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MATHEMATICS

Commutativity of Operators and Characterization of Traces on C^*-Algebras

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This paper continues the author's study begun in [1, 2]; we retain the notation and terminology used there. In Section 2, we give new criteria for the commutativity of a nonnegative operator and a projection in terms of operator inequalities. We show that, in the general case, it is impossible to replace the projection in these inequalities by any nonnegative operator so as to preserve the commutativity of operators. We also obtain new commutativity criteria for projections in terms of operator inequalities.

In Section 3, we obtain a characterization of traces in the class of all positive normal functionals on the von Neumann algebra by using the operator inequalities from Section 2. It is shown that not each characteristic which distinguishes traces among positive normal functionals carries over to normal weights (see Example 3.1). This answers a question which Victor Kaftal (Cincinnati University, U.S.A.) asked the author at the international conference *Operator Theory'23* (Romania, Timisoara, 2010, June 30). We also give a characterization of traces in the class of all weights on von Neumann algebra. In Section 4, we obtain a characterization of traces in the class of all positive functionals on the C^*-algebra in terms of operator inequalities. Moreover, we prove new criteria for the commutativity of C^*-algebras. An information about other characterizations of traces and commutativity criteria for C^*-algebras can be found in the author's survey [3].

1. NOTATION AND DEFINITIONS

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. An operator $X \in \mathcal{B}(\mathcal{H})$ is called a projection if $X = X^2 = X^*$. The commutator of a set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ is defined as $\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H})\colon XY = YX, X^*Y = YX^*\ (X \in \mathcal{X})\}$.

A $*$-subalgebra $\mathcal{M}$ of the algebra $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ if $\mathcal{M} = \mathcal{M}''$. A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. Given a C^*-algebra $\mathcal{A}$, by $\mathcal{A}^{sa}$, $\mathcal{A}^+$, and $\mathcal{A}^{pr}$ we denote its subsets of Hermitian elements, positive elements, and projections, respectively. Given a unital algebra $\mathcal{A}$, its subset of unitary elements is denoted by $\mathcal{A}^u$ and the identity element, by I; we set $P^\perp = I - P$ for $P \in \mathcal{A}^{pr}$.

A weight on a C^*-algebra $\mathcal{A}$ is, by definition, a mapping $\varphi\colon \mathcal{A}^+ \to [0,+\infty]$ with properties

$$\varphi(X+Y) = \varphi(X) + \varphi(Y),$$
$$\varphi(\lambda X) = \lambda\varphi(X)\ (X, Y \in \mathcal{A}^+, \lambda \ge 0)$$

(it is assumed that $0\cdot(+\infty) \equiv 0$).

A weight φ on $\mathcal{A}$ is called a trace if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$; a faithful weight if $\varphi(X) = 0$ implies $X = 0$ for $X \in \mathcal{A}^+$; a finite weight if $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$.

The restriction $\varphi|\{X \in \mathcal{A}^+\colon \varphi(X) < +\infty\}$ admits a well-defined extension by linearity to a functional on $\mathrm{Lin}\{X \in \mathcal{A}^+\colon \varphi(X) < +\infty\}$. Such an extension makes it possible to identify finite weights with positive functionals on $\mathcal{A}$.

Given a von Neumann algebra $\mathcal{M}$ of operators on $\mathcal{H}$, we denote its center by $\mathcal{Z}(\mathcal{M})$. A weight φ on $\mathcal{M}$ is said to be normal if $X_i \nearrow X$ (where $X_i, X \in \mathcal{M}^+$) implies $\varphi(X) = \sup\varphi(X_i)$. Let $\mathcal{M}_*^+$ be the cone of positive normal functionals on $\mathcal{M}$, and let $\mathcal{M}_P = \{PX|P\mathcal{H}\colon X \in \mathcal{M}\}$ be the reduced von Neumann algebra for $P \in \mathcal{M}^{pr}$. Suppose that $\delta \in \mathbb{C}$, $|\delta| = 1$, and $g(t) = \sqrt{t(1-t)}$ for $0 \le t \le 1$. Consider the projection $R^{(\delta,t)}$ in $\mathbb{M}_2(\mathbb{C})$ defined by

$$R^{(\delta,t)} = \begin{pmatrix} t & \delta g(t) \\ \bar{\delta}g(t) & 1-t \end{pmatrix}. \qquad (1)$$

