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METRICS ON PROJECTIONS OF THE VON NEUMANN ALGEBRA ASSOCIATED WITH TRACIAL FUNCTIONALS

A. M. Bikchentaev

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Abstract: Let φ be a positive functional on a von Neumann algebra $\mathscr{A}$ and let $\mathscr{A}^{\mathrm{pr}}$ be the projection lattice in $\mathscr{A}$. Given $P, Q \in \mathscr{A}^{\mathrm{pr}}$, put $\rho_\varphi(P,Q) = \varphi(|P-Q|)$ and $d_\varphi(P,Q) = \varphi(P \vee Q - P \wedge Q)$. Then $\rho_\varphi(P,Q) \le d_\varphi(P,Q)$ and $\rho_\varphi(P,Q) = d_\varphi(P,Q)$ provided that $PQ = QP$. The mapping ρ_φ (or d_φ) meets the triangle inequality if and only if φ is a tracial functional. If τ is a faithful tracial functional then ρ_τ and d_τ are metrics on $\mathscr{A}^{\mathrm{pr}}$. Moreover, if τ is normal then $(\mathscr{A}^{\mathrm{pr}}, \rho_\tau)$ and $(\mathscr{A}^{\mathrm{pr}}, d_\tau)$ are complete metric spaces. Convergences with respect to ρ_τ and d_τ are equivalent if and only if $\mathscr{A}$ is abelian; in this case $\rho_\tau = d_\tau$. We give one more criterion for commutativity of $\mathscr{A}$ in terms of inequalities.

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1. Definitions and Notations

Let $\mathscr{H}$ be a Hilbert space over the field $\mathbb{C}$, let $\mathscr{B}(\mathscr{H})$ be the $*$-algebra of all bounded linear operators in $\mathscr{H}$, let $\mathrm{pr}(X)$ be the projection to the closure of the range of $X \in \mathscr{B}(\mathscr{H})$, and let I be the identity operator on $\mathscr{H}$. If $P, Q \in \mathscr{B}(\mathscr{H})^{\mathrm{pr}}$, then $P^\perp = I - P \in \mathscr{B}(\mathscr{H})^{\mathrm{pr}}$ and the projection $P \wedge Q$ is defined as $(P \wedge Q)\mathscr{H} = P\mathscr{H} \cap Q\mathscr{H}$, while $P \vee Q = (P^\perp \wedge Q^\perp)^\perp$ projects to $\overline{\mathrm{lin}(P\mathscr{H} \cup Q\mathscr{H})}$. The *commutant* of $\mathscr{X} \subset \mathscr{B}(\mathscr{H})$ is

$$\mathscr{X}' = \{Y \in \mathscr{B}(\mathscr{H}) : XY = YX \quad \text{for all } X \in \mathscr{X}\}.$$

A *von Neumann algebra* acting on $\mathscr{H}$ is a $*$-subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ satisfying $\mathscr{A} = \mathscr{A}''$. Given a von Neumann algebra $\mathscr{A}$, we denote the subset of positive elements of A by $\mathscr{A}^+$ and the projection lattice, by $\mathscr{A}^{\mathrm{pr}}$.

Given $P, Q \in \mathscr{A}^{\mathrm{pr}}$, write $P \sim Q$ (the Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathscr{A}$. Projections $P, Q \in \mathscr{A}$ are called *isoclinic* (we write $P \overset{\theta}{\approx} Q$ for the angle $\theta \in (0, \pi/2)$) if $PQP = \cos^2\theta\, P$ and $QPQ = \cos^2\theta\, Q$. If $A \in \mathscr{A}$, then $|A| = \sqrt{A^*A} \in \mathscr{A}^+$ and the projection $\mathrm{pr}(A)$ to the closure of the range of A lies in $\mathscr{A}$. A positive functional φ on a von Neumann algebra $\mathscr{A}$ is called *faithful* if $\varphi(A) = 0$ $(A \in \mathscr{A}^+) \Rightarrow A = 0$; while φ is *tracial* if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathscr{A}$; *normal* if $A_i \nearrow A$ $(A_i, A \in \mathscr{A}^+) \Rightarrow \varphi(A) = \sup_i \varphi(A_i)$; and φ is a *state* if $\varphi(I) = 1$.

