ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2018, Vol. 39, No. 6, pp. 731–741. © Pleiades Publishing, Ltd., 2018.

Paranormal Measurable Operators Affiliated with a Semifinite von Neumann Algebra

A. M. Bikchentaev*

(Submitted by O. E. Tikhonov)

N.I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, ul. Kremlevskaya 18, Kazan, Tatarstan, 420008 Russia

Received March 7, 2017

Abstract—Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, τ be a faithful normal semifinite trace on $\mathcal{M}$. We define two (closed in the topology of convergence in measure τ) classes $\mathcal{P}_1$ and $\mathcal{P}_2$ of τ-measurable operators and investigate their properties. The class $\mathcal{P}_2$ contains $\mathcal{P}_1$. If a τ-measurable operator T is hyponormal, then T lies in $\mathcal{P}_1$; if an operator T lies in $\mathcal{P}_k$, then UTU^* belongs to $\mathcal{P}_k$ for all isometries U from $\mathcal{M}$ and $k = 1, 2$; if an operator T from $\mathcal{P}_1$ admits the bounded inverse T^{-1} then T^{-1} lies in $\mathcal{P}_1$. If a bounded operator T lies in $\mathcal{P}_1$ then T is normaloid, T^n belongs to $\mathcal{P}_1$ and a rearrangement $\mu_t(T^n) \geq \mu_t(T)^n$ for all $t > 0$ and natural n. If a τ-measurable operator T is hyponormal and T^n is τ-compact operator for some natural number n then T is both normal and τ-compact. If an operator T lies in $\mathcal{P}_1$ then T^2 belongs to $\mathcal{P}_1$. If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, then the class $\mathcal{P}_1$ coincides with the set of all paranormal operators on $\mathcal{H}$. If a τ-measurable operator A is q-hyponormal ($1 \geq q > 0$) and $|A^*| \geq \mu_\infty(A)I$ then A is normal. In particular, every τ-compact q-hyponormal (or q-cohyponormal) operator is normal. Consider a τ-measurable nilpotent operator $Z \neq 0$ and numbers $a, b \in \mathbb{R}$. Then an operator $Z^*Z - ZZ^* + a\Re Z + b\Im Z$ cannot be nonpositive or nonnegative. Hence a τ-measurable hyponormal operator $Z \neq 0$ cannot be nilpotent.

DOI: 10.1134/S1995080218060021

Keywords and phrases: *Hilbert space, von Neumann algebra, normal semifinite trace, τ-measurable operator, rearrangement, measure topology, τ-compact operator, integrable operator, hyponormal operator, paranormal operator, nilpotent, projection.*

1. INTRODUCTION

Let $\mathcal{M}$ be a von Neumann operator algebra on a Hilbert space $\mathcal{H}$, τ be a faithful normal semifinite trace on $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be the $*$-algebra of all τ-measurable operators, a number $0 < p < \infty$ and $L_p(\mathcal{M}, \tau)$ be the space of integrable (with respect to τ) in p-th degree operators. Let $\mathcal{M}_1 = \{X \in \mathcal{M} : \ ||X|| = 1\}$, $\mu_t(X)$ be a rearrangement of operator $X \in \widetilde{\mathcal{M}}$ and $\mu_\infty(X) = \lim_{t\to\infty} \mu_t(X)$. In this paper we introduce two classes

$$\mathcal{P}_1 = \{T \in \widetilde{\mathcal{M}} : \ ||T^2A|| \geq ||TA||^2 \ \text{ for all } \ A \in \mathcal{M}_1 \ \text{ with } \ TA \in \mathcal{M}\},$$

$$\mathcal{P}_2 = \{T \in \widetilde{\mathcal{M}} : \ \mu_t(T^2) \geq \mu_t(T)^2 \ \text{ for all } \ t > 0\}$$

of τ-measurable operators and investigate their properties. The classes $\mathcal{P}_1$ and $\mathcal{P}_2$ are closed in the topology of convergence in measure τ and $\mathcal{P}_1 \subset \mathcal{P}_2$ (Propositions 3.5 and 3.30). In Theorem 3.1 we obtain an equivalent definition of the class $\mathcal{P}_1$, that allows us to call $\mathcal{P}_1$ a class of all paranormal τ-measurable operators. If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal then $T \in \mathcal{P}_1$; if an operator $T \in \mathcal{P}_1$ has the inverse $T^{-1} \in \mathcal{M}$ then $T^{-1} \in \mathcal{P}_1$ (Theorem 3.6). If an operator $T \in \mathcal{P}_k$ then $UTU^* \in \mathcal{P}_k$ for all

*E-mail: Airat.Bikchentaev@kpfu.ru

isometries $U \in \mathcal{M}$ and $k = 1, 2$. If an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ then $T^n \in \mathcal{P}_1$ for all $n \in \mathbb{N}$ (Theorem 3.12). Consider an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ and $n \in \mathbb{N}$. Then $\mu_t(T^n) \geq \mu_t(T)^n$ for all $t > 0$ (Theorem 3.16) and we have the equivalences: an operator T is τ-compact $\Leftrightarrow$ an operator T^n is τ-compact; $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$ (Corollary 3.17). Every operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ is normaloid (Corollary 3.18). If an operator $(0 \neq) T \in \mathcal{M}$ is quasinilpotent then $T \notin \mathcal{P}_1$ (Corollary 3.19). If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal and T^n is τ-compact operator for some natural number n then T is both normal and τ-compact (Corollary 3.7); it is a strengthening of item (i) assertion of Corollary 3.2 [2]. If $T \in \mathcal{P}_1$ then $T^2 \in \mathcal{P}_1$ (Theorem 3.21).

The assertions of items (ii)–(iii) of Corollary 3.2, Corollaries 3.4, 3.17 and 3.20, Propositions 3.5, 3.22, 3.27 and Theorems 3.16, 4.1 and 4.6 are new even for $*$-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$. Then the class $\mathcal{P}_1$ coincides with the set of all paranormal operators on $\mathcal{H}$ (Corollary 3.3). The class $\mathcal{P}_1$ is sequentially closed in the strong operator topology (Corollary 3.4) and contains a non-hyponormal operator (Corollary 3.13). The class $\mathcal{P}_2$ is closed in the $||\cdot||$-topology (Corollary 3.31). If $\mathcal{H}$ is separable and infinite-dimensional then $\mathcal{P}_1 \neq \mathcal{P}_2$ (Corollary 3.23). If $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$ and $\tau = \mathrm{tr}_2$ is the canonical trace then $\mathcal{P}_1 = \mathcal{P}_2$ is the set of all normal matrices from $\mathcal{M}$ (Theorem 3.32). Some of these results without proofs were announced in the brief note [4].

Let $1 \geq q > 0$. We prove that if an operator $A \in \widetilde{\mathcal{M}}$ is q-hyponormal and $|A^*| \geq \mu_\infty(A) I$ then A is normal (Theorem 4.1). The proof of Theorem 4.1 is based on a deep result from [7]. Every τ-compact q-hyponormal (or q-cohyponormal) operator is normal (Corollary 4.3; see also [2]). If an operator $A \in \widetilde{\mathcal{M}}$ is hyponormal and $|\lambda I + A^*| \geq \mu_\infty(\lambda I + A^*) I$ for some $\lambda \in \mathbb{C}$ then A is normal (Corollary 4.4). Consider a nilpotent operator $Z \in \widetilde{\mathcal{M}}$, $Z \neq 0$ and numbers $a, b \in \mathbb{R}$. Then an operator $Z^*Z - ZZ^* + a\Re Z + b\Im Z$ cannot be nonpositive or nonnegative (Theorem 4.6). Hence a non-zero hyponormal operator $Z \in \widetilde{\mathcal{M}}$ cannot be nilpotent (Corollary 4.7). If an operator $Q \in \widetilde{\mathcal{M}}$ and $Q^2 = Q \neq Q^*$ then for any number $b \in \mathbb{R}$ the operator $Q^*Q - QQ^* + b\Im Q$ cannot be nonpositive or nonnegative (Corollary 4.8). If an operator $S \in \widetilde{\mathcal{M}}$ and $S^2 = I$, $S \neq S^*$ then for any number $b \in \mathbb{R}$ the operator $S^*S - SS^* + b\Im S$ cannot be nonpositive or nonnegative (Corollary 4.9).

