



# Paranormal measurable operators affiliated with a semifinite von Neumann algebra. II

Airat Bikchentaev<sup>1</sup>

Received: 2 September 2019 / Accepted: 17 February 2020  
© Springer Nature Switzerland AG 2020

## Abstract

Let  $\mathcal{M}$  be a von Neumann algebra of operators on a Hilbert space  $\mathcal{H}$  and  $\tau$  be a faithful normal semifinite trace on  $\mathcal{M}$ . Let  $t_\tau$  be the measure topology on the  $*$ -algebra  $S(\mathcal{M}, \tau)$  of all  $\tau$ -measurable operators. We define three  $t_\tau$ -closed classes  $\mathcal{P}_1$ ,  $\mathcal{P}_2$  and  $\mathcal{P}_3$  of  $\tau$ -measurable operators and investigate their properties. The class  $\mathcal{P}_2$  contains  $\mathcal{P}_1 \cup \mathcal{P}_3$ . If a  $\tau$ -measurable operator  $T$  is hyponormal, then  $T$  lies in  $\mathcal{P}_1 \cap \mathcal{P}_3$ ; if an operator  $T$  lies in  $\mathcal{P}_3$ , then  $UTU^*$  belongs to  $\mathcal{P}_3$  for all isometries  $U$  from  $\mathcal{M}$ . If a bounded operator  $T$  lies in  $\mathcal{P}_1 \cup \mathcal{P}_3$  then  $T$  is normaloid. If an operator  $T \in S(\mathcal{M}, \tau)$  is  $p$ -hyponormal with  $0 < p \leq 1$  then  $T \in \mathcal{P}_1$ . If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  and  $\tau = \text{tr}$  is the canonical trace, then the class  $\mathcal{P}_1$  (resp.,  $\mathcal{P}_3$ ) coincides with the set of all paranormal (resp.,  $*$ -paranormal) operators on  $\mathcal{H}$ . Let  $A, B \in S(\mathcal{M}, \tau)$  and  $A$  be  $p$ -hyponormal with  $0 < p \leq 1$ . If  $AB$  is  $\tau$ -compact then  $A^*B$  is  $\tau$ -compact.

**Keywords** Hilbert space · von Neumann algebra · Trace · Non-commutative integration · Measurable operator · Generalized singular value function · Paranormal operator · Hyponormal operator · Operator inequality

**Mathematics Subject Classification** 46L10 · 47C15 · 46L51

## 1 Introduction

It is well known that bounded hyponormal operators on a Hilbert space  $\mathcal{H}$  have some interesting properties. For example, if  $A$  is a hyponormal operator then  $\|A^n\|_\infty = \|A\|_\infty^n$  for every  $n \in \mathbb{N}$  [20, Problem 162], here  $\|\cdot\|_\infty$  denotes the uniform norm on  $\mathcal{B}(\mathcal{H})$ ; every bounded hyponormal compact operator is normal [20, Problem 163].

---

*This paper is dedicated to Professor P. G. Ovchinnikov.*

✉ Airat Bikchentaev  
Airat.Bikchentaev@kpfu.ru

<sup>1</sup> Kazan Federal University, 18 Kremlyovskaya Str., Kazan, Russia 420008

Fruitful generalizations of the notion of a hyponormal operator are the concepts of  $p$ -hyponormal [1], paranormal [17,23], and  $*$ -paranormal operators [3]. A number of modern authors study properties of such operators (see, for example, [29,30] and references in them).

In this article, we obtain analogs of certain properties of bounded  $p$ -hyponormal, paranormal, and  $*$ -paranormal operators on  $\mathcal{H}$  for some unbounded ones. Let  $\mathcal{M}$  be a von Neumann operator algebra on a Hilbert space  $\mathcal{H}$ ,  $\mathbf{1}$  be the unit of  $\mathcal{M}$ ,  $\tau$  be a faithful normal semifinite trace on  $\mathcal{M}$ ,  $S(\mathcal{M}, \tau)$  be the  $*$ -algebra of all  $\tau$ -measurable operators, a number  $0 < p < +\infty$  and  $L_p(\mathcal{M}, \tau)$  be the space of integrable (with respect to  $\tau$ ) in  $p$ th degree operators. Let  $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\|_\infty = 1\}$ ,  $\mu(\cdot; X)$  be the generalized singular value function of operator  $X \in S(\mathcal{M}, \tau)$  and let  $|X| = \sqrt{X^*X}$ . Assume that  $\|X\|_\infty = +\infty$  for all  $X \in S(\mathcal{M}, \tau) \setminus \mathcal{M}$ .

In papers [6,8] we introduced two classes of  $\tau$ -measurable operators

$$\begin{aligned} \mathcal{P}_1 &= \{T \in S(\mathcal{M}, \tau) : \|T^2 A\|_\infty \geq \|TA\|_\infty^2 \text{ for all } A \in \mathcal{M}_1 \text{ with } TA \in \mathcal{M}\}, \\ \mathcal{P}_2 &= \{T \in S(\mathcal{M}, \tau) : \mu(t; T^2) \geq \mu(t; T)^2 \text{ for all } t > 0\} \end{aligned}$$

and investigated their properties. The classes  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are closed in the topology of convergence in measure  $\tau$  and  $\mathcal{P}_1 \subset \mathcal{P}_2$  (Propositions 3.5 and 3.30 of [6]). In [6, Theorem 3.1] we gave an equivalent definition of the class  $\mathcal{P}_1$  [i.e.,  $T \in \mathcal{P}_1$  if and only if  $|T|^2 \leq (\lambda^{-1}|T|^2 + \lambda\mathbf{1})/2$  for all  $\lambda > 0$ ], that allowed us to call  $\mathcal{P}_1$  a class of all paranormal  $\tau$ -measurable operators. A similar definition of paranormal elements for general normed algebras was introduced and investigated in [7].

If an operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal then  $T \in \mathcal{P}_1$ ; if an operator  $T \in \mathcal{P}_1$  has the inverse  $T^{-1} \in \mathcal{M}$  then  $T^{-1} \in \mathcal{P}_1$  [6, Theorem 3.6]. If an operator  $T \in \mathcal{P}_k$  then  $UTU^* \in \mathcal{P}_k$  for all isometries  $U \in \mathcal{M}$  and  $k = 1, 2$ . If an operator  $T \in \mathcal{P}_1 \cap \mathcal{M}$  then  $T^n \in \mathcal{P}_1$  for all  $n \in \mathbb{N}$  [6, Theorem 3.12]. Consider an operator  $T \in \mathcal{P}_1 \cap \mathcal{M}$  and  $n \in \mathbb{N}$ . Then  $\mu(t, T^n) \geq \mu(t; T)^n$  for all  $t > 0$  [6, Theorem 3.16] and we have the equivalences: an operator  $T$  is  $\tau$ -compact  $\Leftrightarrow$  an operator  $T^n$  is  $\tau$ -compact;  $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau)$ ,  $0 < p < +\infty$  [6, Corollary 3.17]. Every operator  $T \in \mathcal{P}_1 \cap \mathcal{M}$  is normaloid [6, Corollary 3.18]. Each  $\tau$ -compact  $p$ -hyponormal operator is normal [12, Theorem 2.2]. If an operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal and  $T^n$  is  $\tau$ -compact for some natural number  $n$  then  $T$  is both normal and  $\tau$ -compact [6, Corollary 3.7]; it is a strengthening of item (i) of Corollary 3.2 [12]. If  $T \in \mathcal{P}_1$  then  $T^2 \in \mathcal{P}_1$  [6, Theorem 3.21].

Put  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  and  $\tau$  be the canonical trace  $\text{tr}$ . Then the class  $\mathcal{P}_1$  coincides with the set of all paranormal operators on  $\mathcal{H}$  [6, Corollary 3.3], is sequentially closed in the strong operator topology [6, Corollary 3.4] and contains a non-hyponormal operator [6, Corollary 3.13]. If  $\mathcal{H}$  is separable and infinite-dimensional then  $\mathcal{P}_1 \neq \mathcal{P}_2$  [6, Corollary 3.23].

