all sequences $(X_n)_{n=1}^{\infty} \subset \mathcal{U}$ and $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ with $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ [29, Chapter 1, Theorem 1.30]. Verify that $A^{-1} \cdot \mathcal{U}$ is bounded in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$. Take an arbitrary sequence $(Y_n)_{n=1}^{\infty} \subset A^{-1} \cdot \mathcal{U}$ and $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ with $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Then there is $(X_n)_{n=1}^{\infty} \subset \mathcal{U}$ such that $Y_n = A^{-1}X_n$ for all $n \in \mathbb{N}$ and

$$\|\alpha_n Y_n\|_{\mathcal{X}(A)} = \|\alpha_n A^{-1}X_n\|_{\mathcal{X}(A)} = \|\alpha_n A A^{-1}X_n\|_{\mathcal{X}} = \|\alpha_n X_n\|_{\mathcal{X}} \rightarrow 0$$

as $n \rightarrow \infty$. \square

Obviously, if an F -normed space $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ has a Schauder basis then the space is separable. Every topological vector space with a bounded neighborhood of zero is *locally bounded*.

Corollary 4. *An F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ is locally bounded, respectively locally convex, if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.*

Corollary 5. *An F -NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ is normable if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)} \rangle$.*

PROOF. A topological vector space is normable if and only if it includes a bounded convex neighborhood of zero [29, Chapter 1, Theorem 1.39]. \square

Lemma 2. *Consider two F -NIPs \mathcal{X} and \mathcal{Y} on (\mathcal{M}, τ) such that $\langle \mathcal{Y}, \|\cdot\|_{\mathcal{Y}} \rangle$ is complete, and a dense linear subspace \mathcal{Z} of \mathcal{X} . Take $A \in S(\mathcal{M}, \tau)$ with $A\mathcal{Z} \subseteq \mathcal{Y}$ and continuous multiplier $\mathbf{M}_A : \mathcal{Z} \rightarrow \mathcal{Y}$. Then the continuous extension $\overline{\mathbf{M}}_A : \mathcal{X} \rightarrow \mathcal{Y}$ of \mathbf{M}_A to the whole space \mathcal{X} is also a multiplier by A .*

PROOF. Take $X \in \mathcal{X}$ and $X_i \in \mathcal{Z}$ such that $X_i \rightarrow X$ in $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$. Then $\mathbf{M}_A X_i = \overline{\mathbf{M}}_A X_i \rightarrow \overline{\mathbf{M}}_A X$ in $\langle \mathcal{Y}, \|\cdot\|_{\mathcal{Y}} \rangle$. By [5, Theorem 1] we see that $X_i \xrightarrow{w\tau l} X$, as well as $\mathbf{M}_A X_i \xrightarrow{w\tau l} \overline{\mathbf{M}}_A X$. In view of [22, Theorem 1] the multiplier $\mathbf{M}_A : S(\mathcal{M}, \tau) \rightarrow S(\mathcal{M}, \tau)$ is $t_{w\tau l}$ -continuous. Therefore, $\mathbf{M}_A X_i \xrightarrow{w\tau l} \mathbf{M}_A X$, and so $\overline{\mathbf{M}}_A X = \mathbf{M}_A X = AX$. \square

The analogous result holds for $\mathbf{L}_A X = XA$.

Theorem 5. *Consider two F -NIPs \mathcal{X} and \mathcal{Y} on (\mathcal{M}, τ) and suppose that $A\mathcal{X} \subseteq \mathcal{Y}$ for some operator $A \in S(\mathcal{M}, \tau)$. Then the multiplier $\mathbf{M}_A X = AX$, acting as $\mathbf{M}_A : \mathcal{X} \rightarrow \mathcal{Y}$, is continuous. In particular, for $\mathcal{X} \subseteq \mathcal{Y}$ the natural embedding of \mathcal{X} into \mathcal{Y} is continuous.*