Let Φ denote the class of all continuous increasing functions $f\colon [0, 1] \to \mathbb{R}$ satisfying the condition

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$$f(t) > f(0)(1-t) + f(1)t \qquad (2)$$

for all $0 < t < 1$. The class Φ is closed with respect to taking positive linear combinations and adding constants, and it contains all increasing strictly concave functions on [0, 1]. If $f, h \in \Phi$ and f is concave, then the superposition $f \circ h$ belongs to Φ.

2. COMMUTATIVITY OF PROJECTIONS

Proposition 1. *For $A \in \mathcal{B}(\mathcal{H})^+$ and $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, the following conditions are equivalent:*

(i) $AP = PA$;

(ii) $\mathfrak{I}(AP) \le PAP$;

(iii) $\mathfrak{I}(PA) \le PAP$;

(iv) $\mathfrak{I}(AP) \le \mathfrak{R}(AP)$;

(v) $\mathfrak{I}(PA) \le \mathfrak{R}(AP)$;

(vi) $\mathfrak{R}(AP) \le PAP$;

(vii) $PAP \le A^{1/2}PA^{1/2}$.

Take arbitrary vectors $\xi \in P\mathcal{H}$ and $\eta \in P^{\perp}\mathcal{H}$ and any numbers $s, t \in \mathbb{C}$. Let us show that (ii) $\Rightarrow$ (i). We have

$$0 \le \langle (PAP - \mathfrak{I}(AP))(s\xi + t\eta), s\xi + t\eta \rangle$$

$$= |s|^2 \langle A\xi, \xi\rangle + \mathfrak{R}(is\bar{t}\langle A\xi, \eta\rangle),$$

therefore, $\langle A\xi, \eta\rangle = 0$ and $AP = PA$.

The following assertion strengthens (in special cases) two results of [4].

Proposition 2. *Let $\mathcal{A}$ be a C^*-algebra with faithful finite trace φ, and let $A, B \in \mathcal{A}^+$. If* (i) $B^{1/2}AB^{1/2} \le A^{1/2}BA^{1/2}$ *or* (ii) $\mathfrak{R}(AB) \le A^{1/2}BA^{1/2}$, *then $AB = BA$.*

Proposition 3. *There exist noncommuting matrices $A, B \in \mathbb{M}_2(\mathbb{C})^+$ such that* (i) $\mathfrak{I}(AB) \le A^{1/2}BA^{1/2}$; (ii) $\mathfrak{I}(AB) \le B^{1/2}AB^{1/2}$; (iii) $\mathfrak{I}(AB) \le \mathfrak{R}(AB)$.

Theorem 1. *For $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, the following conditions are equivalent:*

(i) $PQ = QP$;

(ii) $f(PQP) \le f(Q)$ *for some function* $f \in \Phi$;

(iii) $\mathfrak{R}(PQ) \le P$;

(iv) $\mathfrak{I}(PQ) \le P$;

(v) $\mathfrak{I}(PQ) \le Q$.

The implication (i) $\Rightarrow$ (ii) is easy to verify by realizing the commutative von Neumann subalgebra $\{P, Q\}''$ in the form of the multiplier algebra $L_\infty(\Omega, \mathfrak{A}, \mu)$ acting on the Hilbert space $L_2(\Omega, \mathfrak{A}, \mu)$, where $(\Omega, \mathfrak{A}, \mu)$ is a localized measure space. There exist $A, B \in \mathfrak{A}$ such that P and Q are the operators of multiplication by the indicators χ_A and χ_B, respectively. Then the operator PQP corresponds to the multiplier by the indicator $\chi_{A \cap B}$, and the inequality in (ii) does hold.

