

IDEAL SPACES OF MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA

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Abstract: Suppose that \mathcal{M} is a von Neumann algebra of operators on a Hilbert space \mathcal{H} and τ is a faithful normal semifinite trace on \mathcal{M} . Let \mathcal{E} , \mathcal{F} and \mathcal{G} be ideal spaces on (\mathcal{M}, τ) . We find when a τ -measurable operator X belongs to \mathcal{E} in terms of the idempotent P of \mathcal{M} . The sets $\mathcal{E} + \mathcal{F}$ and $\mathcal{E} \cdot \mathcal{F}$ are also ideal spaces on (\mathcal{M}, τ) ; moreover, $\mathcal{E} \cdot \mathcal{F} = \mathcal{F} \cdot \mathcal{E}$ and $(\mathcal{E} + \mathcal{F}) \cdot \mathcal{G} = \mathcal{E} \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G}$. The structure of ideal spaces is modular. We establish some new properties of the $L_1(\mathcal{M}, \tau)$ space of integrable operators affiliated to the algebra \mathcal{M} . The results are new even for the $*$ -algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators on \mathcal{H} which is endowed with the canonical trace $\tau = \text{tr}$.

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Introduction

The section of functional analysis, called noncommutative integration theory, is an important part of the theory of operator algebras. This article is devoted to noncommutative analogs of the classical methods for constructing function spaces. The beginning of the development of the corresponding aspect of noncommutative integration theory is related to the names of Segal and Dixmier, who in the early 1950s created a theory of integration with respect to a trace on a semifinite von Neumann algebra [1]. The results of these investigations found spectacular applications in the duality theory for unimodular groups and stimulated the progress of “noncommutative probability theory.” The theory of algebras of measurable and locally measurable operators is rapidly developing and has interesting applications in various areas of functional analysis, mathematical physics, statistical mechanics, and quantum field theory.

In [2–4], Muratov introduced and investigated ideal spaces of measurable operators on a finite von Neumann algebra. They were also studied by Chilin in [5]. In the above-mentioned works, the ideal spaces serve primarily as the object of investigation. Recently, there have appeared publications in which they act as a tool. The foregoing demonstrates the relevance of (1), the search for new methods for constructing ideal spaces of measurable operators; (2), the development of a general theory of these spaces; and (3), the consideration of new particular examples.

Suppose that a von Neumann algebra \mathcal{M} of operators acts on a Hilbert space \mathcal{H} , while τ is a faithful semifinite trace on \mathcal{M} . Let \mathcal{E} , \mathcal{F} , and \mathcal{G} be ideal spaces on (\mathcal{M}, τ) . Let us list the obtained results. Given a normal τ -measurable operator X and an idempotent $P \in \mathcal{M}$, we show that $X \in \mathcal{E} \Leftrightarrow XP + P^\perp X \in \mathcal{E} \Leftrightarrow PXP + P^\perp X \in \mathcal{E} \Leftrightarrow XP + P^\perp XP^\perp \in \mathcal{E}$ (Theorem 1). The condition of normality for X is substantial in Theorem 1 (Example 3). The sets $\mathcal{E} + \mathcal{F}$ and $\mathcal{E} \cdot \mathcal{F}$ are also ideal spaces on (\mathcal{M}, τ) ; moreover, $\mathcal{E} \cdot \mathcal{F} = \mathcal{F} \cdot \mathcal{E}$ and $(\mathcal{E} + \mathcal{F}) \cdot \mathcal{G} = \mathcal{E} \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G}$. The structure of ideal spaces is modular: if $\mathcal{E} \subset \mathcal{G}$ then $(\mathcal{E} + \mathcal{F}) \cap \mathcal{G} = \mathcal{E} + (\mathcal{F} \cap \mathcal{G})$ (Theorems 2 and 3).

Let τ -measurable operators X , Y , and an idempotent $P \in \mathcal{M}$ be such that $XP - PY \in L_1(\mathcal{M}, \tau)$. Then $\tau(XP - PY) = \tau(PXP - PYP)$ and for $X = Y$ we have $\tau([X, P]) = 0$ (Theorem 4). Let $\text{Re } Y$

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and $\Im Y$ be the real and imaginary components of the τ -measurable operator Y and assume that $p \geq 1$ and $X \in L_{2p}(\mathcal{M}, \tau)$. Then (1) $\operatorname{Re} X^2, \Im X^2 \in L_p(\mathcal{M}, \tau)$ and $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \leq \|X^* X\|_p$; if $\tau(I) = 1$ then $\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \leq \sqrt{\|X^* X\|_p}$ (Corollary 6); (2) $\|\operatorname{Re} X^2\|_p \leq 2^{\max\{0, 2-2p\}} \|X\|_{2p}$ (Corollary 7). The results are new even for the $*$ -algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators in \mathcal{H} endowed with the canonical trace $\tau = \operatorname{tr}$.