In this paper we introduce the class

$$\mathcal{P}_3 = \{T \in S(\mathcal{M}, \tau) : \|T^2 A\|_\infty \geq \|T^* A\|_\infty^2 \text{ for all } A \in \mathcal{M}_1 \text{ with } T^* A \in \mathcal{M}\}$$

of  $\tau$ -measurable operators and investigate some properties of  $\mathcal{P}_1$  and  $\mathcal{P}_3$ . In Theorem 3.1 we obtain an equivalent definition of the class  $\mathcal{P}_3$  [i.e.,  $T \in \mathcal{P}_3$  if and only if

$|T^*|^2 \leq (\lambda^{-1}|T|^2 + \lambda\mathbf{1})/2$  for all  $\lambda > 0$ ], that allows us to call  $\mathcal{P}_3$  a class of all  $*$ -paranormal  $\tau$ -measurable operators. The class  $\mathcal{P}_3$  is closed in the measure topology  $t_\tau$  (Corollary 3.2). If an operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal then  $T \in \mathcal{P}_3$ ; if an operator  $T \in \mathcal{P}_3$  then  $UTU^* \in \mathcal{P}_3$  for all isometries  $U \in \mathcal{M}$  (Theorem 3.6). If an operator  $T \in S(\mathcal{M}, \tau)$  is  $p$ -hyponormal with  $0 < p \leq 1$  then  $T \in \mathcal{P}_1$  and  $\mu(t; T^2) \geq \mu(t; T)^2$  for all  $t > 0$  (Theorem 4.4 and Corollary 4.5). It is a strengthening of item (i) of Theorem 3.6 [6] and a generalization of Theorem 3 [28]. Methods of proof are new even for algebra  $\mathcal{B}(\mathcal{H})$ , endowed with the canonical trace  $\text{tr}$ . Let  $A, B \in S(\mathcal{M}, \tau)$  and  $A$  be  $p$ -hyponormal with  $0 < p \leq 1$ . If  $AB$  is  $\tau$ -compact then  $A^*B$  is  $\tau$ -compact (Theorem 5.1). On  $\tau$ -compactness of products of  $\tau$ -measurable operators see [9].

## 2 Notation, definitions and preliminaries

Let  $\mathcal{M}$  be a von Neumann algebra of operators on a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{P}(\mathcal{M})$  be the lattice of projections in  $\mathcal{M}$ ,  $\mathbf{1}$  be the unit of  $\mathcal{M}$ , and let  $P^\perp = \mathbf{1} - P$  for  $P \in \mathcal{P}(\mathcal{M})$ . Also  $\mathcal{M}^+$  denotes the cone of positive elements in  $\mathcal{M}$ , and  $\|\cdot\|_\infty$  denotes the uniform norm on  $\mathcal{M}$ . A mapping  $\varphi : \mathcal{M}^+ \rightarrow [0, +\infty]$  is called a *trace*, if  $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ ,  $\varphi(\lambda X) = \lambda\varphi(X)$  for all  $X, Y \in \mathcal{M}^+, \lambda \geq 0$  [moreover,  $0 \cdot (+\infty) \equiv 0$ ];  $\varphi(Z^*Z) = \varphi(ZZ^*)$  for all  $Z \in \mathcal{M}$ . A trace  $\varphi$  is called *faithful*, if  $\varphi(X) > 0$  for all  $X \in \mathcal{M}^+, X \neq 0$ ; *normal*, if  $X_i \uparrow X (X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$ ; *semifinite*, if  $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$  for every  $X \in \mathcal{M}^+$ .

A linear operator  $X : \mathfrak{D}(X) \rightarrow \mathcal{H}$ , where the domain  $\mathfrak{D}(X)$  of  $X$  is a linear subspace of  $\mathcal{H}$ , is said to be *affiliated* with  $\mathcal{M}$  if  $YX \subseteq XY$  for all  $Y \in \mathcal{M}'$ , where  $\mathcal{M}'$  is the commutant of  $\mathcal{M}$ . A linear operator  $X : \mathfrak{D}(X) \rightarrow \mathcal{H}$  is termed *measurable* with respect to  $\mathcal{M}$  if  $X$  is closed, densely defined, affiliated with  $\mathcal{M}$  and there exists a sequence  $\{P_n\}_{n=1}^\infty$  in the logic of all projections of  $\mathcal{M}, \mathcal{P}(\mathcal{M})$ , such that  $P_n \uparrow \mathbf{1}, P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$  and  $P_n^\perp$  is a finite projection (with respect to  $\mathcal{M}$ ) for all  $n$ . It should be noted that the condition  $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$  implies that  $XP_n \in \mathcal{M}$ . The collection of all measurable operators with respect to  $\mathcal{M}$  is denoted by  $S(\mathcal{M})$ , which is a unital  $*$ -algebra with respect to strong sums and products [denoted simply by  $X + Y$  and  $XY$  for all  $X, Y \in S(\mathcal{M})$ ] [27,31].

Let  $X$  be a self-adjoint operator affiliated with  $\mathcal{M}$ . We denote its spectral measure by  $\{E^X\}$ . It is well known that if  $X$  is a closed operator affiliated with  $\mathcal{M}$  with the polar decomposition  $X = U|X|$ , then  $U \in \mathcal{M}$  and  $E \in \mathcal{M}$  for all projections  $E \in \{E^{|X|}\}$ . Moreover,  $X \in S(\mathcal{M})$  if and only if  $X$  is closed, densely defined, affiliated with  $\mathcal{M}$  and  $E^{|X|}(\lambda, \infty)$  is a finite projection for some  $\lambda > 0$ . It follows immediately that in the case when  $\mathcal{M}$  is a von Neumann algebra of type III or a type I factor, we have  $S(\mathcal{M}) = \mathcal{M}$ . For type II von Neumann algebras, this is no longer true. From now on, let  $\mathcal{M}$  be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace  $\tau$ .

For any closed and densely defined linear operator  $X : \mathfrak{D}(X) \rightarrow \mathcal{H}$ , the *null projection*  $n(X) = n(|X|)$  is the projection onto its kernel  $\text{Ker}(X)$ , the *range projection*  $r(X)$  is the projection onto the closure of its range  $\text{Ran}(X)$  and the *support projection*  $s(X)$  of  $X$  is defined by  $s(X) = \mathbf{1} - n(X)$ .

An operator  $X \in S(\mathcal{M})$  is called  $\tau$ -measurable if there exists a sequence  $\{P_n\}_{n=1}^\infty$  in  $\mathcal{P}(\mathcal{M})$  such that  $P_n \uparrow \mathbf{1}$ ,  $P_n(\mathcal{H}) \subseteq \mathfrak{D}(X)$  and  $\tau(P_n^\perp) < \infty$  for all  $n$ . The collection  $S(\mathcal{M}, \tau)$  of all  $\tau$ -measurable operators is a unital  $*$ -subalgebra of  $S(\mathcal{M})$  denoted by  $S(\mathcal{M}, \tau)$ . It is well known that a linear operator  $X$  belongs to  $S(\mathcal{M}, \tau)$  if and only if  $X \in S(\mathcal{M})$  and there exists  $\lambda > 0$  such that  $\tau(E^{|X|}(\lambda, \infty)) < \infty$ . Alternatively, an unbounded operator  $X$  affiliated with  $\mathcal{M}$  is  $\tau$ -measurable (see [16]) if and only if

$$\tau(E^{|X|}(n, +\infty)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let  $\mathcal{L}^+$  and  $\mathcal{L}_h$  denote the positive and Hermitian parts of a family  $\mathcal{L} \subset S(\mathcal{M}, \tau)$ , respectively. We denote by  $\leq$  the partial order in  $S(\mathcal{M}, \tau)_h$  generated by its proper cone  $S(\mathcal{M}, \tau)^+$ . If  $X \in S(\mathcal{M}, \tau)$ , then  $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$ .