To verify the reverse implications, we associate P and Q with the von Neumann algebra $\{P, Q\}''$ generated by them. There exists a unique projection $Z \in \mathcal{Z}(\mathcal{N})$ such that the algebra $\mathcal{N}_Z$ has type I_2 and the algebra $\mathcal{N}_{Z^{\perp}}$ is Abelian [5, Chapter 5, Theorem 1.41, (ii)]. Clearly, the projections $PZ^{\perp}$ and $QZ^{\perp}$ commute. Gel'fand's theorem on the representation of an Abelian unital C^*-algebra (see, e.g., [5, Chapter 3, Theorem 1.18]) implies that the algebra $\mathcal{Z}(\mathcal{N}_Z)$ is *-isomorphic to the C^*-algebra $C(\Omega)$ of all complex-valued continuous functions on the Stone space Ω of all characters of $\mathcal{Z}(\mathcal{N}_Z)$. Now, according to [6, Theorem 2.3.3], the algebra $\mathcal{N}_Z$ is *-isomorphic to the matrix algebra $\mathbb{M}_2(C(\Omega))$.

For $\tilde{R} \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$, we define the regions of rank constancy (or, equivalently, the regions of constancy of the canonical trace tr) as

$$\Omega_j(\tilde{R}) = \{\omega \in \Omega \mid \tilde{r}_{11}(\omega) + \tilde{r}_{22}(\omega) = j\},$$
$$j \in \{0, 1, 2\}.$$

The projections PZ and QZ are identified with $\tilde{P}$, $\tilde{Q} \in \mathbb{M}_2(C(\Omega))^{\mathrm{pr}}$, respectively. Let

$$\Omega_{ij} = \Omega_i(\tilde{P}) \cap \Omega_j(\tilde{Q}), \quad i, j \in \{0, 1, 2\}.$$

All of the nine sets Ω_{ij} are clopen and form a disjoint cover of the space Ω. If $\omega \in \Omega \backslash \Omega_{11}$, then $\tilde{P}\tilde{Q}(\omega) = \tilde{Q}\tilde{P}(\omega)$.

By virtue of [7, Corollary 3.3], there exists a $U \in \mathbb{M}_2(C(\Omega))^{\mathrm{u}}$ and a closed set $\Omega_1'(\tilde{P}) \subset \Omega_1(\tilde{P})$ such that $U^*(\omega)\tilde{P}(\omega)U(\omega) = \mathrm{diag}(1, 0)$ for all $\omega \in \Omega_1'(\tilde{P})$ and $U^*(\omega)\tilde{P}(\omega)U(\omega) = \mathrm{diag}(0, 1)$ for all $\omega \in \Omega_1(\tilde{P}) \backslash \Omega_1'(\tilde{P})$. Therefore, it is sufficient to consider the case

$$P = \mathrm{diag}(1, 0), \quad Q = R^{(\delta, t)},$$

where $\delta \in \mathbb{C}$, $|\delta| = 1$, and $0 \le t \le 1$ (see (1)). If $t = 0$, then $PQ = QP = 0$. Let $0 < t \le 1$. Then $PQP = tP$.

Let us verify the implication (ii) $\Rightarrow$ (i). We have

$$f(tP) = f(tP + 0P^{\perp}) = f(t)P + f(0)P^{\perp}.$$

Similarly, $f(Q) = f(1)Q + f(0)Q^{\perp}$. By condition,

$$f(t)P + f(0)P^{\perp} \le f(1)Q + f(0)Q^{\perp},$$

which is equivalent to the inequality

$$\begin{pmatrix} f(t) & 0 \\ 0 & f(0) \end{pmatrix} \le \begin{pmatrix} (1-t)f(0) + tf(1) & (f(1) - f(0))\delta g(t) \\ (f(1) - f(0))\bar{\delta} g(t) & tf(0) + (1-t)f(1) \end{pmatrix}.$$

It follows, in particular, that

$$f(t) \le (1-t)f(0) + tf(1),$$

this is possible only at $t = 1$, i.e., for $P = Q$.