2. NOTATION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$, and let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$. Let I be the unit of $\mathcal{M}$ and $P^\perp = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *finite*, if $\varphi(X) < +\infty$ for all $X \in \mathcal{M}^+$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$; *normal*, if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow \varphi(X) = \sup \varphi(X_i)$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated with a von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all the projections from its spectral decomposition of unity belong to $\mathcal{M}$.

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X of everywhere dense in $\mathcal{H}$ domain $\mathcal{D}(X)$ and affiliated with $\mathcal{M}$ is said to be *τ-measurable* if for any $\varepsilon > 0$ there exists such a projection $P \in \mathcal{M}^{\mathrm{pr}}$ that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a $*$-algebra under transition to the adjoint operator, multiplication by a scalar, and strong addition and multiplication operations defined as closure of the usual operations [17, 18]. Let $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{sa}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, respectively. We denote by $\leq$ the partial order in $\widetilde{\mathcal{M}}^{\mathrm{sa}}$ generated by its proper cone $\widetilde{\mathcal{M}}^+$. If an operator $X \in \widetilde{\mathcal{M}}$ then its real and imaginary components $\Re X = (X + X^*)/2$, $\Im X = (X - X^*)/(2i)$ lie in $\widetilde{\mathcal{M}}^{\mathrm{sa}}$.

If X is a closed densely defined linear operator affiliated with $\mathcal{M}$ and $|X| = \sqrt{X^*X}$ then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and X belongs to $\widetilde{\mathcal{M}}$ if and only if there exists a number

$\lambda \in \mathbb{R}$ such that $\tau(P^{|X|}((\lambda,+\infty))) < +\infty$. If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^{+}$. Also if $|X| = \int\limits_0^{+\infty} \lambda P^{|X|}(\mathrm{d}\lambda)$ is a spectral decomposition then $\tau(P^{|X|}((\lambda,+\infty))) \to 0$ as $\lambda \to +\infty$. Let $\mu_t(X)$ denote the *rearrangement* of the operator $X \in \widetilde{\mathcal{M}}$, i. e. the nonincreasing right continuous function $\mu(X)\colon (0,+\infty) \to [0,+\infty)$ given by the formula

$$\mu_t(X) = \inf\{||XP|| : P \in \mathcal{M}^{\mathrm{pr}}, \quad \tau(P^{\perp}) \le t\}, \quad t > 0. \tag{1}$$

The sets $U(\varepsilon,\delta) = \{X \in \widetilde{\mathcal{M}} : (||XP|| \le \varepsilon \text{ and } \tau(P^{\perp}) \le \delta \text{ for some } P \in \mathcal{M}^{\mathrm{pr}})\}$, where $\varepsilon > 0, \delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $\widetilde{\mathcal{M}}$, called *the measure topology* ([17, 20, p. 18]). Equipped with this topology, $\widetilde{\mathcal{M}}$ is a complete topological $*$-algebra in which $\mathcal{M}$ is dense. We will write $X_n \xrightarrow{\tau} X$ if a sequence $\{X_n\}_{n=1}^{\infty}$ converges to $X \in \widetilde{\mathcal{M}}$ in the measure topology on $\widetilde{\mathcal{M}}$.

The set of τ-compact operators $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim\limits_{t\to+\infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathcal{M}}$ [21]. The set of elementary operators $\mathcal{F}(\mathcal{M}) = \{X \in \mathcal{M} : \mu_t(X) = 0 \text{ for some } t > 0\}$ is an ideal in $\mathcal{M}$. Let m be a linear Lebesgue measure on $\mathbb{R}$. A noncommutative L_p-Lebesgue space ($0 < p < +\infty$) affiliated with $(\mathcal{M},\tau)$ can be defined as $L_p(\mathcal{M},\tau) = \{X \in \widetilde{\mathcal{M}} : \mu(X) \in L_p(\mathbb{R}^+, m)\}$ with the F-norm (the norm for $1 \le p < +\infty$) $||X||_p = ||\mu(X)||_p$, $X \in L_p(\mathcal{M},\tau)$. We have $\mathcal{F}(\mathcal{M}) \subset L_p(\mathcal{M},\tau) \subset \widetilde{\mathcal{M}}_0$ for all $0 < p < +\infty$.

If $\tau(\mathbb{I}) < +\infty$ then $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_0$ consists of all closed linear operators on $\mathcal{H}$ affiliated with $\mathcal{M}$ and $\mathcal{F}(\mathcal{M}) = \mathcal{M}$. Furthermore, t_τ is independent of a concrete choice of a trace τ and is minimal among all metrizable topologies which agree with the ring structure of $\widetilde{\mathcal{M}}$ [5, Theorem 2].

Lemma 2.1 (see [9, 21]). *Let $X, Y, Z \in \widetilde{\mathcal{M}}$. Then* 1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$; 2) *if $X, Y \in \mathcal{M}$ then $\mu_t(XZY) \le ||X||||Y||\mu_t(Z)$ for all $t > 0$*; 3) $\mu_t(|X|^p) = \mu_t(X)^p$ *for all $p > 0$ and $t > 0$*; 4) *if $|X| \le |Y|$ then $\mu_t(X) \le \mu_t(Y)$ for all $t > 0$*; 5) $\mu_{s+t}(X+Y) \le \mu_s(X) + \mu_t(Y)$ *for all $s, t > 0$*; 6) $\mu_t(\lambda X) = |\lambda|\mu_t(X)$ *for all $\lambda \in \mathbb{C}$ and $t > 0$*; 7) $\lim\limits_{t\to0+} \mu_t(X) = ||X||$ *if $X \in \mathcal{M}$ and* $\lim\limits_{t\to0+} \mu_t(X) = \infty$ *if $X \notin \mathcal{M}$.*

Lemma 2.2 (see [8], p. 720). *If $X, Y \in \widetilde{\mathcal{M}}^{+}$ and $Z \in \widetilde{\mathcal{M}}$ then the inequality $X \le Y$ implies that $ZXZ^* \le ZYZ^*$.*