PROOF. Verify that the graph of \mathbf{M}_A is closed. Indeed, if $X_n \rightarrow X$ in \mathcal{X} and $AX_n \rightarrow Y$ in \mathcal{Y} then $X_n \xrightarrow{w\tau l} X$ and $AX_n \xrightarrow{w\tau l} Y$ by [22, Theorem 1]. Then $Y = AX$ because of the separate $t_{w\tau l}$ -continuity of multiplication [5, Theorem 1]. It remains to apply the Closed Graph Theorem [29, Chapter 2, Theorem 2.15]. \square

REMARK 1. The topological intersection of an at most countable family of F -NIPs $\langle \mathcal{X}_n, \|\cdot\|_{\mathcal{X}_n} \rangle$ on (\mathcal{M}, τ) is also an F -NIP on (\mathcal{M}, τ) with the F -norm

$$\|X\|_{\mathcal{X}} = \sum_{n \geq 1} 2^{-n} \frac{\|X\|_{\mathcal{X}_n}}{1 + \|X\|_{\mathcal{X}_n}}.$$

Recall that $\mathcal{X} = \bigcap_{n \geq 1} \mathcal{X}_n$; the projective, or initial, topology (see [30, p. 35]) induced by the embeddings $\mathcal{X} \subset \mathcal{X}_n$ for $n \in \mathbb{N}$ is the linear topology whose base of neighborhoods of zero consists of all sets of the form $\mathcal{U}_{n_1} \cap \cdots \cap \mathcal{U}_{n_m} \cap \mathcal{X}$, where \mathcal{U}_n is a neighborhood of zero in \mathcal{X}_n . The space \mathcal{X} endowed with this topology is the *topological intersection* of \mathcal{X}_n .

Theorem 6. Consider a decreasing sequence $(\mathcal{X}_n)_{n \geq 1}$ of F -NIPs on (\mathcal{M}, τ) such that the space $\mathcal{X} = \bigcap_{n \geq 1} \mathcal{X}_n$ is dense in each \mathcal{X}_n , some locally bounded F -NIP \mathcal{Y} on (\mathcal{M}, τ) , and $A \in S(\mathcal{M}, \tau)$. Then $A\mathcal{X} \subseteq \mathcal{Y}$ if and only if $A\mathcal{X}_n \subseteq \mathcal{Y}$ for some $n \in \mathbb{N}$.

PROOF. Assume that $A\mathcal{X} \subseteq \mathcal{Y}$. Theorem 5 and Remark 1 imply that the multiplier $\mathbf{M}_A : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous. Suppose that \mathcal{V} is a bounded neighborhood of zero in \mathcal{Y} . There exists a tuple of neighborhoods of zero $(\mathcal{U}_k \subset \mathcal{X}_k)_{k=1}^n$ such that $\mathbf{M}_A(\mathcal{X} \cap \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n) \subseteq \mathcal{V}$. Since the embeddings $\mathcal{X}_n \subset \mathcal{X}_k$ for $k = 1, \dots, n$ are continuous, there exist neighborhoods \mathcal{W}_k of zero in \mathcal{X}_k such that $\mathcal{W}_k \subset \mathcal{U}_k$ for $k = 1, \dots, n$. Putting $\mathcal{U} = \bigcap_{k=1}^n \mathcal{W}_k$, we obtain a neighborhood of zero in \mathcal{X}_n satisfying $\mathbf{M}_A(\mathcal{U} \cap \mathcal{X}) \subseteq \mathcal{V}$. The boundedness of \mathcal{V} is equivalent to the property that the system of its dilations $\varepsilon\mathcal{V}$ with $\varepsilon > 0$ constitutes a base of neighborhoods of zero in \mathcal{Y} . Since for every $\varepsilon > 0$ we have $\mathbf{M}_A(\varepsilon\mathcal{U} \cap \mathcal{X}) \subseteq \varepsilon\mathcal{V}$, the mapping $\mathbf{M}_A : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous in the topology on \mathcal{X} induced from \mathcal{X}_n . Extend this mapping by continuity to some mapping from \mathcal{X}_n to \mathcal{Y} . By Lemma 2 the extended mapping is also a multiplier by A ; consequently, $A\mathcal{X}_n \subseteq \mathcal{Y}$. \square

Corollary 6. Consider the same $(\mathcal{X}_n)_{n \geq 1}$, \mathcal{X} , and \mathcal{Y} as in Theorem 6. Then $\mathcal{X} \subseteq \mathcal{Y} \iff \mathcal{X}_n \subseteq \mathcal{Y}$ for some $n \in \mathbb{N}$.