Corollary 1 [1, Proposition 1]**.** *For $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, the following conditions are equivalent:*

(i) $PQ = QP$;

(ii) $PQP \le Q$.

The implication (i) $\Rightarrow$ (ii) is obvious; (ii) $\Rightarrow$ (i) follows from the equivalence (ii) $\Rightarrow$ (i) in Theorem 1, because the function $t \mapsto \log(1 + t)$ is operator monotone on the interval $[0, \infty)$ and belongs to the class Φ.

3. TRACES ON THE VON NEUMANN ALGEBRA

The following assertion strengthens Lemma 1 in [8].

Proposition 4. *A weight* φ *on the von Neumann algebra* $\mathcal{M}$ *is a trace if and only if* $\varphi(PAP) \le \varphi(A^{1/2}PA^{1/2})$ *for any* $A \in \mathcal{M}^+$ *and* $P \in \mathcal{M}^{\mathrm{pr}}$.

Proposition 5. *If* φ *is a trace on the von Neumann algebra* $\mathcal{M}$, $P, Q \in \mathcal{M}^{\mathrm{pr}}$, *and* $f \in \Phi$ *with* $f(0) \ge 0$, *then*

$$\varphi(f(PQP)) = \varphi(f(QPQ)) \le \varphi(f(Q)).$$

For a finite trace φ, *the condition* $f(0) \ge 0$ *can be omitted.*

Proof. Suppose that $f \in \Phi$ with $f(0) \ge 0$, and $P, Q \in \mathcal{M}^{\mathrm{pr}}$. There exists a symmetry $S \in \mathcal{M}$ such that $QPQ = SPQPS$ [9, Lemma 3.46]. We have $h(QPQ) = Sh(PQP)S$ for any polynomial $h(t)$. Since the function $f \in \Phi$ is continuous, it follows from Weierstrass' theorem on the uniform approximation of a continuous function on an interval by polynomials and the continuity of functional calculus that $f(QPQ) = Sf(PQP)S$. Therefore, $\varphi(f(QPQ)) = \varphi(Sf(PQP)S)$. Considering the commutative von Neumann subalgebra $\{Q, QPQ\}''$, we obtain the inequality $\varphi(f(QPQ)) \le \varphi(f(Q))$.

Theorem 2. *For a functional* $\varphi \in \mathcal{M}_*^+$, *the following conditions are equivalent:*

(i) φ *is a trace*;

(ii) $\varphi(PQ) \in \mathbb{R}$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(iii) $\varphi(\Re(PQ)) \le \varphi(P)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(iv) $\varphi(\Im(PQ)) \le \varphi(\Re(PQ))$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(v) $\varphi(\Im(PQ)) \le \varphi(P)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(vi) $\varphi(\Im(PQ)) \le \varphi(Q)$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$;

(vii) *there exists a function* $f \in \Phi$ *such that* $\varphi(f(PQP)) \le \varphi(f(Q))$ *for all* $P, Q \in \mathcal{M}^{\mathrm{pr}}$.

Proof. The implications (i) $\Rightarrow$ (iv)–(vi) follow easily from the relation $\varphi(PQ) = \varphi(QP)$. The implication (i) $\Rightarrow$ (vii) is proved in Proposition 5.

As in some other similar cases (see [1] or [2]), the proof of the reverse implications for any von Neumann algebra reduces to the case of the algebra $\mathbb{M}_2(\mathbb{C})$. Each linear functional φ on $\mathbb{M}_2(\mathbb{C})$ can be represented in the form $\varphi(\cdot) = \mathrm{tr}(S_\varphi\, \cdot)$. The matrix $S_\varphi \in \mathbb{M}_2(\mathbb{C})$ is called the density matrix for φ. Without loss of generality, we assume that

$$S_\varphi = \mathrm{diag}\left(\frac{1}{2} + s, \frac{1}{2} - s\right), \quad 0 \le s \le \frac{1}{2}.$$

Thus, $\varphi(X)$ is equal to the quantity $\left(\frac{1}{2} + s\right)x_{11} + \left(\frac{1}{2} - s\right)x_{22}$ for $X = [x_{kj}]_{k,j=1}^2$ from $\mathbb{M}_2(\mathbb{C})$. Let us prove the implication (ii) $\Rightarrow$ (i). If $P = R^{(1, 1/2)}$ and $Q = R^{(i, 1/2)}$, then $\varphi(PQ) = \frac{1}{4} - \frac{is}{2} \in \mathbb{R}$ only at $s = 0$.