An operator $A \in \widetilde{\mathcal{M}}$ is said to be *normal*, if $A^*A = AA^*$; *quasinormal*, if A commute with A^*A, i.e. $A \cdot A^*A = A^*A \cdot A$. Let $1 \ge q > 0$. An operator $A \in \widetilde{\mathcal{M}}$ is said to be *q-hyponormal* if $(A^*A)^q \ge (AA^*)^q$. If $q = 1$ then A is said to be *hyponormal*. An operator $A \in \widetilde{\mathcal{M}}$ is said to be *q-cohyponormal* if A^* is q-hyponormal; *nilpotent* if $A^n = 0$ for some $n \in \mathbb{N}$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e. the $*$-algebra of all linear bounded operators on $\mathcal{H}$, and $\tau = \mathrm{tr}$ is the canonical trace then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$. In this case $\widetilde{\mathcal{M}}_0$ is the compact operators ideal on $\mathcal{H}$, $\mathcal{F}(\mathcal{M})$ is the finite-dimensional operators ideal on $\mathcal{H}$ and

$$\mu_t(X) = \sum_{n=1}^{+\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{+\infty}$ is a sequence of the operator X s-numbers [11, Ch. 1]; here χ_A is the indicator function of a set $A \subset \mathbb{R}$. Then the space $L_p(\mathcal{M},\tau)$ is a Shatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < +\infty$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *quasinilpotent*, if its spectrum $\sigma(T) = \{0\}$; *paranormal*, if $||T^2x||_{\mathcal{H}} \ge ||Tx||^2_{\mathcal{H}}$ for all $x \in \mathcal{H}_1 = \{y \in \mathcal{H} : ||y||_{\mathcal{H}} = 1\}$, see [14, 10]; *normaloid*, if $||T|| = \sup\limits_{y\in\mathcal{H}_1} |\langle Tx, x\rangle|$. It is known that T is normaloid $\Leftrightarrow$ its spectral radius equals $||T||$, or, equivalently, $||T^n|| = ||T||^n$ for all $n \in \mathbb{N}$ [12]. It is shown in [15, Problem 9.5] that an operator $T \in \mathcal{B}(\mathcal{H})$ is paranormal $\Leftrightarrow$

$$|T|^2 \le \frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda I) \text{ for all } \lambda > 0. \tag{2}$$

Let (Ω, ν) be a measure space and $\mathcal{M}$ be the von Neumann algebra of multiplicator operators M_f by functions f from $L_\infty(\Omega, \nu)$ on a space $L_2(\Omega, \nu)$. The algebra $\mathcal{M}$ contains no compact operators $\Leftrightarrow$ the measure ν has no atoms [1, Theorem 8.4].

3. TWO CLASSES OF τ-MEASURABLE OPERATORS

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Assume that $||X|| = +\infty$ for all $X \in \widetilde{\mathcal{M}} \setminus \mathcal{M}$. Put $\mathcal{M}_1 = \{X \in \mathcal{M} : ||X|| = 1\}$. We introduce two classes of τ-measurable operators:

$$\mathcal{P}_1 = \{T \in \widetilde{\mathcal{M}} : ||T^2A|| \geq ||TA||^2 \text{ for all } A \in \mathcal{M}_1 \text{ with } TA \in \mathcal{M}\},$$

$$\mathcal{P}_2 = \{T \in \widetilde{\mathcal{M}} : \mu_t(T^2) \geq \mu_t(T)^2 \text{ for all } t > 0\}.$$

It is obvious that

$$T \in \mathcal{P}_k \Leftrightarrow \lambda T \in \mathcal{P}_k \text{ for all } \lambda \in \mathbb{C} \setminus \{0\}, k = 1, 2. \tag{3}$$

Theorem 3.1. *For an operator $T \in \widetilde{\mathcal{M}}$ the following conditions are equivalent*: (i) $T \in \mathcal{P}_1$; (ii) T *meets condition* (2).

Proof. (i) $\Rightarrow$ (ii). Assume that for an operator $T \in \mathcal{P}_1$ condition (2) does not hold. Then there exists a number $\lambda > 0$ such that

$$\frac{1}{2}(\lambda^{-1}|T^2|^2 + \lambda I) - |T|^2 = X - Y, \tag{4}$$

where $X, Y \in \widetilde{\mathcal{M}}^+$, $XY = 0$ and $Y \neq 0$. Let $Y = \int_0^\infty t P^Y(\mathrm{d}t)$ be the spectral decomposition and $n \in \mathbb{N}$ be such that a projection $P = P^Y((n^{-1}, n)) \neq 0$. Then $PXP = 0$ and $PYP \geq n^{-1}P$. Relation (4) multiplication by the projection P from the left and the right-hand sides, leads us to

$$P|T|^2P = \frac{1}{2}(\lambda^{-1}P|T^2|^2P + \lambda P) + PYP \geq \frac{1}{2}(\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P).$$

Since P is a unit in the reduced von Neumann algebra $\mathcal{M}_P$, we have

$$||TP||^2 = ||P|T|^2P|| \geq \frac{1}{2}||\lambda^{-1}P|T^2|^2P + (\lambda + 2n^{-1})P|| = \frac{1}{2}(\lambda^{-1}||T^2P||^2 + (\lambda + 2n^{-1})).$$

If $T^2P = 0$ then $||TP||^2 \geq \lambda 2^{-1} + n^{-1} > ||T^2P|| = 0$. If $T^2P \neq 0$ then by the inequality $a^2 + b^2 \geq 2|ab|$ for all $a, b \in \mathbb{R}$ we have

$$||TP||^2 \geq \frac{1}{2} \cdot 2\sqrt{\lambda^{-1}(\lambda + 2n^{-1})} \cdot ||T^2P|| > ||T^2P||.$$

Thus, in both cases $T \notin \mathcal{P}_1$ — a contradiction.

(ii) $\Rightarrow$ (i). Consider an operator $A \in \mathcal{M}_1$ such that $TA \in \mathcal{M}$. Then $A^*A \leq I$ and $|T|A \in \mathcal{M}$. If $T^2A \notin \mathcal{M}$ then the assertion is met. Let $T^2A \in \mathcal{M}$. Inequality (2) multiplication from the left-hand side by the operator A^* and from the right-hand side by the operator A, leads us to

$$A^*|T|^2A \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda A^*A) \leq \frac{1}{2}(\lambda^{-1}A^*|T^2|^2A + \lambda I) \text{ for all } \lambda > 0.$$

Therefore $||A^*|T|^2A|| = ||TA||^2 \leq \frac{1}{2}(\lambda^{-1}||T^2A||^2 + \lambda)$ for all $\lambda > 0$. Put here $\lambda = ||T^2A||$ and obtain $||TA||^2 \leq ||T^2A||$. Theorem is proved. □

Corollary 3.2. *Consider operators $T \in \mathcal{P}_1$, $A \in \widetilde{\mathcal{M}}$ and numbers $k \in \mathbb{N}$, $0 < p, q, r < \infty$ with $1/p + 1/q = 1/r$. Then*

(i) *if $T^kA, T^{k+2}A \in \mathcal{M}$ then $T^{k+1}A \in \mathcal{M}$*;

(ii) *if $T^kA \in \mathcal{M}$, $T^{k+2}A \in \mathcal{F}(\mathcal{M})$ or $T^kA \in \mathcal{F}(\mathcal{M})$, $T^{k+2}A \in \mathcal{M}$ then $T^{k+1}A \in \mathcal{F}(\mathcal{M})$*;

(iii) *if $T^{k+2}A \in \widetilde{\mathcal{M}}_0$ then $T^{k+1}A \in \widetilde{\mathcal{M}}_0$*;

(iv) *if* $T^kA \in L_p(\mathcal{M},\tau)$, $T^{k+2}A \in L_q(\mathcal{M},\tau)$ *then* $T^{k+1}A \in L_{2r}(\mathcal{M},\tau)$.