**Definition 2.1** Let a semifinite von Neumann algebra  $\mathcal{M}$  be equipped with a faithful normal semifinite trace  $\tau$  and let  $X \in S(\mathcal{M}, \tau)$ . The generalized singular value function  $\mu(X) : t \rightarrow \mu(t; X)$  of the operator  $X$  is defined by setting

$$\mu(s; X) = \inf\{\|XP\|_\infty : P \in \mathcal{P}(\mathcal{M}) \text{ such that } \tau(P^\perp) \leq s\}. \quad (1)$$

An equivalent definition in terms of the distribution function of the operator  $X$  is the following. For every self-adjoint operator  $X \in S(\mathcal{M}, \tau)$ , setting

$$d_X(t) = \tau(E^X(t, \infty)), \quad t > 0,$$

we have (see e.g. [16] and [26])

$$\mu(t; X) = \inf\{s \geq 0 : d_{|X|}(s) \leq t\}.$$

Note that  $d_X(\cdot)$  is a right-continuous function (see e.g. [16]).

For convenience of the reader, we also recall the definition of the *measure topology*  $t_\tau$  on the algebra  $S(\mathcal{M}, \tau)$ . For every  $\varepsilon, \delta > 0$ , we define the set

$$V(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \exists P \in \mathcal{P}(\mathcal{M}) \text{ such that } \|XP\|_\infty \leq \varepsilon, \tau(P^\perp) \leq \delta\}.$$

The topology generated by the sets  $V(\varepsilon, \delta)$ ,  $\varepsilon, \delta > 0$ , is called the *measure topology*  $t_\tau$  on  $S(\mathcal{M}, \tau)$  [16, 27]. It is well-known that the algebra  $S(\mathcal{M}, \tau)$  equipped with the measure topology is a complete metrizable topological algebra [27]. We note that a sequence  $\{X_n\}_{n=1}^\infty \subset S(\mathcal{M}, \tau)$  converges to zero with respect to measure topology  $t_\tau$  (i.e.  $X_n \xrightarrow{t_\tau} 0$ ) if and only if  $\tau(E^{|X_n|}(\varepsilon, \infty)) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\varepsilon > 0$ .

The space  $S_0(\mathcal{M}, \tau)$  of  $\tau$ -compact operators is the space associated to the algebra of functions from  $S(0, \infty)$  vanishing at infinity, that is,

$$S_0(\mathcal{M}, \tau) = \left\{ X \in S(\mathcal{M}, \tau) : \lim_{t \rightarrow +\infty} \mu(t; X) = 0 \right\}.$$

The two-sided ideal  $\mathcal{F}(\tau)$  in  $\mathcal{M}$  consisting of all elements of  $\tau$ -finite range is defined by

$$\mathcal{F}(\tau) = \{X \in \mathcal{M} : \tau(r(X)) < +\infty\} = \{X \in \mathcal{M} : \tau(s(X)) < +\infty\}.$$

Equivalently,  $\mathcal{F}(\tau) = \{X \in \mathcal{M} : \mu(t; X) = 0 \text{ for some } t > 0\}$ . Clearly,  $S_0(\mathcal{M}, \tau)$  is the closure of  $\mathcal{F}(\tau)$  with respect to the measure topology [14], which is a two-sided ideal in  $S(\mathcal{M}, \tau)$ .

Let  $m$  be Lebesgue measure on  $\mathbb{R}$ . The noncommutative  $L_p$ -Lebesgue space ( $0 < p < +\infty$ ) affiliated with  $(\mathcal{M}, \tau)$  is defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \mu(X) \in L_p(\mathbb{R}^+, m)\}$$

with the quasi-norm  $\|X\|_p = \|\mu(X)\|_p$ ,  $X \in L_p(\mathcal{M}, \tau)$ . In particular,  $\|\cdot\|_p$  is a norm when  $1 \leq p < +\infty$ . We have  $\mathcal{F}(\tau) \subset L_p(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$  for all  $0 < p < +\infty$ .

If  $\tau(\mathbf{1}) < +\infty$  then  $S(\mathcal{M}, \tau) = S_0(\mathcal{M}, \tau)$  consists of all closed linear operators on  $\mathcal{H}$  affiliated with  $\mathcal{M}$  and  $\mathcal{F}(\tau) = \mathcal{M}$ . Furthermore,  $t_\tau$  is independent of a concrete choice of a trace  $\tau$  and is minimal among all metrizable topologies which agree with the ring structure of  $S(\mathcal{M}, \tau)$  [13, Theorem 2].

**Lemma 2.2** [16] *Let  $X, Y, Z \in S(\mathcal{M}, \tau)$ . Then*

- (1)  $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$  for all  $t > 0$ ;
- (2) if  $X, Y \in \mathcal{M}$  then  $\mu(t; XZY) \leq \|X\|_\infty \|Y\|_\infty \mu(t; Z)$  for all  $t > 0$ ;
- (3)  $\mu(t; |X|^p) = \mu(t, X)^p$  for all  $p > 0$  and  $t > 0$ ;
- (4) if  $|X| \leq |Y|$  then  $\mu(t; X) \leq \mu(t; Y)$  for all  $t > 0$ ;
- (5)  $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$  for all  $s, t > 0$ ;
- (6)  $\mu(t; \lambda X) = |\lambda| \mu(t; X)$  for all  $\lambda \in \mathbb{C}$  and  $t > 0$ ;
- (7)  $\lim_{t \rightarrow 0^+} \mu(t; X) = \|X\|_\infty$  if  $X \in \mathcal{M}$  and  $\lim_{t \rightarrow 0^+} \mu(t; X) = +\infty$  if  $X \notin \mathcal{M}$ .

An operator  $A \in S(\mathcal{M}, \tau)$  is said to be  $p$ -hyponormal with  $0 < p \leq 1$ , if  $(A^*A)^p \geq (AA^*)^p$ ; hyponormal, if it is 1-hyponormal; cohyponormal, if  $A^*$  is hyponormal; quasinormal, if  $A$  commutes with  $A^*A$ , i.e.  $A \cdot A^*A = A^*A \cdot A$ .

**Lemma 2.3** (See [15], p. 720) *If  $X, Y \in S(\mathcal{M}, \tau)^+$  and  $Z \in S(\mathcal{M}, \tau)$  then the inequality  $X \leq Y$  implies that  $ZXZ^* \leq ZYZ^*$ .*

If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$ , i.e. the  $*$ -algebra of all linear bounded operators on  $\mathcal{H}$ , and  $\tau = \text{tr}$  is the canonical trace then  $S(\mathcal{M}, \tau)$  coincides with  $\mathcal{B}(\mathcal{H})$ . In this case the measure topology coincides with the  $\|\cdot\|_\infty$ -topology,  $S_0(\mathcal{M}, \tau)$  is the ideal of all compact operators on  $\mathcal{H}$ ,  $\mathcal{F}(\tau)$  is the finite-dimensional operator ideal on  $\mathcal{H}$  and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1, n)}(t), \quad t > 0,$$

where  $\{s_n(X)\}_{n=1}^{\infty}$  is the sequence of  $s$ -numbers of an operator  $X$  [19, Chap. 1]; here  $\chi_A$  is the indicator function of a set  $A \subset \mathbb{R}$ . In this case, the space  $L_p(\mathcal{M}, \tau)$  is a Schatten-von Neumann ideal  $C_p(\mathcal{H})$ ,  $0 < p < +\infty$ .