2. Metrics on $\mathscr{A}^{\mathrm{pr}}$ Associated with a Tracial State

Lemma 1 [1, Proposition 4.4]. *The real function $\lambda \mapsto \sqrt{\lambda}$ is operator monotone on $\mathbb{R}^+$.*

Let $\mathscr{A}$ be a von Neumann algebra. Given $P, Q \in \mathscr{A}^{\mathrm{pr}}$, put

$$P \circ Q = 2^{-1}(PQ + QP), \quad P \ominus Q = P \vee Q - P \wedge Q.$$

We have $P \circ Q \in \mathscr{A}^+ \Leftrightarrow PQ = QP$ (see the lemma in [2]). If $U \in \mathscr{A}$ is unitary, then

$$U(P \circ Q)U^* = (UPU^*) \circ (UQU^*), \quad U(P \ominus Q)U^* = (UPU^*) \ominus (UQU^*).$$

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Lemma 2. *Let $\mathscr{A}$ be a von Neumann algebra and $P, Q \in \mathscr{A}^{\mathrm{pr}}$. Then*
(i) $|P \circ Q|^2 \leq 4^{-1}(P+Q)^2$;
(ii) $P \wedge Q \leq |P \circ Q| \leq 2^{-1}(P+Q) \leq P \vee Q$;
(iii) $P \ominus Q = Q \ominus P = \mathrm{pr}(|P-Q|)$;
(iv) $P \ominus Q = P^{\perp} \ominus Q^{\perp}$;
(v) $|P-Q|^a \leq P \ominus Q$ *for all* $a > 0$;
(vi) $I \ominus P = P^{\perp}$, $0 \ominus P = P$, $P \ominus P^{\perp} = I$, *and* $P \ominus P = 0$;
(vii) *if* $PQ = QP$, *then* $P \ominus Q = |P-Q|$.

PROOF. It suffices to verify the inequality $P \wedge Q \leq |P \circ Q|$ and item (i) for rank one projection $P, Q \in \mathbb{M}_2(\mathbb{C})$ (see the proof of Theorem 1 in [3] or Theorem 1 in [4]). Without loss of generality, put

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t & \delta\sqrt{t(1-t)} \\ \bar{\delta}\sqrt{t(1-t)} & 1-t \end{pmatrix}$$

for $\delta \in \mathbb{C}$ with $|\delta| = 1$ and $0 \leq t \leq 1$. Now, the inequality $P \wedge Q \leq |P \circ Q|$ is obvious and

$$(PQ+QP)^2 = t(P+Q)^2 \leq (P+Q)^2. \tag{1}$$

By Lemma 1, from (1) we obtain $|P \circ Q| \leq 2^{-1}(P+Q) \leq P \vee Q$. Items (iii) and (v) were established in Theorem 2 of [5]. Since $P - Q = Q^{\perp} - P^{\perp}$, (iv) follows from (iii). Item (vii) is established in Proposition 1(iii) of [5]. The lemma is proven. □

REMARK 1. Let us give a simple proof of the inequality $|P \circ Q| \leq P \vee Q$. We have $(P \pm Q)^2 \geq 0$ and $-2\,P \vee Q \leq -P - Q \leq PQ + QP \leq P + Q \leq 2\,P \vee Q$, i.e., $-P \vee Q \leq P \circ Q \leq P \vee Q$. Then $|P \circ Q| \leq P \vee Q$ by Theorem 2.4 of [6].

DEFINITION. Given a positive functional φ on a von Neumann algebra $\mathscr{A}$, we introduce the mappings $\rho_\varphi, d_\varphi : \mathscr{A}^{\mathrm{pr}} \times \mathscr{A}^{\mathrm{pr}} \to \mathbb{R}^+$ by the formulas

$$\rho_\varphi(P,Q) = \varphi(|P-Q|) \quad \text{and} \quad d_\varphi(P,Q) = \varphi(P \ominus Q) \quad \text{for all } P, Q \in \mathscr{A}^{\mathrm{pr}}.$$