Proof. For all $t, \lambda > 0$ and $k \in \mathbb{N}$ by Theorem 3.1, items 3)–5), 6), and 7) of Lemma 2.1, Lemma 2.2 and inequality (2) we have the following estimates for the rearrangements:

$$\begin{aligned}2\mu_t(T^{k+1}A)^2 &= 2\mu_t(A^*(T^*)^{k+1}T^{k+1}A) = 2\mu_t(A^*T^{*k}\cdot T^*T\cdot T^kA)\\ &\le \mu_t(A^*T^{*k}(\lambda^{-1}T^{*2}T^2+\lambda I)T^kA) \le \lambda^{-1}\mu_{t/2}(A^*(T^*)^{k+2}T^{k+2}A)\\ &\quad + \lambda\mu_{t/2}(A^*T^{*k}T^kA) = \lambda^{-1}\mu_{t/2}(T^{k+2}A)^2 + \lambda\mu_{t/2}(T^kA)^2.\end{aligned}$$

Note that $\inf_{\lambda>0} \lambda^{-1}a + \lambda b = 2\sqrt{ab}$ for all $a, b \ge 0$. Hence

$$\mu_t(T^{k+1}A)^2 \le \mu_{t/2}(T^{k+2}A)\mu_{t/2}(T^kA) \ \text{ for all } \ t > 0.$$

In order to check item (i) we apply item 6) of Lemma 2.1. The assertion is proved. □

Corollary 3.3. *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then the class $\mathcal{P}_1$ coincides with the class of all paranormal operators on $\mathcal{H}$.*

Since the product operation is sequentially jointly continuous in the strong operator topology in $\mathcal{B}(\mathcal{H})$ [12, Problem 93], Corollary 3.3 implies

Corollary 3.4. *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then the class $\mathcal{P}_1$ is sequentially closed in the strong operator topology.*

Proposition 3.5. *Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. Then $\mathcal{P}_1 \subset \mathcal{P}_2$.*

Proof. Let $t > 0$ be fixed. From relation (1) for $X = T^2$ we have

$$\forall\varepsilon > 0\ \exists P_\varepsilon \in \mathcal{M}^{\mathrm{pr}}(\tau(P_\varepsilon^\perp) \le t, \varepsilon + \mu_t(T^2) > ||T^2P_\varepsilon|| \ge \mu_t(T^2)),$$

thereby $||TP_\varepsilon||^2 \le \varepsilon + \mu_t(T^2)$. Note that a projection P_ε is included in the right-hand side of (1) for $X = T$. Therefore $\mu_t(T) \le ||TP_\varepsilon||$ and because of the arbitrariness of the number $\varepsilon > 0$ we get $\mu_t(T^2) \ge \mu_t(T)^2$. Proposition is proved. □

If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal or cohyponormal then $\mu_t(T^2) = \mu_t(T)^2$ for all $t > 0$ [2, Theorem 3.1] and $T \in \mathcal{P}_2$. If $T \in \widetilde{\mathcal{M}}$ is nilpotent of second order ($T \ne 0 = T^2$) then $T \notin \mathcal{P}_2$.

Theorem 3.6. (i) *If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal then $T \in \mathcal{P}_1$.*

(ii) *If an operator $T \in \mathcal{P}_1$ then $UTU^* \in \mathcal{P}_1$ for all isometries $U \in \mathcal{M}$.*

(iii) *If an operator $T \in \mathcal{P}_1$ has an inverse $T^{-1} \in \mathcal{M}$ then $T^{-1} \in \mathcal{P}_1$.*

Proof. (i). Consider a hyponormal operator $T \in \widetilde{\mathcal{M}}$ and $A \in \mathcal{M}_1$ such that $TA \in \mathcal{M}$. If $T^2A \notin \mathcal{M}$ then the assertion is obvious. For $T^2A \in \mathcal{M}$ by Lemma 2.2 we have

$$\begin{aligned}||T^2A|| &= ||A^*T^{2*}T^2A||^{1/2} = ||A^*T^*\cdot T^*T\cdot TA||^{1/2} \ge ||A^*T^*\cdot TT^*\cdot TA||^{1/2}\\ &= |||T|^2\cdot A|| \ge ||A^*\cdot|T|^2\cdot A|| = ||TA||^2.\end{aligned}$$

(ii). Consider $A \in \mathcal{M}_1$ such that $UTU^*\cdot A \in \mathcal{M}$. If $(UTU^*)^2\cdot A \notin \mathcal{M}$ or $U^*A = 0$ then the assertion is obvious. Let $(UTU^*)^2\cdot A \in \mathcal{M}$ and $U^*A \ne 0$. Then $0 < ||U^*A|| \le 1$ and

$$||(UTU^*)^2\cdot A|| = ||UT^2U^*\cdot A|| \ge ||U^*\cdot UT^2U^*\cdot A|| = ||T^2U^*A||$$

$$= \Big|\Big|T^2\frac{U^*A}{||U^*A||}\Big|\Big|\cdot||U^*A|| \ge \Big|\Big|T\frac{U^*A}{||U^*A||}\Big|\Big|^2\cdot||U^*A|| = \frac{||T\cdot U^*A||^2}{||U^*A||} \ge ||T\cdot U^*A||^2 \ge ||UTU^*\cdot A||^2.$$

(iii). Consider $A \in \mathcal{M}_1$, it is necessary to prove that $||T^{-2}A|| \ge ||T^{-1}A||^2$. If $T^{-2}A = 0$ then $T\cdot T^{-2}A = T^{-1}A = 0$ and the assertion holds. If $T^{-2}A \ne 0$ then

$$\Big|\Big|T^2\frac{T^{-2}A}{||T^{-2}A||}\Big|\Big| \ge \Big|\Big|T\frac{T^{-2}A}{||T^{-2}A||}\Big|\Big|^2,$$

i.e. $\frac{||A||}{||T^{-2}A||} = \frac{1}{||T^{-2}A||} \geq \frac{||T^{-1}A||^2}{||T^{-2}A||^2}$ and the assertion is proved. □

Corollary 3.7. *If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal and $T^n \in \widetilde{\mathcal{M}}_0$ for some $n \in \mathbb{N}$ then $T \in \widetilde{\mathcal{M}}_0$ and is normal.*

Proof. By item (i) of Theorem 3.6 we have $T \in \mathcal{P}_1$. Applying $n-1$ times item (iii) of Corollary 3.2 with the operator $A = I$, we obtain $T \in \widetilde{\mathcal{M}}_0$ and can apply Theorem 3.2 from [2]. □

Corollary 3.8. *If an operator $T \in \widetilde{\mathcal{M}}$ is quasinormal then $T \in \mathcal{P}_1$.*

Proof. Every quasinormal operator $T \in \widetilde{\mathcal{M}}$ is hyponormal [3, Theorem 2.9]. □

If an operator $T \in \widetilde{\mathcal{M}}$ is quasinormal then T^n is also quasinormal [6, Proposition 2.10] and $\mu_t(T^n) = \mu_t(T)^n$ for all $t > 0$ and $n \in \mathbb{N}$ [6, Theorem 2.6]. Similarly to Lemma 1 from [19] one can prove

Proposition 3.9. *If an operator $T \in \widetilde{\mathcal{M}}$ is hyponormal and $(T - zI)^{-1} \in \mathcal{M}$ for some $z \in \mathbb{C}$ then $(T - zI)^{-1}$ is hyponormal.*

Lemma 3.10. *If an operator $T \in \mathcal{P}_1$ then*

$$||T^3A|| \geq ||T^2A|| \cdot ||TA|| \ \text{ for all } \ A \in \mathcal{M}_1 \ \text{ with } \ TA \in \mathcal{M}. \tag{5}$$

Proof. If $T^3A \in \widetilde{\mathcal{M}} \setminus \mathcal{M}$ then the assertion is obvious. Let $T^3A \in \mathcal{M}$. Without loss of generality, assume that $TA \neq 0$. Then

$$\begin{aligned} ||T^3A|| &= ||TA|| \cdot \left|\left|T^2\frac{TA}{||TA||}\right|\right| \geq ||TA|| \cdot \left|\left|T\frac{TA}{||TA||}\right|\right|^2 \\ &= \frac{||T^2A||^2}{||TA||} \geq \frac{||T^2A|| \cdot ||TA||^2}{||TA||} = ||T^2A|| \cdot ||TA|| \end{aligned}$$

and Lemma is proved. □

Lemma 3.11. *If an operator $T \in \mathcal{P}_1$ then*

$$||T^{k+1}A||^2 \geq ||T^kA||^2 \cdot ||T^2A|| \ \text{ for all } \ A \in \mathcal{M}_1 \ \text{ with } \ TA \in \mathcal{M} \ \text{ and } \ k \in \mathbb{N}. \tag{6_k}$$