An operator  $T \in \mathcal{B}(\mathcal{H})$  is said to be *paranormal* (*\*-paranormal*), if  $\|T^2x\|_{\mathcal{H}} \geq \|Tx\|_{\mathcal{H}}^2$  (respectively,  $\|T^2x\|_{\mathcal{H}} \geq \|T^*x\|_{\mathcal{H}}^2$ ) for all  $x \in \mathcal{H}_1 = \{y \in \mathcal{H} : \|y\|_{\mathcal{H}} = 1\}$ , see [17,24]; *normaloid*, if  $\|T\|_{\infty} = \sup_{y \in \mathcal{H}_1} |\langle Tx, x \rangle|$ . It is known that  $T$  is normaloid  $\Leftrightarrow$  its spectral radius equals  $\|T\|_{\infty}$ , or, equivalently,  $\|T^n\|_{\infty} = \|T\|_{\infty}^n$  for all  $n \in \mathbb{N}$  [20]. It is shown in [25, Problem 9.5] that an operator  $T \in \mathcal{B}(\mathcal{H})$  is paranormal  $\Leftrightarrow |T|^2 \leq (\lambda^{-1}|T^2|^2 + \lambda\mathbf{1})/2$  for all  $\lambda > 0$ . It is shown in [4] that an operator  $T \in \mathcal{B}(\mathcal{H})$  is *\*-paranormal*  $\Leftrightarrow$

$$|T^*|^2 \leq \frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda\mathbf{1}) \quad \text{for all } \lambda > 0. \quad (2)$$

Let  $(\Omega, \nu)$  be a measure space and  $\mathcal{M}$  be the von Neumann algebra of multiplier operators  $M_f$  by functions  $f$  from  $L_{\infty}(\Omega, \nu)$  on a space  $L_2(\Omega, \nu)$ . The algebra  $\mathcal{M}$  contains no compact operators  $\Leftrightarrow$  the measure  $\nu$  has no atoms [2, Theorem 8.4].

### 3 Three classes of $\tau$ -measurable operators

Let  $\tau$  be a faithful normal semifinite trace on a von Neumann algebra  $\mathcal{M}$ . It is obvious that

$$T \in \mathcal{P}_k \Leftrightarrow \lambda T \in \mathcal{P}_k \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}, \quad k = 1, 2, 3.$$

**Theorem 3.1** *For an operator  $T \in S(\mathcal{M}, \tau)$  the following conditions are equivalent:*

- (i)  $T \in \mathcal{P}_3$ ;
- (ii)  $T$  meets condition (2).

**Proof** (i)  $\Rightarrow$  (ii). Assume that for an operator  $T \in \mathcal{P}_3$  condition (2) does not hold. Then there exists a number  $\lambda > 0$  such that

$$\frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda\mathbf{1}) - |T^*|^2 = X - Y, \quad (3)$$

where  $X, Y \in S(\mathcal{M}, \tau)^+$ ,  $XY = 0$  and  $Y \neq 0$ . Let  $Y = \int_0^{+\infty} t E^Y(dt)$  be the spectral decomposition and  $n \in \mathbb{N}$  be such that the projection

$$P = E^Y((n^{-1}, n)) \neq 0.$$

Then  $PXP = 0$  and  $PYP \geq n^{-1}P$ .

Multiplying relation (3) by the projection  $P$  on both sides, leads us to

$$P|T^*|^2P = \frac{1}{2}(\lambda^{-1}P|T^2|^2P + \lambda P) + PYP \geq \frac{1}{2}(\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P).$$

Since  $P$  is a unit in the reduced von Neumann algebra  $\mathcal{M}_P$ , we have

$$\begin{aligned} \|T^*P\|_\infty^2 &= \|P|T^*|^2P\|_\infty \geq \frac{1}{2}\|\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P\|_\infty \\ &= \frac{1}{2}(\lambda^{-1}\|T^2P\|_\infty^2 + (\lambda + 2n^{-1})). \end{aligned}$$

If  $T^2P = 0$  then  $\|T^*P\|_\infty^2 \geq \lambda 2^{-1} + n^{-1} > \|T^2P\|_\infty = 0$ . If  $T^2P \neq 0$  then by the inequality  $a^2 + b^2 \geq 2|ab|$  for all  $a, b \in \mathbb{R}$  we have

$$\|T^*P\|_\infty^2 \geq \frac{1}{2} \cdot 2\sqrt{\lambda^{-1}(\lambda + 2n^{-1})} \cdot \|T^2P\|_\infty > \|T^2P\|_\infty.$$

Thus, in both cases  $T \notin \mathcal{P}_3$ —a contradiction.

(ii)  $\Rightarrow$  (i). Consider an operator  $A \in \mathcal{M}_1$  such that  $T^*A \in \mathcal{M}$ . Then  $A^*A \leq \mathbf{1}$  and  $|T^*|A \in \mathcal{M}$ . If  $T^2A \notin \mathcal{M}$  then the assertion is met. Let  $T^2A \in \mathcal{M}$ . Multiplying inequality (2) from the left-hand side by the operator  $A^*$  and from the right-hand side by the operator  $A$ , leads us to

$$A^*|T^*|^2A \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda A^*A) \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda \mathbf{1}) \quad \text{for all } \lambda > 0.$$

Therefore  $\|A^*|T^*|^2A\|_\infty = \|T^*A\|_\infty^2 \leq \frac{1}{2}(\lambda^{-1}\|T^2A\|_\infty^2 + \lambda)$  for all  $\lambda > 0$ . Put here  $\lambda = \|T^2A\|_\infty$  and obtain  $\|T^*A\|_\infty^2 \leq \|T^2A\|_\infty$ . □

**Corollary 3.2** *The class  $\mathcal{P}_3$  is closed in the measure topology  $t_\tau$ .*

**Proof** Condition (2) is equivalent to the condition  $T^{2*}T^2 - 2\lambda TT^* + \lambda^2\mathbf{1} \geq 0$  for all  $\lambda > 0$ . Hence  $t_\tau$ -closedness of the class  $\mathcal{P}_3$  follows from Theorem 3.1,  $t_\tau$ -continuity of the involution,  $t_\tau$ -continuity of the product operation on  $S(\mathcal{M}, \tau)$  and  $t_\tau$ -closedness of the cone  $S(\mathcal{M}, \tau)^+$  in  $S(\mathcal{M}, \tau)$ . □

**Corollary 3.3** *Consider operators  $T \in \mathcal{P}_3$ ,  $A \in S(\mathcal{M}, \tau)$  and numbers  $k \in \mathbb{N}$ ,  $0 < p, q, r < +\infty$  with  $1/p + 1/q = 1/r$ . Then*

- (i) *if  $T^2T^{*k}A, T^{*k}A \in \mathcal{M}$  then  $(T^*)^{k+1}A \in \mathcal{M}$ ;*
- (ii) *if  $T^2T^{*k}A \in \mathcal{M}$ ,  $T^{*k}A \in \mathcal{F}(\tau)$  or  $T^2T^{*k}A \in \mathcal{F}(\tau)$ ,  $T^{*k}A \in \mathcal{M}$  then  $(T^*)^{k+1}A \in \mathcal{F}(\tau)$ ;*
- (iii) *if  $T^2T^{*k}A \in L_p(\mathcal{M}, \tau)$ ,  $T^{*k}A \in L_q(\mathcal{M}, \tau)$  then  $(T^*)^{k+1}A \in L_{2r}(\mathcal{M}, \tau)$ .*

**Proof** A slight modification of the proof of [6, Corollary 3.1] leads to the goal. □

**Corollary 3.4** *Every operator  $T \in \mathcal{M} \cap \mathcal{P}_3$  is  $*$ -paranormal, hence it is normaloid. If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  then the class  $\mathcal{P}_3$  coincides with the class of all  $*$ -paranormal operators on  $\mathcal{H}$  and is closed in  $\|\cdot\|_\infty$ -topology.*

**Proof** Every  $*$ -paranormal operator is normaloid [3, Theorem 1.1]. □

**Remark 3.5** If an operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal or cohyponormal then  $\mu(t; T^2) = \mu(t; T)^2$  for all  $t > 0$  [12, Theorem 3.1] and  $T \in \mathcal{P}_2$ . If  $T \in S(\mathcal{M}, \tau)$  is nilpotent of second order ( $T \neq 0 = T^2$ ) then  $T \notin \mathcal{P}_2$ .