Proposition 1. *The following are valid:*
(i) $\rho_\varphi(P,Q) \leq d_\varphi(P,Q)$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$;
(ii) $d_\varphi(P,Q) \leq \sin^{-2}\theta\, \rho_\varphi(P,Q)$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$ *with* $P \overset{\theta}{\approx} Q$;
(iii) *if* $PQ = QP$, *then* $\rho_\varphi(P,Q) = d_\varphi(P,Q)$;
(iv) $\rho_\varphi(Q,P) = \rho_\varphi(P,Q) = \rho_\varphi(P^{\perp}, Q^{\perp})$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$;
(v) $d_\varphi(Q,P) = d_\varphi(P,Q) = d_\varphi(P^{\perp}, Q^{\perp})$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$;
(vi) $\rho_\varphi(P,0) = d_\varphi(P,0) = \varphi(P)$ *for all* $P \in \mathscr{A}^{\mathrm{pr}}$;
(vii) $\rho_\varphi(P,Q) = d_\varphi(P,Q) = \varphi(P+Q)$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$ *with* $PQ = 0$;
(viii) $\varphi(||P \circ Q| - P \wedge Q|) \leq d_\varphi(P,Q)$ *for all* $P, Q \in \mathscr{A}^{\mathrm{pr}}$;
(ix) $\rho_\varphi(P,Q) + \rho_\varphi(Q,R) = \rho_\varphi(P,R)$ *for all* $P, Q, R \in \mathscr{A}^{\mathrm{pr}}$ *with* $P \leq Q \leq R$;
(x) $d_\varphi(P,Q) + d_\varphi(Q,R) = d_\varphi(P,R)$ *for all* $P, Q, R \in \mathscr{A}^{\mathrm{pr}}$ *with* $P \leq Q \leq R$.

PROOF. Items (i), (iii), and (iv) follow from (v), (vii), and (iv) of Lemma 2 respectively. In particular, if $\mathscr{A}$ is abelian, then $\rho_\varphi(P,Q) = d_\varphi(P,Q)$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$.

Show (ii). By (iii) of Theorem 10.5 of [1] $P \wedge Q = 0$ and $P \vee Q = \sin^{-2}\theta\,(P-Q)^2$ for $P, Q \in \mathscr{A}^{\mathrm{pr}}$ with $P \overset{\theta}{\approx} Q$. Since $\|P-Q\| \leq 1$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$; therefore, $(P-Q)^2 \leq \sqrt{(P-Q)^2} = |P-Q|$.

By Lemma 2(ii), for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$ we obtain $0 \leq |P \circ Q| - P \wedge Q \leq P \ominus Q$, which implies (viii). The proposition is proven. □

Theorem 1. *Let τ be a faithful tracial functional on a von Neumann algebra $\mathscr{A}$. Then ρ_τ and d_τ are metrics on $\mathscr{A}^{\mathrm{pr}}$.*

PROOF. Take $P, Q, R \in \mathscr{A}^{\mathrm{pr}}$. For each pair of operators $X, Y \in \mathscr{A}$ there exist some partial isometries $U, V \in \mathscr{A}$ such that $|X+Y| \leq U|X|U^* + V|Y|V^*$ [7, Theorem 2.2]. Letting $X = P - R$ and $Y = R - Q$, we obtain the triangle inequality for ρ_τ.

From (iii) of Lemma 2 it follows that $d_\tau(P,Q) = d_\tau(Q,P)$ and $d_\tau(P,Q) = 0 \Leftrightarrow P = Q$. If $A, B \in \mathscr{A}^{\mathrm{pr}}$, then $A \vee B - B \sim B - A \wedge B$ [8, Chapter III, Theorem 1.1.3]. Consequently,

$$\tau(A \vee B) + \tau(A \wedge B) = \tau(A) + \tau(B). \tag{2}$$

Prove the triangle inequality for d_τ; i.e.,

$$\tau(P \vee Q) - \tau(P \wedge Q) \le \tau(P \vee R) - \tau(P \wedge R) + \tau(R \vee Q) - \tau(R \wedge Q). \tag{3}$$

From (2) $\tau(A \wedge B) = \tau(A) + \tau(B) - \tau(A \vee B)$; therefore, (3) can be rewritten as

$$\tau(P \vee Q) \le \tau(P \vee R) + \tau(R \vee Q) - \tau(R). \tag{4}$$

Putting $A = P \vee R$ and $B = R \vee Q$ in (2), we have

$$\tau(P \vee R) + \tau(R \vee Q) = \tau(P \vee Q \vee R) + \tau((P \vee R) \wedge (R \vee Q)). \tag{5}$$

Rewrite (4) using (5):

$$\tau(P \vee Q) \le \tau(P \vee Q \vee R) + \tau((P \vee R) \wedge (R \vee Q)) - \tau(R). \tag{6}$$

Since $P \vee Q \vee R \ge P \vee Q$ and $(P \vee R) \wedge (R \vee Q) \ge R$; therefore, (6) is valid by monotonicity of τ on $\mathscr{A}^+$. Consequently, (3) is also valid. The theorem is proven. $\square$