PROOF. Apply Theorem 6 to the operator $A = I$. \square

Theorem 7 (cf. [20, Chapter 6, § 6.5]). If \mathcal{X} , $\mathcal{Y}_1, \dots, \mathcal{Y}_n, \dots$ is a sequence of F -NIPs on (\mathcal{M}, τ) with $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}} \rangle$ complete then $\mathcal{X} \subseteq \bigcup_{n=1}^{\infty} \mathcal{Y}_n$ if and only if $\mathcal{X} \subseteq \mathcal{Y}_n$ for some $n \in \mathbb{N}$.

PROOF. For $\mathcal{X}_n = \mathcal{X} \cap \mathcal{Y}_n$ we have $\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$. The Baire Category Theorem implies that for some $n \in \mathbb{N}$ the space \mathcal{X}_n is a nonmeager set in \mathcal{X} . The space \mathcal{X}_n regarded as the topological intersection of \mathcal{X} and \mathcal{Y}_n is an F -NIP on (\mathcal{M}, τ) and the embedding $\mathcal{X}_n \subset \mathcal{X}$ is continuous. The Banach Theorem [31, Chapter III, Theorem 3] yields $\mathcal{X}_n = \mathcal{X}$; i.e., $\mathcal{X} \subseteq \mathcal{Y}_n$. \square

References

1. Ciach L.J., “Some remarks on the convergence in measure and on a dominated sequence of operators measurable with respect to a semifinite von Neumann algebra,” *Colloq. Math.*, vol. 55, no. 1, 109–121 (1988).
2. Dodds P.G., Dodds T.K.-Y., and Pagter de B., “Noncommutative Köthe duality,” *Trans. Amer. Math. Soc.*, vol. 339, no. 2, 717–750 (1993).
3. Skvortsova G.Sh. and Tikhonov O.E., “Convex sets in noncommutative L^1 -spaces that are closed in the topology of local convergence in measure,” *Russian Math. (Iz. VUZ)*, vol. 42, no. 8, 46–52 (1998).
4. Skvortsova G.Sh., “On the weak sequential completeness of quotient spaces of the space of integrable operators,” *Russian Math. (Iz. VUZ)*, vol. 46, no. 9, 68–71 (2002).
5. Bikhentaev A.M., “The continuity of multiplication for two topologies associated with a semifinite trace on von Neumann algebra,” *Lobachevskii J. Math.*, vol. 14, 17–24 (2004).
6. Dodds P.G., Dodds T.K.-Y., Sukochev F.A., and Tikhonov O.Ye., “A non-commutative Yosida–Hewitt theorem and convex sets of measurable operators closed locally in measure,” *Positivity*, vol. 9, no. 3, 457–484 (2005).
7. Bikhentaev A.M., “Local convergence in measure on semifinite von Neumann algebras,” *Proc. Steklov Inst. Math.*, vol. 255, 35–48 (2006).
8. Bikhentaev A.M., “Local convergence in measure on semifinite von Neumann algebras. II,” *Math. Notes*, vol. 82, no. 5, 703–707 (2007).
9. Bikhentaev A. and Sukochev F., “When weak and local measure convergence implies norm convergence,” *J. Math. Anal. Appl.*, vol. 473, no. 2, 1414–1431 (2019).
10. Bikhentaev A.M., “Convergence in measure and τ -compactness of τ -measurable operators, affiliated with a semifinite von Neumann algebra,” *Russian Math. (Iz. VUZ)*, vol. 64, no. 5, 79–82 (2020).
11. Cordero E., de Gosson M., and Nicola F., “On the positivity of trace class operators,” *Adv. Theor. Math. Phys.*, vol. 23, no. 8, 2061–2091 (2020).
12. Moslehian M.S., Kian M., and Xu Q., “Positivity of 2×2 block matrices of operators,” *Banach J. Math. Anal.*, vol. 13, no. 3, 726–743 (2019).
13. Bikhentaev A.M., “Invertibility of the operators on Hilbert spaces and ideals in C^* -algebras,” *Math. Notes*, vol. 112, no. 3, 360–368 (2022).
14. Akemann C.A., Anderson J., and Pedersen G.K., “Triangle inequalities in operator algebras,” *Linear Multilinear Algebra*, vol. 11, no. 2, 167–178 (1982).
15. Ber A.F., Levitina G.B., and Chilin V.I., “Derivations with values in quasi-normed bimodules of locally measurable operators,” *Siberian Adv. Math.*, vol. 25, no. 3, 169–178 (2015).