Proof. The proof is by induction. For $k = 1$ we have

$$||T^2A||^2 = ||T^2A|| \cdot ||T^2A|| \geq ||TA||^2 \cdot ||T^2A||$$

and (6_1) is met. Let (6_k) hold for k and $TA \neq 0$, then

$$\begin{aligned} ||T^{k+2}A||^2 &= ||TA||^2 \cdot \left|\left|T^{k+1}\frac{TA}{||TA||}\right|\right|^2 \geq ||TA||^2 \cdot \left|\left|T^k\frac{TA}{||TA||}\right|\right|^2 \cdot \left|\left|T^2\frac{TA}{||TA||}\right|\right| \\ &= ||T^{k+1}A||^2\frac{||T^3A||}{||TA||} \geq ||T^{k+1}A||^2 \cdot ||T^2A|| \end{aligned}$$

by item (5) of Lemma 3.10 and (6_k). Therefore (6_{k+1}) holds and Lemma is proved. □

Theorem 3.12. *If an operator $T \in \mathcal{P}_1 \cap \mathcal{M}$ then $T^n \in \mathcal{P}_1$ for all $n \in \mathbb{N}$.*

Proof. The proof is by induction. It suffices to show that if $T, T^k \in \mathcal{P}_1 \cap \mathcal{M}$ then $T^{k+1} \in \mathcal{P}_1$. Let $A \in \mathcal{M}_1$ and $T^2A \neq 0$. Then

$$\begin{aligned} ||T^{2(k+1)}A|| &= \left|\left|T^{2k}\frac{T^2A}{||T^2A||}\right|\right| \cdot ||T^2A|| \geq \left|\left|T^k\frac{T^2A}{||T^2A||}\right|\right|^2 \cdot ||T^2A|| \\ &= \frac{||T^{k+2}A||^2}{||T^2A||} \geq \frac{||T^{k+1}A||^2 \cdot ||T^2A||}{||T^2A||} = ||T^{k+1}A||^2 \end{aligned} \tag{7}$$

by (6_{k+1}) of Lemma 3.11. Theorem is proved. □

Corollary 3.13. *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then $\mathcal{P}_1$ possesses a non-hyponormal operator.*

Proof. P. Halmos ([12, Problem 164]) presented an example of a hyponormal operator $T \in \mathcal{M}$ such that T^2 is non-hyponormal. We have $T \in \mathcal{P}_1$ by item (i) of Theorem 3.5, hence $T^2 \in \mathcal{P}_1$ by Theorem 3.12. □

Proposition 3.14. *The set* $\mathcal{P}_1 \cap \mathcal{M}$ is $||\cdot||$*-closed in* $\mathcal{M}$.

Proof. Consider $T_n \in \mathcal{P}_1 \cap \mathcal{M}$, $T \in \mathcal{M}$ and $A \in \mathcal{M}_1$. If $||T_n - T|| \to 0$ as $n \to \infty$ then $||T_n A - TA|| \to 0$ and $||T_n^2 A - T^2 A|| \to 0$ as $n \to \infty$ via $||\cdot||$-continuity of the product operation in $\mathcal{M}$. Therefore $||T_n A|| \to ||TA||$ and $||T_n^2 A|| \to ||T^2 A||$ as $n \to \infty$. □

Lemma 3.15. *Let a sequence* $\{a_n\}_{n=1}^{\infty}$ *of positive numbers be so that* $a_2 \ge a_1^2$ *and* $a_n a_{n-2} \ge a_{n-1}^2$ *for all* $n \ge 3$. *Then* $a_n \ge a_1^n$ *for all* $n \ge 2$.

Proof. If $k > 2$ then $a_k a_{k-2} \ge a_{k-1}^2, a_{k-1}a_{k-3} \ge a_{k-2}^2, \dots, a_4 a_2 \ge a_3^2, a_3 a_1 \ge a_2^2$. Multiplying all the left-hand sides and all the right-hand sides of these inequalities, after obvious contractions, we obtain $a_k a_1 \ge a_{k-1} a_2$, hence $a_k / a_{k-1} \ge a_2 / a_1 \ge a_1$ and $a_n \ge a_1 a_{n-1} \ge a_1^2 a_{n-2} \ge \dots \ge a_1^n$. Lemma is proved. □

Theorem 3.16. *If an operator* $T \in \mathcal{P}_1 \cap \mathcal{M}$ *then* $\mu_t(T^n) \ge \mu_t(T)^n$ *for all* $t > 0$ *and* $n \in \mathbb{N}$.

Proof. Let $t > 0$ and $n \in \mathbb{N}$ be fixed. From (1) for $X = T^n$ we have

$$\forall \varepsilon > 0\ \exists P_\varepsilon \in \mathcal{M}^{\mathrm{pr}}(\tau(P_\varepsilon^\perp) \le t, \varepsilon + \mu_t(T^n) > ||T^n P_\varepsilon|| \ge \mu_t(T^n)).$$

Since

$$||T^k P_\varepsilon|| = \left|\left|T^2 \frac{T^{k-2}P_\varepsilon}{||T^{k-2}P_\varepsilon||}\right|\right| \cdot ||T^{k-2}P_\varepsilon|| \ge \left|\left|T \frac{T^{k-2}P_\varepsilon}{||T^{k-2}P_\varepsilon||}\right|\right|^2 \cdot ||T^{k-2}P_\varepsilon|| = \frac{||T^{k-1}P_\varepsilon||^2}{||T^{k-2}P_\varepsilon||}$$

and $||T^2 P_\varepsilon|| \ge ||TP_\varepsilon||^2$, for a number sequence $a_k = ||T^k P_\varepsilon||$, $k \in \mathbb{N}$, all the conditions of Lemma 3.15 are met. Hence $a_n \ge a_1^n$, i.e. $||T^n P_\varepsilon|| \ge ||TP_\varepsilon||^n$ for all $n \in \mathbb{N}$. Thus, $\varepsilon + \mu_t(T^n) > ||TP_\varepsilon||^n \ge \mu_t(T)^n$ and Theorem is proved. □

Corollary 3.17. *Consider an operator* $T \in \mathcal{P}_1 \cap \mathcal{M}$ *and* $n \in \mathbb{N}$. *We have the equivalences*: (i) $T \in \mathcal{F}(\mathcal{M}) \Leftrightarrow T^n \in \mathcal{F}(\mathcal{M})$; (ii) $T \in \widetilde{\mathcal{M}}_0 \Leftrightarrow T^n \in \widetilde{\mathcal{M}}_0$; (iii) $T \in L_{pn}(\mathcal{M}, \tau) \Leftrightarrow T^n \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$.

Corollary 3.18. *Every operator* $T \in \mathcal{P}_1 \cap \mathcal{M}$ *is normaloid.*

Corollary 3.19. *If an operator* $(0 \ne) T \in \mathcal{M}$ *is quasinilpotent then* $T \notin \mathcal{P}_1$.

Corollary 3.3 and Theorem 3.16 put together imply

Corollary 3.20. *If an operator* $T \in \mathcal{B}(\mathcal{H})$ *is paranormal then* $s_n(T^k) \ge s_n(T)^k$ for all $n, k \in \mathbb{N}$.