Theorem 2. *If in the conditions of Theorem 1 τ is a normal state, then $(\mathscr{A}^{\mathrm{pr}}, \rho_\tau)$ and $(\mathscr{A}^{\mathrm{pr}}, d_\tau)$ are complete metric spaces.*

Proof. A von Neumann algebra $\mathscr{A}$ possesses the topology t_τ of convergence in measure $\mathscr{A}$ (see [9]) whose fundamental system of neighborhoods of zero is constituted by the sets

$$U(\varepsilon, \delta) = \{X \in \mathscr{A} : \exists P \in \mathscr{A}^{\mathrm{pr}} \ (\|XP\| \le \varepsilon \text{ and } \tau(P^\perp) \le \delta)\}, \quad \varepsilon > 0,\ \delta > 0.$$

It is well known that $\langle \mathscr{A},\ t_\tau \rangle$ is a metrizable topological $*$-algebra. Define the L_1-norm on $\mathscr{A}$ by putting $\|X\|_1 = \tau(|X|)$ for all $X \in \mathscr{A}$. Let $\mathscr{A}_1 = \{X \in \mathscr{A} : \|X\| \le 1\}$.

For $(\mathscr{A}^{\mathrm{pr}}, \rho_\tau)$, the claim follows from the $\|\cdot\|_1$-completeness of $(\mathscr{A}_1, \|\cdot\|_1)$, the continuity of the embedding $(\mathscr{A}_1, \|\cdot\|_1)$ in the topological $*$-algebra $(\mathscr{A}, t_\tau)$, and the t_τ-closedness of $\mathscr{A}^{\mathrm{pr}}$.

For $(\mathscr{A}^{\mathrm{pr}}, d_\tau)$, the claim follows from coincidence of d_τ to the restriction to $\mathscr{A}^{\mathrm{pr}}$ of the well-known metric $d_{s,\tau}(A,B) = \tau(\mathrm{pr}(|A-B|))$, with $A, B \in \mathscr{A}$, the continuity of the embedding $(\mathscr{A}, d_{s,\tau})$ in $(\mathscr{A}, t_\tau)$ (see [10, 11]), and the t_τ-closedness of $\mathscr{A}^{\mathrm{pr}}$. The theorem is proven. $\square$

Theorem 3. *Let τ be a faithful tracial functional on a von Neumann algebra $\mathscr{A}$. Convergences with respect to the metrics ρ_τ and d_τ are equivalent if and only if $\mathscr{A}$ is abelian.*

Proof. If $\mathscr{A}$ is abelian, then $\rho_\tau(P,Q) = d_\tau(P,Q)$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$ by assertion (iii) of Proposition 1.

If $\mathscr{A}$ is not abelian, then $\mathscr{A}$ has a $*$-subalgebra $*$-isomorphic to the complete matrix algebra $\mathbb{M}_2(\mathbb{C})$. Consider the sequence of rank one projections

$$P_n = \begin{pmatrix} \frac{1}{n} & \sqrt{\frac{1}{n}\left(1 - \frac{1}{n}\right)} \\ \sqrt{\frac{1}{n}\left(1 - \frac{1}{n}\right)} & 1 - \frac{1}{n} \end{pmatrix}, \quad n \in \mathbb{N},$$

in $\mathbb{M}_2(\mathbb{C})$. Then $P_n \to \mathrm{diag}(0,1)$ as $n \to \infty$ in ρ_τ, but $\{P_n\}_{n=1}^\infty$ is not fundamental in the metric d_τ, since $P_n \vee P_m = I$ and $P_n \wedge P_m = 0$ for $n \ne m$ for all $n \in \mathbb{N}$. The theorem is proven. $\square$

3. Characterization of Tracial Functionals

Show that ρ_φ (or d_φ) satisfies the triangle inequality if and only if φ is tracial. Let $P, Q, R \in \mathscr{A}^{\mathrm{pr}}$. The triangle inequalities for ρ_φ and d_φ with $R = 0$ take the form $\rho_\varphi(P, Q) \le \varphi(P+Q)$ and $d_\varphi(P, Q) \le \varphi(P+Q)$ respectively.

Theorem 4. *For a positive normal functional φ on a von Neumann algebra $\mathscr{A}$ the following are equivalent:*

(i) *φ is tracial;*

(ii) *$\rho_\varphi(P, Q) \le \varphi(P + Q)$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$;*

(iii) *$d_\varphi(P, Q) \le \varphi(P + Q)$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$.*

Proof. (i) $\Longrightarrow$ (ii) is established in the proof of Theorem 1.

(i) $\Longrightarrow$ (iii) follows from (2).