**Theorem 3.6** (i) If an operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal then  $T \in \mathcal{P}_3$ .  
 (ii) If an operator  $T \in \mathcal{P}_3$  then  $UTU^* \in \mathcal{P}_3$  for all isometries  $U \in \mathcal{M}$ .

**Proof** (i) For a hyponormal operator  $T \in S(\mathcal{M}, \tau)$  and every number  $\lambda > 0$  by Lemma 2.3 we have

$$T^* \cdot T^*T \cdot T - 2\lambda TT^* + \lambda^2 \mathbf{1} \geq T^* \cdot TT^* \cdot T - 2\lambda T^*T + \lambda^2 \mathbf{1} = (T^*T - \lambda \mathbf{1})^2 \geq 0.$$

Applying Theorem 3.1 we conclude the proof.

(ii) Consider operators  $T \in \mathcal{P}_3$  and  $A \in \mathcal{M}_1$  such that  $(UTU^*)^* \cdot A \in \mathcal{M}$  for an isometry  $U \in \mathcal{M}$ . If  $(UTU^*)^2 \cdot A \notin \mathcal{M}$  or  $U^*A = 0$  then the assertion is obvious. Let  $(UTU^*)^2 \cdot A \in \mathcal{M}$  and  $U^*A \neq 0$ . Then  $0 < \|U^*A\|_\infty \leq 1$  and

$$\begin{aligned} \|(UTU^*)^2 \cdot A\|_\infty &= \|UT^2U^* \cdot A\|_\infty \geq \|U^* \cdot UT^2U^* \cdot A\|_\infty = \|T^2U^*A\|_\infty \\ &= \left\| T^2 \frac{U^*A}{\|U^*A\|_\infty} \right\|_\infty \cdot \|U^*A\|_\infty \\ &\geq \left\| T \frac{U^*A}{\|U^*A\|_\infty} \right\|_\infty^2 \cdot \|U^*A\|_\infty = \frac{\|T \cdot U^*A\|_\infty^2}{\|U^*A\|_\infty} \\ &\geq \|T^* \cdot U^*A\|_\infty^2 \geq \|UT^*U^* \cdot A\|_\infty^2 = \|(UTU^*)^* \cdot A\|_\infty^2. \end{aligned}$$

□

**Corollary 3.7** If an operator  $T \in S(\mathcal{M}, \tau)$  is quasinormal then  $T \in \mathcal{P}_3$ .

**Proof** Every quasinormal operator  $T \in S(\mathcal{M}, \tau)$  is hyponormal [11, Theorem 2.9]. □

**Remark 3.8** If an operator  $T \in S(\mathcal{M}, \tau)$  is quasinormal then  $T^n$  is also quasinormal [10, Proposition 2.10] and  $\mu(t; T^n) = \mu(t; T)^n$  for all  $t > 0$  and  $n \in \mathbb{N}$  [10, Theorem 2.6].

**Proposition 3.9** Let  $\tau$  be a faithful normal semifinite trace on a von Neumann algebra  $\mathcal{M}$ . Then  $\mathcal{P}_3 \subset \mathcal{P}_2$ . If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  for separable and infinite dimensional  $\mathcal{H}$  then  $\mathcal{P}_3 \neq \mathcal{P}_2$ .

**Proof** Let  $t > 0$  be fixed. From relation (1) for  $X = T^2$  we have

$$\forall \varepsilon > 0 \exists P_\varepsilon \in \mathcal{P}(\mathcal{M}) \ (\tau(P_\varepsilon^\perp) \leq t, \ \varepsilon + \mu(t; T^2) > \|T^2 P_\varepsilon\|_\infty \geq \mu(t; T^2)),$$

thereby  $\|T^* P_\varepsilon\|_\infty^2 \leq \varepsilon + \mu(t; T^2)$ . Note that a projection  $P_\varepsilon$  is included in the right-hand side of (1) for  $X = T^*$ . Therefore  $\mu(t; T) = \mu(t; T^*) \leq \|T^* P_\varepsilon\|_\infty$  and because of the arbitrariness of the number  $\varepsilon > 0$  we get  $\mu(t; T^2) \geq \mu(t; T)^2$ . Thus  $\mathcal{P}_3 \subset \mathcal{P}_2$ .

For  $T \in \widetilde{\mathcal{M}}$  we have  $T \in \mathcal{P}_2 \Leftrightarrow T^* \in \mathcal{P}_2$  [6, Proposition 3.22]. Let  $\{e_n\}_{n=0}^\infty$  be an orthonormal basis in  $\mathcal{H}$ . The unilateral shift  $Te_n = e_{n+1}$  ( $n = 0, 1, 2, \dots$ ) is



a hyponormal operator (an isometry) and  $T \in \mathcal{P}_3$  by item (i) of Theorem 3.5. The null-space  $\text{Ker}T^*$  is generated by vector  $e_0$ , and the null-space  $\text{Ker}(T^*)^2$  is generated by vectors  $e_0$  and  $e_1$ . We have for the one-dimensional projection  $A = \langle \cdot, e_1 \rangle e_1$  the relations

$$0 = \|(T^*)^2 A\|_\infty < \|(T^*)^* A\|_\infty^2 = \|T^* A\|_\infty^2 = 1$$

and  $T^* \notin \mathcal{P}_3$ . The assertion is proved. □

Now by Proposition 3.24 of [6] we have

**Corollary 3.10** *For  $T \in \mathcal{P}_3$  we have the equivalences:*

- (i)  $T \in \mathcal{M} \Leftrightarrow T^2 \in \mathcal{M}$ ;
- (ii)  $T \in \mathcal{F}(\tau) \Leftrightarrow T^2 \in \mathcal{F}(\tau)$ ;
- (iii)  $T \in \mathcal{S}_0(\mathcal{M}, \tau) \Leftrightarrow T^2 \in \mathcal{S}_0(\mathcal{M}, \tau)$ ;
- (iv)  $T \in L_{2p}(\mathcal{M}, \tau) \Leftrightarrow T^2 \in L_p(\mathcal{M}, \tau), 0 < p < +\infty$ .

**Proposition 3.11** *If a  $\tau$ -measurable operator  $T$  belongs to  $\mathcal{P}_k$  and  $P \in \mathcal{P}(\mathcal{M})$  is such that  $TP = PTP$  then the restriction  $T|_{P\mathcal{H}}$  belongs to  $\mathcal{P}_k, k = 1, 3$ .*

**Proof** For  $k = 1, P \in \mathcal{P}(\mathcal{M})$  and  $A \in \mathcal{M}_1$  with  $PA \neq 0$  we have  $0 < \|PA\|_\infty \leq 1$  and

$$\begin{aligned} & \|(T|_{P\mathcal{H}})^2 A\|_\infty \\ &= \|(PTP)^2 A\|_\infty = \|T^2 PA\|_\infty = \left\| T^2 \frac{PA}{\|PA\|_\infty} \right\|_\infty \cdot \|PA\|_\infty \geq \\ & \geq \left\| T \frac{PA}{\|PA\|_\infty} \right\|_\infty^2 \cdot \|PA\|_\infty = \|T|_{P\mathcal{H}} A\|_\infty^2 \cdot \frac{1}{\|PA\|_\infty} \geq \|T|_{P\mathcal{H}} A\|_\infty^2. \end{aligned}$$

For  $k = 3$  we apply the equality  $(T|_{P\mathcal{H}})^* = PT^*P$ . □

**Proposition 3.12** *Let  $T \in S(\mathcal{M}, \tau)$  and a unitary operator  $S \in \mathcal{M}_h$  be so that  $ST = TS$ . Then  $T \in \mathcal{P}_3 \Leftrightarrow ST \in \mathcal{P}_3$ .*