1. Notations and Definitions

Suppose that \mathcal{M} is the von Neumann algebra of operators on a Hilbert space \mathcal{H} , while $\mathcal{M}^{\operatorname{id}}$ and $\mathcal{M}^{\operatorname{pr}}$ are the subset of idempotents ($P = P^2$) and the lattice of projections ($P = P^2 = P^*$) in \mathcal{M} respectively, I is the identity of \mathcal{M} , $P^\perp = I - P$ for $P \in \mathcal{M}^{\operatorname{id}}$, and \mathcal{M}^+ is the cone of positive elements in \mathcal{M} . The formula $S_P = 2P - I$ establishes a bijection between $\mathcal{M}^{\operatorname{id}}$ and the set of symmetries ($S^2 = I$).

A mapping $\varphi : \mathcal{M}^+ \rightarrow [0, +\infty]$ is called a *trace* if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^* Z) = \varphi(Z Z^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$; *normal* if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow \varphi(X) = \sup \varphi(X_i)$ (see [6, Chapter V, § 2]). For a trace φ , put $\mathfrak{M}_\varphi^+ = \{X \in \mathcal{M}^+ : \varphi(X) < +\infty\}$, and $\mathfrak{M}_\varphi = \operatorname{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+$.

An operator \mathcal{H} (not necessarily bounded or densely defined) is called *affiliated to a von Neumann algebra* \mathcal{M} if \mathcal{H} commutes with every unitary operator in the commutant \mathcal{M}' of \mathcal{M} . From now on, τ is a faithful normal semifinite trace on \mathcal{M} . A closed operator X affiliated to \mathcal{M} and having everywhere dense domain of definition $\mathcal{D}(X)$ in \mathcal{H} is called τ -*measurable* if for every $\varepsilon > 0$ there exists $P \in \mathcal{M}^{\operatorname{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ -measurable operators is a $*$ -algebra with respect to passing to the adjoint operator, multiplication by a scalar, and the operations of strong addition and multiplication that are obtained by closing the usual operations [1, 7]. Given a family $\mathcal{L} \subset \widetilde{\mathcal{M}}$, denote by \mathcal{L}^+ and $\mathcal{L}^{\operatorname{sa}}$ its positive and Hermitian parts respectively. The partial order in $\widetilde{\mathcal{M}}^{\operatorname{sa}}$, generated by the proper cone $\widetilde{\mathcal{M}}^+$, will be denoted by \leq . If $X \in \widetilde{\mathcal{M}}$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathcal{M}$ and $|X| \in \widetilde{\mathcal{M}}^+$.

Denote by $\mu_t(X)$ the *rearrangement* of $X \in \widetilde{\mathcal{M}}$, i.e., the nonincreasing right continuous function $\mu(X) : (0, \infty) \rightarrow [0, \infty)$ defined by the formula

$$\mu_t(X) = \inf\{\|XP\|_\infty : P \in \mathcal{M}^{\operatorname{pr}}, \tau(P^\perp) \leq t\}, \quad t > 0,$$

where $\|\cdot\|_\infty$ is the uniform operator norm on the $*$ -algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators in \mathcal{H} . The set of τ -compact operators $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim_{t \rightarrow \infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathcal{M}}$ (see [8]).

Let m be the linear Lebesgue measure on \mathbb{R} . The noncommutative Lebesgue L_p -space ($0 < p < \infty$) associated with (\mathcal{M}, τ) can be defined as $L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \mu_t(X) \in L_p(\mathbb{R}^+, m)\}$ with the F -norm (the norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu_t(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. The restriction $\tau|_{\mathfrak{M}_\tau^+}$ extends to a bounded linear functional on $L_1(\mathcal{M}, \tau)$, which we will denote by the same symbol τ . We have $\mathfrak{M}_\tau = \mathcal{M} \cap L_1(\mathcal{M}, \tau)$, $L_p(\mathcal{M}, \tau) \subset \widetilde{\mathcal{M}}_0$, and $\|X\|_p = \tau(|X|^p)^{1/p}$ for all $0 < p < \infty$.