Let us prove the implication (iii) $\Rightarrow$ (i). If $P = R^{(1, 1/2 - \varepsilon)}$ and $Q = R^{(1, 1/2 + \varepsilon)}$ for $0 < \varepsilon \le \frac{1}{2}$, then $\varphi(PQ + QP) = 1 - 4\varepsilon^2$ and $2\varphi(P) = 1 - 4\varepsilon s$. Therefore, inequality (iii) holds for all $0 < \varepsilon \le \frac{1}{2}$ only at $s = 0$.

Let us prove the implication (iv) $\Rightarrow$ (i). We set $P = R^{(u, t)}$ and $Q = R^{(v, d)}$ with $u, v \in C$, $|u| = |v| = 1$, and $t, d \in [0, 1]$. Let $u\overline{v} = x + iy$, where $x^2 + y^2 = 1$, and let $0 < \varepsilon \le \frac{1}{2}$. Then

$$2\varphi(\Im(PQ)) = 4yg(t)g(d)s,$$
$$2\varphi(\Re(PQ))$$
$$= 2xg(t)g(d) + 1 - t - d + 2td - 2s + 2st + 2sd.$$

Assuming that $y \in [0, 1]$ and $x = \sqrt{1 - y^2}$, we have the following inequality for $F(y) = 4ys + \sqrt{1 - y^2}$:

$$F(y)g(t)g(d) \le 1 - t - d + 2td - 2s + 2st + 2sd.$$

It is easy to see that

$$\max_{y \in [0, 1]} F(y) = F\left(\frac{2s}{\sqrt{1 + 4s^2}}\right) = 2\sqrt{1 + 4s^2},$$

therefore, at $t = d = \frac{1}{2} - s + \varepsilon$, the inequality in (iv) takes the form

$$\sqrt{1 + 4s^2}\left(\frac{1}{2} - 2s^2 - 2\varepsilon^2 + 2s\varepsilon\right) \le \frac{1}{2} - 6s^2 + 2\varepsilon^2.$$

This inequality holds for all $0 < \varepsilon \le \frac{1}{2}$ only at $s = 0$.

To prove the implication (vii) $\Rightarrow$ (i), we set $P = R^{(1, 1/2 + \varepsilon)}$, $Q = R^{(1, 1/2 - \varepsilon)}$, where $0 < \varepsilon \le \frac{1}{2}$. Then $PQP = hP$ for $h = 1 - 4\varepsilon^2$. By assumption,

$$\varphi(f(PQP)) = f(h)\varphi(P) + f(0)\varphi(P^\perp) \le \varphi(f(Q))$$
$$= f(1)\varphi(Q) + f(0)\varphi(Q^\perp).$$

Hence,

$$f(0)(\varphi(Q^\perp) - \varphi(P^\perp)) \ge f(h)\varphi(P) - f(1)\varphi(Q).$$

By virtue of (2), we have $f(h) > f(0)(1 - h) + f(1)h$. It follows that

$$f(0)(\varphi(P) - \varphi(Q))$$
$$\ge (f(0) \cdot 4\varepsilon^2 + f(1)(1 - 4\varepsilon^2))\varphi(P) - f(1)\varphi(Q),$$

which is equivalent to the inequality

$$f(0)\left(s - \frac{\varepsilon}{2} - 2s\varepsilon^2\right) \ge f(1)\left(s - \frac{\varepsilon}{2} - 2s\varepsilon^2\right).$$

Since f increases, this inequality can be valid for all $0 < \varepsilon \le \frac{1}{2}$ only at $s = 0$.