**Proof** We have  $S^2 = \mathbf{1}$  and  $(ST)^2 = T^2, ST^* = T^*S$ .

( $\Rightarrow$ ) Let  $A \in \mathcal{M}_1$  be so that  $(ST)^* A \in \mathcal{M}$ . Then

$$\begin{aligned} \|(ST)^2 A\|_\infty &= \|T^2 A\|_\infty \geq \|T^* A\|_\infty^2 = \|A^* T T^* A\|_\infty = \|A^* T S^2 T^* A\|_\infty \\ &= \|ST^* A\|_\infty^2 = \|T^* S A\|_\infty^2 = \|(ST)^* A\|_\infty^2. \end{aligned}$$

( $\Leftarrow$ ) If  $ST \in \mathcal{P}_3$  then by the above proved result  $T = S \cdot ST \in \mathcal{P}_3$ . □

### 4 Every $p$ -hyponormal $\tau$ -measurable operator lies in $\mathcal{P}_1$

**Lemma 4.1** *For all operators  $Y \in S(\mathcal{M}, \tau)^+, X \in \mathcal{M}_1$  and  $1 \leq r \leq 2$  we have  $(X^* Y X)^r \leq X^* Y^r X$ .*

**Proof** Let  $Y = \int_0^{+\infty} t E^Y (dt)$  be the spectral decomposition. Put  $Y_n = \int_0^n t E^Y (dt)$  for all  $n \in \mathbb{N}$ . Since the function  $f(t) = t^r (t \geq 0)$  is operator convex, we apply [22, Theorem 2.1] and obtain  $(X^* Y_n X)^r \leq X^* Y_n^r X$  for all  $n \in \mathbb{N}$ . By  $t_\tau$ -continuity of operator functions [33] and the product operation we have  $(X^* Y_n X)^r \xrightarrow{\tau} (X^* Y X)^r$  and  $X^* Y_n^r X \xrightarrow{\tau} X^* Y^r X$  as  $n \rightarrow \infty$ . Finally, we apply the  $t_\tau$ -closedness of the cone  $S(\mathcal{M}, \tau)^+$  in  $S(\mathcal{M}, \tau)$ .  $\square$

**Lemma 4.2** For all operators  $Y \in S(\mathcal{M}, \tau)^+$ ,  $X \in \mathcal{M}_1$  and  $t > 0, q \geq 1$  we have  $\mu(t; X^* Y^q X) \geq \mu(t; X^* Y X)^q$ . In particular, we have  $\|X^* Y^q X\|_\infty \geq \|X^* Y X\|_\infty^q$ .

**Proof** Let  $1 < q = p_1 p_2 \dots p_k$  with some  $1 < p_n \leq 2, n = 1, 2, \dots, k$ . By Lemma 4.1 and by items (3), (4) of Lemma 2.2 for all  $t > 0$  we have

$$\begin{aligned} \mu(t; X^* Y^q X) &= \mu(t; X^* (Y^{q/p_1})^{p_1} X) \geq \mu(t; (X^* Y^{q/p_1} X)^{p_1}) \\ &= \mu(t; X^* Y^{q/p_1} X)^{p_1} = \mu(t; X^* (Y^{q/p_1 p_2})^{p_2} X)^{p_1} \geq \dots \\ &\geq \mu(t; X^* Y^{q/p_1 p_2 p_3 \dots p_k} X)^{p_1 p_2 p_3 \dots p_k} = \mu(t; X^* Y X)^q. \end{aligned}$$

We apply item (7) of Lemma 2.2 and obtain  $\|X^* Y^q X\|_\infty \geq \|X^* Y X\|_\infty^q$ .  $\square$

**Lemma 4.3** Let an operator  $T \in S(\mathcal{M}, \tau)$  be  $p$ -hyponormal with  $0 < p \leq 1$  and  $T = U|T|$  be the polar decomposition of  $T$ . Then

- (i)  $U^*|T|^{1/2^n} U \geq |T|^{1/2^n} \geq U|T|^{1/2^n} U^*$  for some  $n \in \mathbb{N}$ ;
- (ii) the operator  $T_p = U|T|^p$  is hyponormal.

**Proof** (i) We have  $|T^*| = U|T|U^*$  and  $|T|^{2p} = (T^*T)^p \geq (TT^*)^p = |T^*|^{2p} = U|T|^{2p}U^*$ . Let  $n \in \mathbb{N}$  be such that  $q = \frac{1}{p2^{n-1}} \in (0, 1)$ . Then by Hansen’s Theorem ([21]; [5, Lemma 3.1.1]), we have the relations

$$|T|^{1/2^n} = (|T|^{2p})^q \geq (U|T|^{2p}U^*)^q \geq U|T|^{2pq}U^* = U|T|^{1/2^n}U^*,$$

i.e.,  $|T|^{1/2^n} \geq U|T|^{1/2^n}U^*$ . Multiplication of this relation from the left-hand side by the operator  $U^*$  and from the right-hand side by the operator  $U$  and Lemma 2.3 lead us to

$$U^*|T|^{1/2^n}U \geq U^*U|T|^{1/2^n}U^*U = |T|^{1/2^n}.$$

- (ii) We have  $U|T|^{2p}U^* = (U|T|^2U^*)^p \leq |T|^{2p} \leq U^*|T|^{2p}U$ .  $\square$

The following statement strengthens item (i) of Theorem 3.6 [6] and is a generalization of Theorem 3 [28].

**Theorem 4.4** If an operator  $T \in S(\mathcal{M}, \tau)$  is  $p$ -hyponormal with  $0 < p \leq 1$  then  $T \in \mathcal{P}_1$ .

**Proof** Let  $T = U|T|$  be the polar decomposition of the  $p$ -hyponormal operator  $T \in S(\mathcal{M}, \tau)$  with  $0 < p \leq 1$  and  $n \in \mathbb{N}$  be as in item (i) of Lemma 4.3. For  $A \in \mathcal{M}_1$  with  $TA \in \mathcal{M} \setminus \{0\}$  we have

$$\begin{aligned} \|T^2A\|_\infty^2 &= \|A^*T^{*2}T^2A\|_\infty = \|A^*T^* \cdot |T|^2 \cdot TA\|_\infty \\ &= \left\| \frac{A^*T^*}{\|A^*T^*\|_\infty} \cdot (|T|^{1/2^n})^{2^{n+1}} \cdot \frac{TA}{\|TA\|_\infty} \right\|_\infty \cdot \|TA\|_\infty^2. \end{aligned}$$

Then by Lemma 4.2 we obtain

$$\begin{aligned} \|T^2A\|_\infty^2 &\geq \left\| \frac{A^*T^*}{\|A^*T^*\|_\infty} \cdot |T|^{1/2^n} \cdot \frac{TA}{\|TA\|_\infty} \right\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2 \\ &= \frac{\|A^*T^* \cdot |T|^{1/2^n} \cdot TA\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \\ &= \frac{\|A^*|T|U^* \cdot |T|^{1/2^n} \cdot U|TA\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}}. \end{aligned}$$

Therefore by item (i) of Lemmas 4.3 and 2.3 we have

$$\begin{aligned} \|T^2A\|_\infty^2 &\geq \frac{\|A^*|T| \cdot |T|^{1/2^n} \cdot |TA\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \\ &= \frac{\|A^* \cdot |T|^{1/2^n+2} \cdot A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \\ &= \frac{\|A^* \cdot (|T|^2)^{1/2^{n+1}+1} \cdot A\|_\infty^{2^{n+1}} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \end{aligned}$$

Thus by Lemma 4.2 we obtain

$$\begin{aligned} \|T^2A\|_\infty^2 &\geq \frac{\|A^* \cdot |T|^2 \cdot A\|_\infty^{2^{n+1}+1} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \\ &= \frac{\||T| \cdot A\|_\infty^{2^{2n+2}+2} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} \\ &\geq \frac{\|U|T| \cdot A\|_\infty^{2^{2n+2}+2} \cdot \|TA\|_\infty^2}{\|TA\|_\infty^{2^{2n+2}}} = \|TA\|_\infty^4 \end{aligned}$$

and Theorem 4.4 is proved. □

**Corollary 4.5** *If an operator  $T \in S(\mathcal{M}, \tau)$  is  $p$ -hyponormal with  $0 < p \leq 1$  then  $\mu(t; T^2) \geq \mu(t; T)^2$  for all  $t > 0$ .*

**Proof** We have  $\mathcal{P}_1 \subset \mathcal{P}_2$  by Proposition 3.5 of [6]. □

**Corollary 4.6** *If an operator  $T \in \mathcal{M}$  is  $p$ -hyponormal with  $0 < p \leq 1$  then  $\mu(t; T^n) \geq \mu(t; T)^n$  for all  $t > 0$  and  $n \in \mathbb{N}$ .*

**Proof** We apply Theorem 3.16 of [6]. □

## 5 On $p$ -hyponormal $\tau$ -measurable operators

Let a semifinite von Neumann algebra  $\mathcal{M}$  be equipped with a faithful normal semifinite trace  $\tau$ .