Theorem 3.21. *If an operator* $T \in \mathcal{P}_1$ *then* $T^{2^n} \in \mathcal{P}_1$ *for all* $n \in \mathbb{N}$. *Moreover,* $\mu_t(T^{2^n}) \ge \mu_t(T)^{2^n}$ *for all* $t > 0$ *and* $n \in \mathbb{N}$.

Proof. It suffices to verify that if $T \in \mathcal{P}_1$ then $T^2 \in \mathcal{P}_1$. Let $A \in \mathcal{M}_1$ and $T^2 A \in \mathcal{M}$. It is necessary to show that $||T^4 A|| \ge ||T^2 A||^2$. If $T^4 A \notin \mathcal{M}$ or $T^2 A = 0$ then the inequality is satisfied. If $T^4 A \in \mathcal{M}$ and $T^2 A \ne 0$ then $T^3 A \in \mathcal{M}$ by item (i) of Corollary 3.2 with $k = 1$ and repeating the calculations (7) with $k = 1$ we obtain $T^2 \in \mathcal{P}_1$. Applying successively n times Proposition 3.5 and the fact established above, we have

$$\mu_t(T^{2^n}) = \mu_t((T^{2^{n-1}})^2) \ge \mu_t(T^{2^{n-1}})^2 = \mu_t((T^{2^{n-2}})^2)^2 \ge \mu_t(T^{2^{n-2}})^4 \ge \dots \ge \mu_t(T)^{2^n}.$$

Theorem is proved. □

Proposition 3.22. *For* $T \in \widetilde{\mathcal{M}}$ *we have* $T \in \mathcal{P}_2 \Leftrightarrow T^* \in \mathcal{P}_2$.

Proof. ($\Rightarrow$). For all $T \in \mathcal{P}_2$ and $t > 0$ by item 1) of Lemma 2.1 we have

$$\mu_t((T^*)^2) = \mu_t((T^2)^*) = \mu_t(T^2) \ge \mu_t(T)^2 = \mu_t(T^*)^2. \tag{8}$$

($\Leftarrow$). Holds by the equality $(T^*)^* = T$ for all $T \in \widetilde{\mathcal{M}}$ and (8). □

Corollary 3.23. *If* $\mathcal{M} = \mathcal{B}(\mathcal{H})$ *for separable and infinite dimensional* $\mathcal{H}$ *then* $\mathcal{P}_1 \ne \mathcal{P}_2$.

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. The unilateral shift $Te_n = e_{n+1}$ $(n = 0, 1, 2, \dots)$ is a hyponormal operator (an isometry) and $T \in \mathcal{P}_1$ by item (i) of Theorem 3.6. The null-space KerT^* is generated by vector e_0, and the null-space Ker$(T^*)^2$ is generated by vectors e_0 and e_1. We have

$$0 = ||(T^*)^2 A|| < ||T^* A||^2 = 1$$

and $T^* \notin \mathcal{P}_1$ for the one-dimensional projection $A = \langle \cdot, e_1 \rangle e_1$. The assertion is proved. □

Proposition 3.24. *For $T \in \mathcal{P}_2$ we have the equivalences*: (i) $T \in \mathcal{M} \Leftrightarrow T^2 \in \mathcal{M}$; (ii) $T \in \mathcal{F}(\mathcal{M}) \Leftrightarrow T^2 \in \mathcal{F}(\mathcal{M})$; (iii) $T \in \widetilde{\mathcal{M}}_0 \Leftrightarrow T^2 \in \widetilde{\mathcal{M}}_0$; (iv) $T \in L_{2p}(\mathcal{M}, \tau) \Leftrightarrow T^2 \in L_p(\mathcal{M}, \tau)$, $0 < p < +\infty$.

Lemma 3.25. *If $T \in \widetilde{\mathcal{M}}$ and operators $U, V \in \mathcal{M}$ are isometries then $\mu_t(UTV^*) = \mu_t(T)$ for all $t > 0$.*

Proof. For all $t > 0$ by item 2) of Lemma 2.1 we have

$$\mu_t(T) = \mu_t(U^* \cdot UTV^* \cdot V) \leq ||U^*||||V|| \cdot \mu_t(UTV^*) = \mu_t(UTV^*) \leq ||U||||V^*|| \cdot \mu_t(T) = \mu_t(T)$$

and Lemma is proved. □

Proposition 3.26. *If $T \in \mathcal{P}_2$ and an operator $U \in \mathcal{M}$ is an isometry then $UTU^* \in \mathcal{P}_2$.*

Proof. Double application of Lemma 3.25 for all $t > 0$ yields

$$\mu_t((UTU^*)^2) = \mu_t(UT^2U^*) = \mu_t(T^2) \geq \mu_t(T)^2 = \mu_t(UTU^*)^2.$$

The assertion is proved. □

Proposition 3.27. *Let $T \in \widetilde{\mathcal{M}}$ and a unitary operator $S \in \mathcal{M}^{\mathrm{sa}}$ be so that $ST = TS$. Then $T \in \mathcal{P}_k \Leftrightarrow ST \in \mathcal{P}_k$, $k = 1, 2$.*

Proof. We have $S^2 = I$ and $(ST)^2 = T^2$.

($\Rightarrow$). Let $k = 1$ and $A \in \mathcal{M}_1$ be so that $TA \in \mathcal{M}$. Then

$$||(ST)^2 A|| = ||T^2 A|| \geq ||TA||^2 = ||A^*T^*TA|| = ||A^*T^*S^2TA|| = ||STA||^2.$$

If $k = 2$ then for all $t > 0$ by Lemma 3.25 we obtain $\mu_t((ST)^2) = \mu_t(T^2) \geq \mu_t(T)^2 = \mu_t(ST)^2$.

($\Leftarrow$). If $ST \in \mathcal{P}_k$ then by the above proved results $T = S \cdot ST \in \mathcal{P}_k$, $k = 1, 2$. □

Example 3.28. Assume that $T \in \widetilde{\mathcal{M}}$ and $T^2 = I$. If $T \in \mathcal{P}_2$ then T belongs to $\mathcal{M}^{\mathrm{sa}}$ and is unitary. Indeed, the equality $T^2 = I$ implies that $T = 2P - I$ with $P = P^2 \in \widetilde{\mathcal{M}}$. Since $T \in \mathcal{P}_2$, we have $\mu_t(I) = 1 \geq \mu_t(2P - I)^2$, i.e. $\mu_t(2P - I) \in [0, 1]$ for all $t > 0$. Therefore, $||2P - I|| \leq 1$ and $||2P|| = ||(2P - I) + I|| \leq ||2P - I|| + ||I|| \leq 2$. Thus $P = P^* \in \mathcal{M}^{\mathrm{pr}}$ and T both belongs to $\mathcal{M}^{\mathrm{sa}}$ and is unitary.

Example 3.29. Consider $T \in \widetilde{\mathcal{M}}$ and $T^2 = T$. If $T \in \mathcal{P}_2$ then $T \in \mathcal{M}^{\mathrm{pr}}$. Indeed, we have $\mu_t(T^2) = \mu_t(T) \geq \mu_t(T)^2$, i.e. $\mu_t(T) \in [0, 1]$ for all $t > 0$. Therefore, $||T|| \leq 1$ by item 7) of Lemma 2.1 and $T = T^* \in \mathcal{M}^{\mathrm{pr}}$.

Proposition 3.30. *The classes $\mathcal{P}_1$ and $\mathcal{P}_2$ are closed in the measure topology t_τ.*

Proof. Condition (2) is equivalent to the condition $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ for all $\lambda > 0$. Hence t_τ-closedness of the class $\mathcal{P}_1$ follows from Theorem 3.1, t_τ-continuity of the involution, t_τ-continuity of the product operation on $\widetilde{\mathcal{M}}$ and t_τ-closedness of the cone $\widetilde{\mathcal{M}}^+$ in $\widetilde{\mathcal{M}}$.