16. Takesaki M., *Theory of Operator Algebras. I.* Operator Algebras and Non-Commutative Geometry V, Springer, Berlin (2002) (Encyclopaedia of Mathematical Sciences; vol. 124).
17. Takesaki M., *Theory of Operator Algebras. II.* Operator Algebras and Non-Commutative Geometry VI, Springer, Berlin (2003) (Encyclopaedia of Mathematical Sciences; vol. 125).
18. Fack T. and Kosaki H., “Generalized s -numbers of τ -measurable operators,” *Pacific J. Math.*, vol. 123, no. 2, 269–300 (1986).
19. Stroh A. and West G. P., “ τ -Compact operators affiliated to a semifinite von Neumann algebra,” *Proc. Roy. Irish Acad. Sect. A*, vol. 93, no. 1, 73–86 (1993).
20. Edwards R.D., *Functional Analysis. Theory and Applications*, Holt Rinehart and Winston, New York, Chicago, San Francisco, Toronto, and London (1965).
21. Conrădie J. and Crowther Ch., “The algebra of τ -measurable operators,” in: *Operator algebras, Operator Theory and Applications*, Birkhäuser, Basel (2010), 103–119 (Oper. Theory Adv. Appl.; vol. 195).
22. Bikchentaev A.M., “Minimality of convergence in measure topologies on finite von Neumann algebras,” *Math. Notes*, vol. 75, no. 3, 315–321 (2004).
23. Bikchentaev A.M., “Ideal spaces of measurable operators affiliated to a semifinite von Neumann algebra,” *Sib. Math. J.*, vol. 59, no. 2, 243–251 (2018).
24. Bikchentaev A.M., “Renormalizations of measurable operator ideal spaces affiliated to semi-finite von Neumann algebra,” *Ufa Math. J.*, vol. 11, no. 3, 3–10 (2019).
25. Yeadon F.J., “Convergence of measurable operators,” *Proc. Cambridge Phil. Soc.*, vol. 74, no. 2, 257–268 (1973).
26. Tembo I.D., “Invertibility in the algebra of τ -measurable operators,” in: *Operator Algebras, Operator Theory and Applications*, Birkhäuser, Basel (2010), 245–256 (Oper. Theory Adv. Appl.; vol. 195).
27. Bikchentaev A.M., “Essentially invertible measurable operators affiliated to a semifinite von Neumann algebra and commutators,” *Sib. Math. J.*, vol. 63, no. 2, 224–232 (2022).
28. Rolewicz S., *Metric Linear Spaces*. Second Edition, PWN–Polish Sci. and Reidel, Warsaw and Dordrecht (1984).
29. Rudin W., *Functional Analysis*, McGraw-Hill, New York (1991).
30. Bourbaki N., *General Topology. Fundamental Structures*, Nauka, Moscow (1968) [Russian translation].
31. Banach S., *Theory of Linear Operators*, North-Holland, Amsterdam etc. (1987) (North-Holland Math. Library; vol. 38).

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