**Theorem 5.1** *Let  $A, B \in S(\mathcal{M}, \tau)$  and  $A$  be  $p$ -hyponormal with  $0 < p \leq 1$ .*

- (i) *If  $AB \in S_0(\mathcal{M}, \tau)$  then  $A^*B \in S_0(\mathcal{M}, \tau)$ .*
- (ii) *If  $A, B \in \mathcal{M}$  and  $AB \in \mathcal{F}(\tau)$  then  $A^*B \in \mathcal{F}(\tau)$ .*
- (iii) *If  $A, B \in \mathcal{M}$  and  $AB \in L_q(\mathcal{M}, \tau)$  then  $A^*B \in L_{q/p}(\mathcal{M}, \tau)$ .*

**Proof** (i) Let  $A^* = U|A^*|$  be the polar decomposition of an operator  $A^*$ . Every operator  $B \in S(\mathcal{M}, \tau)$  can be represented as a sum  $B = S + T$  with  $S \in \mathcal{M}$  and  $T \in S_0(\mathcal{M}, \tau)$ , see [32]. Hence we may assume that  $B \in \mathcal{M}$ . By items (1), (2), (3), (4) and (6) of Lemma 2.2 and by the Hansen's inequality ([21]; [5, Lemma 3.1.1]) for  $B_1 = B/\|B\|_\infty$  for all  $t > 0$  we have

$$\begin{aligned} \mu(t; AB)^2 &= \mu(t; B^*A^*)^2 = \mu(t; B^*A^*AB) = \|B\|_\infty^2 \mu(t; B_1^*A^*AB_1) \\ &= \|B\|_\infty^2 \mu(t; (B_1^*A^*AB_1)^p)^{1/p} \geq \|B\|_\infty^2 \mu(t; B_1^*(A^*A)^p B_1)^{1/p} \quad (3) \\ &\geq \|B\|_\infty^2 \mu(t; B_1^*(AA^*)^p B_1)^{1/p} = \|B\|_\infty^2 \mu(t; |A^*|^p B_1)^{2/p}. \end{aligned}$$

Therefore  $|A^*|^p B \in S_0(\mathcal{M}, \tau)$  and  $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in S_0(\mathcal{M}, \tau)$ .

(ii) We apply (3) and conclude that  $|A^*|^p B \in \mathcal{F}(\tau)$ . Thus  $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in \mathcal{F}(\tau)$ .

(iii) For  $q > 0$  by (3) we have  $|A^*|^p B \in L_{q/p}(\mathcal{M}, \tau)$ . Thus  $A^*B = U|A^*|^{1-p} \cdot |A^*|^p B \in L_{q/p}(\mathcal{M}, \tau)$ . Moreover, for all  $t > 0$  and for  $C = \|B\|_\infty^q \cdot \|U|A^*|^{1-p}\|_\infty^{-q/p}$  by (3) and items (2), (3) and (4) of Lemma 2.2 we have

$$\begin{aligned} \mu(t; AB)^q &\geq \|B\|^q \mu(t; |A^*|^p B_1)^{q/p} = C \|U|A^*|^{1-p}\|_\infty^{q/p} \mu(t; |A^*|^p B_1)^{q/p} \\ &\geq C \mu(t; U|A^*|^{1-p} \cdot |A^*|^p B_1)^{q/p} = C \mu(t; A^* B_1)^{q/p} \quad (4) \\ &= C \|B\|_\infty^{-q/p} \mu(t; A^* B)^{q/p}. \end{aligned}$$

Theorem is proved. □

**Corollary 5.2** *Let  $A, B \in \mathcal{M}$  and  $A, B^*$  be  $p$ -hyponormal with  $1/2 < p \leq 1$ .*

- (i) *If  $AB \in S_0(\mathcal{M}, \tau)$  then  $BA \in S_0(\mathcal{M}, \tau)$ .*
- (ii) *If  $AB \in \mathcal{F}(\tau)$  then  $BA \in \mathcal{F}(\tau)$ .*
- (iii) *If  $AB \in L_q(\mathcal{M}, \tau)$  then  $BA \in L_{2q/p}(\mathcal{M}, \tau)$ .*

**Proof** (i), (ii). Dividing suitably if need be, we may assume that  $A, B \in \{X \in \mathcal{M} : \|X\|_\infty \leq 1\}$ . Also, by Löwner’s inequality, both  $A$  and  $B^*$  are  $\frac{1}{2}$ -hyponormal. Hence, by Hansen’s inequality [21] we conclude that

$$A^*|B|^2A \leq A^*|B|A \leq A^*|B^*|A = A^*(|B^*|^2)^{1/2}A \leq (A^*|B^*|^2A)^{1/2}. \tag{5}$$

Then we apply items (1), (3) and (4) of Lemma 2.2 and Theorem 5.1.

(iii) We have  $A^*B \in L_{q/p}(\mathcal{M}, \tau)$  by Theorem 5.1. Hence  $B^*A = (A^*B)^* \in L_{q/p}(\mathcal{M}, \tau)$ . Inequality (5) yields  $|BA|^2 \leq |B^*A|$  and we apply items (1), (3) and (4) of Lemma 2.2. Moreover, for all  $t > 0$  and for  $C = \|B\|_\infty^q \cdot \|U|A^*|^{1-p}\|_\infty^{-q/p}$  by (4), (5) and items (1), (2), (3) and (4) of Lemma 2.2 we have

$$\begin{aligned} \mu(t; AB)^q &\geq C\|B\|_\infty^{-q/p} \mu(t; A^*B)^{q/p} = C\|B\|_\infty^{-q/p} \mu(t; B^*A)^{q/p} \\ &\geq C\|B\|_\infty^{-q/p} \mu(t; BA)^{2q/p}. \end{aligned}$$

The assertion is proved. □

**Theorem 5.3** (cf [1, Theorem 1]) *Let  $T \in S(\mathcal{M}, \tau)$  be  $p$ -hyponormal with  $\frac{1}{2} \leq p < 1$  and  $T = U|T|$  be the polar decomposition of  $T$ . Then the operator  $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$  is hyponormal.*

**Proof** We have  $|T|^{2p} \geq (U|T|^2U^*)^p$ . By operator monotonicity of the function  $t \mapsto t^{\frac{1}{2p}}$  ( $t \geq 0$ ) and by Hansen’s inequality ([21]; [5, Lemma 3.1.1]) we obtain

$$|T| \geq (U|T|^2U^*)^{p \cdot \frac{1}{2p}} = (U|T|^2U^*)^{\frac{1}{2}} \geq U|T|U^*.$$

Thus by Lemma 2.3 we have

$$\begin{aligned} \tilde{T}^*\tilde{T} &= |T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}} \geq |T|^{\frac{1}{2}}U^* \cdot U|T|U^* \cdot U|T|^{\frac{1}{2}} = |T|^2 = |T|^{\frac{1}{2}} \cdot |T| \cdot |T|^{\frac{1}{2}} \\ &\geq |T|^{\frac{1}{2}} \cdot U|T|U^* \cdot |T|^{\frac{1}{2}} = \tilde{T}\tilde{T}^*. \end{aligned}$$

Theorem is proved. □

**Lemma 5.4** *Let  $A, B \in \tilde{\mathcal{M}}^+$  with  $A \geq B$ . Then for each  $r > 0$ ,*

$$(B^r A^p B^r)^{\frac{1}{q}} \geq B^{(p+2r)/q} \tag{6}$$

and

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{\frac{1}{q}} \tag{7}$$

hold for each  $p$  and  $q$  such that  $p \geq 0, q \geq 1$ , and  $(1 + 2r)q \geq p + 2r$ .

