On convex closed sets of measurable operators

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Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let t_τ be the measure topology on the *-algebra $\widetilde{\mathcal{M}}$ of all τ-measurable operators.

We prove that for $B \in \widetilde{\mathcal{M}}^+$ the sets $I_B = \{A = A^* \in \widetilde{\mathcal{M}} : \ -B \leq A \leq B\}$ and $K_B = \{A \in \widetilde{\mathcal{M}} : A^*A \leq B\}$ are convex and t_τ-closed in $\widetilde{\mathcal{M}}$. The set $M_B = \{A \in \widetilde{\mathcal{M}} : \ |A| \leq B\}$ is convex for every operator $B \in \widetilde{\mathcal{M}}^+$ if and only if $\mathcal{M}$ is abelian.

Let an algebra $\mathcal{M}$ contain a sequence $(P_n)_{n=1}^\infty$ of pairwise orthogonal nonzero equivalent projections. If $B \in \widetilde{\mathcal{M}}^+$ and $bP_1 \leq B$ for some number $0 < b < 1$ then the sets K_B and M_B cannot be t_τ-compact. In particular, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = \infty$ and $B \in \mathcal{M}^+ \setminus \{0\}$ then the sets K_B and M_B cannot be $\|\cdot\|$-compact. An operator $B \in \mathcal{B}(\mathcal{H})^+$ is compact if and only if the set I_B is $\|\cdot\|$-compact.

Let $\mathfrak{S}_p(\mathcal{H})$ be a Shatten–von Neumann ideal, $0 < p < +\infty$. If an operator $B \in \mathfrak{S}_p(\mathcal{H})^+$ then I_B is a $\|\cdot\|_p$-compact subset of $\mathfrak{S}_p(\mathcal{H})^{\mathrm{sa}}$.

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