We show t_τ-closedness of the class $\mathcal{P}_2$ in $\widetilde{\mathcal{M}}$. Let $T_n \in \mathcal{P}_2$, $T \in \widetilde{\mathcal{M}}$ and $T_n \xrightarrow{\tau} T$ as $n \to \infty$. Then $T_n^2 \xrightarrow{\tau} T^2$ as $n \to \infty$ via t_τ-continuity of the product operation on $\widetilde{\mathcal{M}}$. Now we note that if $X_n, X \in \widetilde{\mathcal{M}}$ and $X_n \xrightarrow{\tau} X$ as $n \to \infty$, then $\mu_t(X_n) \to \mu_t(X)$ as $n \to \infty$ in every continuity point t of the function $\mu(X)$ [9]. The assertion is proved. □

Corollary 3.31. *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then the class $\mathcal{P}_2$ is closed in $||\cdot||$-topology.*

Theorem 3.32. *If $\mathcal{M} = \mathbb{M}_2(\mathbb{C})$ and $\tau = \mathrm{tr}_2$ is the canonical trace then $\mathcal{P}_1 = \mathcal{P}_2$ is the set $\mathcal{M}^{\mathrm{nor}}$ of all normal matrices in $\mathcal{M}$.*

Proof. By Proposition 3.5 and item (i) of Theorem 3.6 we have $\mathcal{M}^{\mathrm{nor}} \subset \mathcal{P}_1 \subset \mathcal{P}_2$. We show that if $T \in \mathcal{M}$ and $T \notin \mathcal{M}^{\mathrm{nor}}$ then $T \notin \mathcal{P}_2$. Recall that every matrix $A \in \mathbb{M}_n(\mathbb{C})$ is unitarily similar to upper triangular matrix B via Shur decomposition $A = UBU^*$ [13, Theorem 2.3.1]. Wherein $s_k(A) = s_k(B)$, $k = 1, 2, \ldots, n$, see Lemma 3.25. If $A \in \mathcal{P}_2$ then by items 1) and 3) of Lemma 2.1 we have

$$s_k(A^2)^2 \geq s_k(A)^4 = s_k(|A|)^4 = s_k((A^*A)^2), \quad k = 1, 2, \ldots, n.$$

Without loss of generality we assume that the matrix $T \notin \mathcal{M}^{\text{nor}}$ has the form $T = \begin{pmatrix} c & a \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{C}$, $a \neq 0$. If $c = 0$ then $T^2 = bT$ and $s_1(T)^2 = |a|^2 + |b|^2$. Therefore $s_1(T^2) = |b| s_1(T) < s_1(T)^2$ and $T \notin \mathcal{P}_2$. If $c \neq 0$ then with allowance for (3), we can assume that $c = 1$. Put

$$f(a, b) = 1 + 2|a|^2 + |a|^4 + 2|a|^2|b|^2 + |b|^4, g(a, b) = 1 + |a|^2|1 + b|^2 + |b|^4$$

for $a, b \in \mathbb{C}$, $a \neq 0$. Since

$$(T^*T)^2 = \begin{pmatrix} 1 + |a|^2 & a(1 + |a|^2 + |b|^2) \\ \bar{a}(1 + |a|^2 + |b|^2) & |a|^2 + (|a|^2 + |b|^2)^2 \end{pmatrix},$$

we have

$$s_1((T^*T)^2) = \frac{1}{2}(f(a, b) + \sqrt{f(a, b)^2 - 4|b|^2}). \tag{9}$$

Since

$$T^{2*}T^2 = \begin{pmatrix} 1 & a(1 + b) \\ \bar{a}(1 + \bar{b}) & |a|^2||1 + b|^2 + |b|^4 \end{pmatrix},$$

we have

$$s_1((T^2)^2) = \frac{1}{2}(g(a, b) + \sqrt{g(a, b)^2 - 4|b|^2}). \tag{10}$$

We show that $s_1(T^2) < s_1(T)^2$, i.e. $T \notin \mathcal{P}_2$. It suffices to establish the inequality $g(a, b) < f(a, b)$ for all $a, b \in \mathbb{C}$, $a \neq 0$, and use monotonocity of the real function $t \mapsto \sqrt{t} (t \geq 0)$, see (9), (10). By the triangle inequality and the Cauchy–Bunyakovsky inequality we obtain $|1 + b|^2 \leq 1 + |b|^2 + 2|b| \leq 2 + 2|b|^2$, hence $g(a, b) < f(a, b)$ for all $a, b \in \mathbb{C}$, $a \neq 0$, and Theorem is proved. □

Example 3.33. For $T \in \mathcal{B}(\mathcal{H})$ the inequality

$$s_k(T^2) \leq s_k(T)^2 \tag{11}$$

holds for $k = 1$; for $k = 2$ in the general case relation (11) does not hold true. Indeed,

$$s_1(T^2) = ||T^2|| \leq ||T|| \cdot ||T|| = ||T||^2 = s_1(T)^2$$

by submultiplicativity of the C^*-norm. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Then $T^2 = I$ and $T \notin \mathcal{P}_2$ via Example 3.28. By items 1) and 3) of Lemma 2.1 we have $s_2(T^2) = 1 > (3 - \sqrt{5})/2 = s_2(T^*T) = s_2(|T|)^2 = s_2(T)^2$.

4. HYPONORMAL τ-MEASURABLE OPERATORS

Theorem 4.1. *Let $1 \geq q > 0$, an operator $A \in \widetilde{\mathcal{M}}$ be q-hyponormal and $|A^*| \geq \mu_\infty(A)I$. Then A is normal.*

Proof. By items 1) and 3) of Lemma 2.1 for $A \in \widetilde{\mathcal{M}}$ we have

$$\mu_t((A^*A)^q) = \mu_t(|A|^{2q}) = \mu_t(A)^{2q} = \mu_t(|A^*|)^{2q} = \mu_t((AA^*)^q) \text{ for all } t > 0. \tag{12}$$

Let a q-hyponormal operator A be not normal. Then there exists $0 \neq B \in \widetilde{\mathcal{M}}^+$ such that $(A^*A)^q = (AA^*)^q + B$. If $X, Y \in \widetilde{\mathcal{M}}^+$, $Y \neq 0$ and $X \geq \mu_\infty(X)I$ then there exists a number $s > 0$ such that

$$\mu_s(X) < \mu_s(X + Y), \tag{13}$$

see Proposition 2.2 [7]. From the inequality $|A^*| \geq \mu_\infty(A)I$ by monotonocity of the real function $f(\lambda) = \lambda^{2q}(\lambda \geq 0)$ we obtain $(AA^*)^q \geq \mu_\infty((AA^*)^q)I$, see items 1) and 3) of Lemma 2.1. For $X = (AA^*)^q$, $Y = B$ via (12) we have

$$\mu_t(X) = \mu_t((AA^*)^q) = \mu_t((A^*A)^q) = \mu_t(X+Y) \text{ for all } t > 0.$$

We have a contradiction with (13). Thus $Y = B = 0$ and $(A^*A)^q = (AA^*)^q$. Therefore $A^*A = AA^*$ and Theorem is proved. □

Corollary 4.2. *Let an operator $A \in \widetilde{\mathcal{M}}$ be q-cohyponormal and $|A| \geq \mu_\infty(A)I$. Then A is normal.*

Corollary 4.3 ([2]). *Every τ-compact q-hyponormal (or q-cohyponormal) operator is normal.*

Corollary 4.4. *Let an operator $A \in \widetilde{\mathcal{M}}$ be hyponormal and $|\lambda I + A^*| \geq \mu_\infty(\lambda I + A^*)I$ for some $\lambda \in \mathbb{C}$. Then A is normal.*

Proof. *An operator $\bar{\lambda}I + A$ is also hyponormal (the bar sign over a symbol stands for complex conjugation).* □

Example 4.5. If $A = XY$ with $X, Y \in \mathcal{B}(\mathcal{H})^{\text{sa}}$ is hyponormal then A is normal (see Corollary on p. 49 in [16]). There exists a nonnormal hyponormal operator $A = XYZ$ with $X, Y, Z \in \mathcal{B}(\mathcal{H})^{\text{sa}}$, see p. 51 in [16]. Therefore the condition $|A^*| \geq \mu_\infty(A)I$ does not hold for such an operator A by Theorem 4.1.