A subspace \mathcal{E} in $\widetilde{\mathcal{M}}$ is called an *ideal space on* (\mathcal{M}, τ) (see [9, 10]) if (1) $X \in \mathcal{E}$ implies that $X^* \in \mathcal{E}$; (2) $X \in \mathcal{E}$, $Y \in \widetilde{\mathcal{M}}$, and $|Y| \leq |X|$ imply that $Y \in \mathcal{E}$. Such are, for example, the algebra \mathcal{M} , the set of elementary operators $\mathcal{F}(\mathcal{M})$, $\widetilde{\mathcal{M}}_0$, $(L_1 + L_\infty)(\mathcal{M}, \tau)$, and $L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$. The real and imaginary components $\operatorname{Re} X = (X + X^*)/2$ and $\Im X = (X - X^*)/(2i)$ of $X \in \mathcal{E}$ lie in \mathcal{E} as well.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \operatorname{tr}$ is the canonical trace then $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_0$ coincide with $\mathcal{B}(\mathcal{H})$ and the ideal of compact operators in \mathcal{H} respectively. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1, n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of the s -numbers of a compact operator X [11, Chapter 2, § 2]; χ_A is the indicator of a set $A \subset \mathbb{R}$. Then the $L_p(\mathcal{M}, \tau)$ space is the Schatten-von Neumann ideal \mathfrak{S}_p , $0 < p < \infty$.

Let (Ω, ν) be a measure space and let \mathcal{M} be the von Neumann algebra of multiplication by functions in $L_\infty(\Omega, \nu)$ in $L_2(\Omega, \nu)$. The algebra \mathcal{M} contains no nonzero compact operators if and only if the measure ν has no atoms [12, Theorem 8.4].

2. Lemmas and Examples

Lemma 1 [13, p. 720]. *If $X, Y \in \widetilde{\mathcal{M}}^{\text{sa}}$ and $Z \in \widetilde{\mathcal{M}}$ then from $X \leq Y$ it follows that $ZXZ^* \leq ZYZ^*$.*

Lemma 2 [14, p. 261]. *If $X, Y \in \widetilde{\mathcal{M}}^+$ and $X \leq Y$ then there exists $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$ such that $\sqrt{X} = Z\sqrt{Y}$ and $X = ZYZ^*$.*

Lemma 3 [15, Theorem 17]. *If $X, Y \in \widetilde{\mathcal{M}}$ and $XY, YX \in L_1(\mathcal{M}, \tau)$ then $\tau(XY) = \tau(YX)$.*

Lemma 4. *If $X \in L_1(\mathcal{M}, \tau)$ then $\tau(X) = \tau(S_P X S_P)$ for all $P \in \mathcal{M}^{\text{id}}$.*

PROOF. If $X \in L_1(\mathcal{M}, \tau)$ then $AXB \in L_1(\mathcal{M}, \tau)$ for all $A, B \in \mathcal{M}$. Since $X \in L_1(\mathcal{M}, \tau)$, we have $X S_P \in L_1(\mathcal{M}, \tau)$. Since the operators $S_P \cdot X S_P$ and $X S_P \cdot S_P = X$ lie in $L_1(\mathcal{M}, \tau)$, by Lemma 3, $\tau(X) = \tau(S_P X S_P)$. The lemma is proved. \square

EXAMPLE 1. For $P \in \mathcal{M}^{\text{id}}$ and $X \in \mathcal{M}^{\text{pr}}$, the equality $\mu_t(X) = \mu_t(S_P X S_P)$ in general fails. In $\mathbb{M}_2(\mathbb{C})^{\text{id}}$, choose

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $s_1(X) = 1 < \sqrt{5} = s_1(S_P X S_P)$.