**Proof** Let  $A = \int_0^{+\infty} \lambda E^A(d\lambda)$  and  $B = \int_0^{+\infty} \lambda E^B(d\lambda)$  be the spectral decompositions of  $A$  and  $B$ . Put

$$A_n = \int_0^n \lambda E^A(d\lambda) \quad \text{and} \quad B_n = \int_0^n \lambda E^B(d\lambda)$$

for all  $n \in \mathbb{N}$ . Then  $A_n, B_n$  belong to  $\mathcal{M}^+$  and meet inequalities (6) and (7) for all  $n \in \mathbb{N}$  by [18]. We have  $A_n \xrightarrow{\tau} A, B_n \xrightarrow{\tau} B$  as  $n \rightarrow \infty$  and apply  $t_\tau$ -continuity of real continuous functions [33] and  $t_\tau$ -continuity of the product operation. Inequalities (6), (7) follow by  $t_\tau$ -closedness of the cone  $\widetilde{\mathcal{M}}^+$  in  $\widetilde{\mathcal{M}}$ .  $\square$

**Theorem 5.5** (cf [1, Theorem 2]) *Let  $T \in S(\mathcal{M}, \tau)$  be  $p$ -hyponormal with  $0 < p < \frac{1}{2}$  and  $T = U|T|$  be the polar decomposition of  $T$ . Then the operator  $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$  is  $(p + \frac{1}{2})$ -hyponormal.*

**Proof** We apply Lemma 5.4 and repeat the proof of [1, Theorem 2]. Theorem is proved.  $\square$

**Corollary 5.6** *If an operator  $T \in S(\mathcal{M}, \tau)$  is  $p$ -hyponormal with  $0 < p < \frac{1}{2}$  and has the polar decomposition  $T = U|T|$ , then the  $\tau$ -measurable operator  $|\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$  is hyponormal, where  $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$  and  $\widetilde{T} = \widetilde{U}|\widetilde{T}|$  is the polar decomposition of  $\widetilde{T}$ .*

**Acknowledgements** The research was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities, Ministry of Education and Science of the Russian Federation Project 1.13556.2019/13.1.

## References

1. Aluthge, A.: On  $p$ -hyponormal operators for  $0 < p < 1$ . Integral Equ. Oper. Theory **13**(3), 307–315 (1990)
2. Antonevich, A.B.: Linear Fractional Equations. Operator Approach. Birkhäuser, Basel (1996)
3. Arora, S.C., Thukral, J.K.: On a class of operators. Glas. Mat. Ser. III **21**(2), 381–386 (1986)
4. Arora, S.C., Thukral, J.K.:  $M^*$ -Paranormal operators. Glas. Mat. Ser. III **22**(1), 123–129 (1987)
5. Bekjan, T.N., Dautbek, D.: Submajorization inequalities of  $\tau$ -measurable operators for concave and convex functions. Positivity **19**(2), 341–345 (2015)
6. Bikchentaev, A.M.: Paranormal measurable operators affiliated with a semifinite von Neumann algebra. Lobachevskii J. Math. **39**(6), 731–741 (2018)
7. Bikchentaev, A.M., Abed, S.A.: Paranormal elements in normed algebra. Russ. Math. **62**(5), 10–15 (2018)
8. Bikchentaev, A.M.: Two classes of  $\tau$ -measurable operators affiliated with a von Neumann algebra. Russ. Math. **61**(1), 76–80 (2017)
9. Bikchentaev, A.M.: On  $\tau$ -compactness of products of  $\tau$ -measurable operators. Int. J. Theor. Phys. **56**(12), 3819–3830 (2017)
10. Bikchentaev, A.M.: Integrable products of measurable operators. Lobachevskii J. Math. **37**(4), 397–403 (2016)
11. Bikchentaev, A.M.: On idempotent  $\tau$ -measurable operators affiliated to a von Neumann algebra. Math. Notes **100**(3–4), 515–525 (2016)
12. Bikchentaev, A.M.: On normal  $\tau$ -measurable operators affiliated with semifinite von Neumann algebras. Math. Notes **96**(3–4), 332–341 (2014)
13. Bikchentaev, A.M.: Minimality of the convergence in measure topologies on finite von Neumann algebras. Math. Notes **75**(3–4), 315–321 (2004)

14. Dodds, P.G., de Pagter, B.: Normed Köthe spaces: a non-commutative viewpoint. *Indag. Math. (N.S.)* **25**(2), 206–249 (2014)
15. Dodds, P.G., Dodds, T.K.-Y., de Pagter, B.: Noncommutative Köthe duality. *Trans. Am. Math. Soc.* **339**(2), 717–750 (1993)
16. Fack, T., Kosaki, H.: Generalized  $s$ -numbers of  $\tau$ -measurable operators. *Pac. J. Math.* **123**(2), 269–300 (1986)
17. Furuta, T.: On the class of paranormal operators. *Proc. Jpn. Acad.* **43**(7), 594–598 (1967)
18. Furuta, T.:  $A \geq B \geq 0$  assures  $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$  for  $r \geq 0$ ,  $p \geq 0$ ,  $q \geq 1$  with  $(1 + 2r)q \geq p + 2r$ . *Proc. Am. Math. Soc.* **101**(1), 85–88 (1987)
19. Gohberg, I.C., Kreĭn, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. *Transl. Mathem. Monographs*, Vol. 18, Amer. Math. Soc., Providence, R.I. (1969)
20. Halmos, P.R.: A Hilbert Space Problem Book. D. van Nostrand Company Inc., Princeton (1967)
21. Hansen, F.: An operator inequality. *Math. Ann.* **246**(3), 249–250 (1980)
22. Hansen, F., Pedersen, G.K.: Jensen's inequality for operators and Löwner's theorem. *Math. Ann.* **258**(3), 229–241 (1982)
23. Istrăţescu, V., Saitô, T., Yoshino, T.: On a class of operators. *Tôhoku Math. J. (2)* **18**(4), 410–413 (1966)
24. Istrăţescu, V.: On some hyponormal operators. *Pac. J. Math.* **22**(3), 413–417 (1967)
25. Kubrusly, C.S.: Hilbert Space Operators. A Problem Solving Approach. Birkhäuser Boston Inc., Boston (2003)
26. Lord, S., Sukochev, F., Zanin, D.: Singular Traces. Theory and Applications, de Gruyter Studies in Mathematics, Vol. 46 (2013)
27. Nelson, E.: Notes on non-commutative integration. *J. Funct. Anal.* **15**(2), 103–116 (1974)
28. Patel, S.M.: A note on  $p$ -hyponormal operators for  $0 < p < 1$ . *Integral Equ. Oper. Theory* **21**(4), 498–503 (1995)
29. Ramesh, G.: Absolutely norm attaining paranormal operators. *J. Math. Anal. Appl.* **465**(1), 547–556 (2018)
30. Rashid, M.H.M.: Some results on totally  $n$ -\*-paranormal operators. *Gulf J. Math.* **5**(3), 1–17 (2017)
31. Segal, I.E.: A non-commutative extension of abstract integration. *Ann. Math. (2)* **57**(3), 401–457 (1953)
32. Stroh, A., West, P.G.:  $\tau$ -compact operators affiliated to a semifinite von Neumann algebra. *Proc. Roy. Irish Acad. Sect. A* **93**(1), 73–86 (1993)
33. Tikhonov, O.E.: Continuity of operator functions in topologies connected with a trace on a von Neumann algebra. *Sov. Math. (Iz. VUZ)* **31**(1), 110–114 (1987)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.