Example 1. Following the scheme of proof of Proposition 5, we can show that, for each trace φ on the von Neumann algebra $\mathcal{M}$, we have

$$\varphi((I+Q)^{-1}) \le \varphi((I+PQP)^{-1}) \text{ for all } P, Q \in \mathcal{M}^{\mathrm{pr}}. \tag{3}$$

The function $f(t) = -(1+ t^q)^{-1}$ (where $0 < q \le 1$) belongs to the class Φ, and (3) characterizes traces in the class $\mathcal{M}_*^+$; see the equivalence (vii) $\Rightarrow$ (i) in Theorem 2. For each invertible $A \in \mathcal{M}^+$, there exists a number $\varepsilon > 0$ such that $A \ge \varepsilon I$; therefore, inequality (3) transforms into the equality $+\infty = +\infty$ for all weights φ on $\mathcal{M}$ with $\varphi(I) = +\infty$. Thus, not every characteristic which distinguishes traces among positive normal functionals carries over to normal weights.

4. TRACES ON C^*-ALGEBRAS

Theorem 3. *For a positive functional* φ *on a* C^*-*algebra* $\mathcal{A}$, *the following conditions are equivalent:*

(i) φ *is a trace*;

(ii) $\varphi(AB) \in \mathbb{R}$ *for all* $A, B \in \mathcal{A}^+$;

(iii) $\varphi(\Re(AB)) \le \varphi(A^{1/2}BA^{1/2})$ *for all* $A, B \in \mathcal{A}^+$;

(iv) $\varphi(\Im(AB)) \le \varphi(\Re(AB))$ *for all* $A, B \in \mathcal{A}^+$;

(v) $\varphi(\Im(AB)) \le \varphi(A^{1/2}BA^{1/2})$ *for all* $A, B \in \mathcal{A}^+$;

(vi) $\varphi(\Im(AB)) \le \varphi(B^{1/2}AB^{1/2})$ *for all* $A, B \in \mathcal{A}^+$;

(vii) $\varphi(A^{1/2}BA^{1/2}) = \varphi(B^{1/2}AB^{1/2})$ *for all* $A, B \in \mathcal{A}^+$.

Corollary 2. *For a* C^*-*algebra* $\mathcal{A}$, *the following conditions are equivalent:*

(i) $\mathcal{A}$ *is commutative*;

(ii) $\Re(AB) \le A^{1/2}BA^{1/2}$ *for all* $A, B \in \mathcal{A}^+$;

(iii) $\Im(AB) \le \Re(AB)$ *for all* $A, B \in \mathcal{A}^+$;

(iv) $\Im(AB) \le A^{1/2}BA^{1/2}$ *for all* $A, B \in \mathcal{A}^+$;

(v) $\Im(AB) \le B^{1/2}AB^{1/2}$ *for all* $A, B \in \mathcal{A}^+$.

REFERENCES

1. A. M. Bikchentaev, Sib. Math. J. **51** (6), 971–977 (2010).
2. A. M. Bikchentaev, Math. Notes **89** (4), 461–471 (2011).
3. A. M. Bikchentaev, in *Proceedings of International Conference Operator Theory'23, Timisoara, Romania, 2010* (Theta, Bucharest, 2012), pp. 1–12.
4. S. Gudder and G. Nagy, J. Math. Phys. **42** (11), 5212–5222 (2001).
5. M. Takesaki, *Theory of Operator Algebras* (Springer, New York, 1979), Vol. 1.
6. S. Sakai, *C*-Algebras and W*-Algebras* (Springer, New York, 1971).
7. D. Deckard and C. Pearcy, Proc. Amer. Math. Soc. **14** (2), 322–328 (1963).
8. A. M. Bikchentaev, Math. Notes **64** (2), 159–163 (1998).
9. E. M. Alfsen and F. W. Shultz, *Geometry of State Spaces of Operator Algebras* (Birkhäuser, Boston, 2003).