(ii) $\Longrightarrow$ (i) is established in item (v) of Theorem 3.4 of [12].

Below, we show that the proof of (iii) $\Longrightarrow$ (i) for an arbitrary von Neumann algebra is reduced to the case of the algebra $\mathbb{M}_2(\mathbb{C})$, in the same manner as it was done in a series of other similar cases (see [13] or [14]).

It is well known [13] that a positive normal functional φ on a von Neumann algebra $\mathscr{A}$ is tracial if and only if $\varphi(P) = \varphi(Q)$ for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$ with $PQ = 0$ and $P \sim Q$ (see also [14, Lemma 2]). Let the $*$-algebra $\mathscr{B}$ in the reduced algebra $(P + Q)\mathscr{A}(P + Q)$ be generated by the partial isometry $V \in \mathscr{A}$ realizing the equivalence of P and Q. Then $\mathscr{B}$ is $*$-isomorphic to $\mathbb{M}_2(\mathbb{C})$ and the inequality in (ii) remains valid for the operators in $\mathscr{B}$ and the restriction $\varphi|\mathscr{B}$. Show that this restriction is a tracial functional on $\mathscr{B}$ which implies that $\varphi(P) = \varphi(Q)$.

It is well known that each linear functional φ on $\mathbb{M}_2(\mathbb{C})$ can be presented as $\varphi(\cdot) = \mathrm{tr}(S_\varphi\,\cdot)$. The matrix $S_\varphi \in \mathbb{M}_2(\mathbb{C})$ is called the *density matrix* for φ. Following the proof of Theorem 4 of [15], suppose that S_φ has two eigenvalues λ and μ, while u and v are the corresponding mutually orthogonal eigenvectors. Let P_w be an orthogonal projection to the straight line $\mathbb{C}w$ and $\varepsilon > 0$ is an arbitrary positive real. Choose linearly independent vectors x and y so that $|\varphi(P_x - P_v)| < \varepsilon$ and $|\varphi(P_y - P_v)| < \varepsilon$. Note that $P_x \vee P_y = I$ and $P_x \wedge P_y = 0$. Since

$$\lambda + \mu = \mathrm{tr}(S_\varphi) = \varphi(I) = \varphi(P_x \vee P_y - P_x \wedge P_y) \le \varphi(P_x) + \varphi(P_y) \le 2\varphi(P_v) + 2\varepsilon = 2\mu + 2\varepsilon,$$

we obtain $\lambda \le \mu + 2\varepsilon$. Since ε is arbitrary while λ and μ are interchangeable, $\lambda = \mu$. The theorem is proven. $\square$

For other characterizations of a trace, see [16–18] and the references therein.

Remark 2. It is well known that $d_\varphi(P, Q) \le \varphi(P + Q)$ for every positive normal functional φ on a von Neumann algebra $\mathscr{A}$ and every pair of commutative projections $P, Q \in \mathscr{A}^{\mathrm{pr}}$ [19, p. 168]. By (iii) of Proposition 1, $\rho_\varphi(P, Q) \le \varphi(P+Q)$ for every positive normal functional φ on a von Neumann algebra $\mathscr{A}$ and each pair of commutative projections $P, Q \in \mathscr{A}^{\mathrm{pr}}$. Theorem 4 demonstrates that each of these inequalities holds for all pairs $P, Q \in \mathscr{A}^{\mathrm{pr}}$ if and only if φ is tracial.

Corollary. *For a von Neumann algebra $\mathscr{A}$ the following are equivalent:*

(i) *$\mathscr{A}$ is abelian;*

(ii) *$\rho_\varphi(P, Q) \le \varphi(P + Q)$ for all normal states φ on $\mathscr{A}$ and $P, Q \in \mathscr{A}^{\mathrm{pr}}$;*

(iii) *$d_\varphi(P, Q) \le \varphi(P + Q)$ for all normal states φ on $\mathscr{A}$ and $P, Q \in \mathscr{A}^{\mathrm{pr}}$.*

Proof. By Theorem 4, every normal state on $\mathscr{A}$ is tracial. The set of normal states on $\mathscr{A}$ separates the points of $\mathscr{A}$ [8, Chapter III, Theorem 2.4.5]; therefore, $\mathscr{A}$ is commutative. $\square$

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A. M. Bikchentaev
Lobachevskii Institute of Mathematics and Mechanics
of Kazan (Volga Region) Federal University, Kazan, Russia
E-mail address: Airat.Bikchentaev@kpfu.ru