Theorem 4.6. *Consider a nilpotent operator $Z \in \widetilde{\mathcal{M}}$, $Z \neq 0$ and numbers $a, b \in \mathbb{R}$. Then the operator*

$$T_{Z,a,b} = Z^*Z - ZZ^* + a\Re Z + b\Im Z \tag{14}$$

cannot be nonpositive or nonnegative.

Proof. Let a number $n \in \mathbb{N}$ be such that $Z^{n-1} \neq 0 = Z^n$.

Step 1. Assume that $T_{Z,a,b} \geq 0$ for some pair $a, b \in \mathbb{R}$. We multiply both sides of equality (14) by the operator $(Z^*)^{n-1}$ from the left and by the operator Z^{n-1} from the right, and achieve

$$(Z^*)^{n-1}T_{Z,a,b}Z^{n-1} = -(Z^*)^{n-1}ZZ^*Z^{n-1} = -|Z^*Z^{n-1}|^2.$$

By Lemma 2.2 we have $(Z^*)^{n-1}T_{Z,a,b}Z^{n-1} \geq 0$, and at the same time $-|Z^*Z^{n-1}|^2 \leq 0$. Hence $|Z^*Z^{n-1}| = 0$ and $Z^*Z^{n-1} = 0$. If $n = 2$ then $Z^{n-1} = Z = 0$; if $n > 2$ then $0 = (Z^*)^{n-2} \cdot Z^*Z^{n-1} = |Z^{n-1}|^2$. Consequently $Z^{n-1} = 0$, which is a contradiction.

Step 2. Assume now that $T_{Z,a,b} \leq 0$ for some pair $a, b \in \mathbb{R}$. Then the nilpotent $V = -Z^*$ is subject to the conditions $V^{n-1} \neq 0 = V^n$ and $T_{V,a,-b} = -T_{Z,a,b} \geq 0$. By Step 1 we have $V^{n-1} = 0$, which is a contradiction. This completes the proof. □

For $a = b = 0$ we have

Corollary 4.7 ([6], Theorem 2.4). *A non-zero hyponormal operator $Z \in \widetilde{\mathcal{M}}$ cannot be nilpotent.*

Assume that an operator $Q \in \widetilde{\mathcal{M}}$ and $Q^2 = Q$. Then there exists a unique projection $P \in \mathcal{M}^{\text{pr}}$ such that $QP = P$, $PQ = Q$ and $P\widetilde{\mathcal{M}} = Q\widetilde{\mathcal{M}}$ (see Theorem 2.21 in [3]). There is a unique decomposition $Q = P + Z$, where $Z^2 = 0 = ZP$ and $PZ = Z$ (see Theorem 2.23 in [3]). Therefore $Q \in \widetilde{\mathcal{M}}_0$ if and only if $P \in \widetilde{\mathcal{M}}_0$. By Theorem 4.6 for $a = 2$ by using the above mentioned decomposition we have

Corollary 4.8. *If an operator $Q \in \widetilde{\mathcal{M}}$ and $Q^2 = Q \neq Q^*$ then for any number $b \in \mathbb{R}$ the operator $Q^*Q - QQ^* + b\Im Q$ cannot be nonpositive or nonnegative.*

Corollary 4.9. *If an operator $S \in \widetilde{\mathcal{M}}$ and $S^2 = I$, $S \neq S^*$, then for any number $b \in \mathbb{R}$ the operator $S^*S - SS^* + b\Im S$ cannot be nonpositive or nonnegative.*

Proof. The formula $S = 2Q - I$ defines a one-to-one correspondence between the symmetries S ($S^2 = I$) and the idempotents $Q = Q^2$. □

ACKNOWLEDGMENTS

This work was supported by the subsidies allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (1.1515.2017/4.6, 1.9773.2017/8.9).

REFERENCES

1. A. B. Antonevich, *Linear Functional Equations. The Operator Approach* (Universitetskoe, Minsk, 1988) [in Russian].
2. A. M. Bikchentaev, "On normal τ-measurable operators affiliated with semifinite von Neumann algebras," Math. Notes **96**, 332–341 (2014).
3. A. M. Bikchentaev, "On idempotent τ-measurable operators affiliated to a von Neumann algebra," Math. Notes **100**, 515–525 (2016).
4. A. M. Bikchentaev, "Two classes of τ-measurable operators affiliated with a von Neumann algebra," Russ. Math. **61**, 76–80 (2017).
5. A. M. Bikchentaev, "Minimality of convergence in measure topologies on finite von Neumann algebras," Math. Notes **75** (3–4), 315–321 (2004).
6. A. M. Bikchentaev, "Integrable products of measurable operators," Lobachevskii J. Math. **37**, 397–403 (2016).
7. V. I. Chilin, A. V. Krygin, and F. A. Sukochev, "Extreme points of convex fully symmetric sets of measurable operators," Integral Equat. Operator Theory **15** (2), 186–226 (1992).
8. P. G. Dodds, T. K.-Y. Dodds, and B. de Pagter, "Noncommutative Köthe duality," Trans. Am. Math. Soc. **339**, 717–750 (1993).
9. T. Fack and H. Kosaki, "Generalized s-numbers of τ-measurable operators," Pacif. J. Math. **123**, 269–300 (1986).
10. T. Furuta, "On the class of paranormal operators," Proc. Jpn. Acad. **43**, 594–598 (1967).
11. I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space* (Nauka, Moscow, 1965) [in Russian].
12. P. R. Halmos, *A Hilbert Space Problem Book* (D. van Nostrand, Princeton, New Jersey, Toronto, London, 1967).
13. R. A. Horn and C. R. Johnson, *Matrix Analysis,* 2nd ed. (Cambridge Univ. Press, Cambridge, 2013).
14. V. Istrăţescu, "On some hyponormal operators," Pacif. J. Math. **22**, 413–417 (1967).
15. C. S. Kubrusly, *Hilbert Space Operators. A Problem Solving Approach* (Birkhäuser, Boston, MA, 2003).
16. C. S. Lin and M. Radjabalipour, "On intertwining and factorization by self-adjoint operators," Canad. Math. Bull. **21**, 47–51 (1978).
17. E. Nelson, "Notes on non-commutative integration," J. Funct. Anal. **15**, 103–116 (1974).
18. I. E. Segal, "A non-commutative extension of abstract integration," Ann. Math. **57**, 401–457 (1953).
19. J. G. Stampfli, "Hyponormal operators and spectral density," Trans. Am. Math. Soc. **117**, 469–476 (1965).
20. M. Terp, *L^p-Spaces Associated with von Neumann Algebras* (Copenhagen Univ. Press, Copenhagen, 1981).
21. F. J. Yeadon, "Non-commutative L^p-spaces," Math. Proc. Cambridge Phil. Soc. **77**, 91–102 (1975).