Lemma 5. *If \mathcal{E} is an ideal space on (\mathcal{M}, τ) , $X \in \mathcal{E}$, and $Y, Z \in \mathcal{M}$ then $YXZ \in \mathcal{E}$. Therefore, $\mathcal{E} \cap \mathcal{M}$ is an ideal in \mathcal{M} . If $A \in \widetilde{\mathcal{M}}$ and $A^*A \in \mathcal{E}$ then $AA^* \in \mathcal{E}$.*

PROOF. If $Y \in \mathcal{M}$ and $X \in \mathcal{E}$ then $Y^*Y \leq \|Y\|_\infty^2 \cdot I$ and $|YX| = \sqrt{X^*Y^*YX} \leq \|Y\|_\infty \cdot |X|$ by Lemma 2 and the operator monotonicity of $t \mapsto \sqrt{t}$ ($t \geq 0$). Consequently, $YX \in \mathcal{E}$. Using this, the equality $(XZ)^* = Z^*X^*$, and the definition of ideal space, we see that $XZ \in \mathcal{E}$ for all $Z \in \mathcal{M}$ and $X \in \mathcal{E}$. If $A = U|A|$ is the polar decomposition of A then $U \in \mathcal{M}$ and $AA^* = UA^*AU^*$. The lemma is proved. \square

Lemma 6. *Let \mathcal{E} be an ideal space on (\mathcal{M}, τ) . The following are equivalent for $X \in \widetilde{\mathcal{M}}$ and $P \in \mathcal{M}^{\text{id}}$:*

- (i) $XP + P^\perp X \in \mathcal{E}$;
- (ii) $PXP + P^\perp X \in \mathcal{E}$;
- (iii) $XP + P^\perp X P^\perp \in \mathcal{E}$.

PROOF. (i) \Rightarrow (ii): We have

$$XP = \frac{1}{2}((XP + P^\perp X)P + P(XP + P^\perp X)) \in \mathcal{E}, \tag{1}$$

and so $PXP, P^\perp X \in \mathcal{E}$ by Lemma 5 and $PXP + P^\perp X \in \mathcal{E}$.

(ii) \Rightarrow (i): The operators $PXP = P(PXP + P^\perp X)$ and $P^\perp X = P^\perp(PXP + P^\perp X)$ lie in \mathcal{E} by Lemma 5; therefore, $XP = P^\perp XP + PXP \in \mathcal{E}$. Thus, $XP + P^\perp X \in \mathcal{E}$.

(iii) \Rightarrow (i): The operators $XP = (XP + P^\perp X P^\perp)P$ and $P^\perp X P^\perp = (XP + P^\perp X P^\perp)P^\perp$ lie in \mathcal{E} by Lemma 5; therefore, $PXP \in \mathcal{E}$ and

$$P^\perp X = X - PX = P^\perp X P^\perp + XP - PXP \in \mathcal{E}.$$

Thus, $XP + P^\perp X \in \mathcal{E}$.

(i) \Rightarrow (iii): We have $P^\perp X P^\perp = (XP + P^\perp X)P^\perp \in \mathcal{E}$ by Lemma 5. Now, (1) gives $XP \in \mathcal{E}$. The lemma is proved. \square

Under the equivalent conditions of Lemma 6, we have $PXP + P^\perp X P^\perp \in \mathcal{E}$.

Lemma 7. Let \mathcal{E} be an ideal space on (\mathcal{M}, τ) . The following are equivalent for $X \in \widetilde{\mathcal{M}}^+$ and $P \in \mathcal{M}^{\text{pr}}$:

- (i) $X \in \mathcal{E}$;
- (ii) $PXP + P^\perp XP^\perp \in \mathcal{E}$.

PROOF. (ii) \Rightarrow (i): For the selfadjoint symmetry $S_P = 2P - I$, by Lemma 1, we have $S_PXS_P \geq 0$ and

$$0 \leq \frac{1}{2}X \leq \frac{1}{2}(X + S_PXS_P) = PXP + P^\perp XP^\perp. \quad (2)$$

The implication (i) \Rightarrow (ii) follows, for example, from Lemma 5. The lemma is proved. \square

In [16, Theorem 4.8], it was proved that if $\tau(I) = 1$ then the following are equivalent for $X \in L_1(\mathcal{M}, \tau)$: (i) $\tau(X) = 0$; and (ii) $\|I + zX\|_1 \geq 1$ for all $z \in \mathbb{C}$. In particular, if $\tau(I) = 1$ and $A, B \in \mathcal{M}$ then $\|I + z[A, B]\|_1 \geq 1$ for all $z \in \mathbb{C}$. If $\tau(I) = 1$ and $X \in L_1(\mathcal{M}, \tau)$ then $\|I + z(X - S_PXS_P)\|_1 \geq 1$ for all $P \in \mathcal{M}^{\text{id}}$ and $z \in \mathbb{C}$ (see Lemma 4).

Lemma 8 [17, Theorem 2.23]. Let $P = P^2 \in \widetilde{\mathcal{M}}$. There exists a unique decomposition $P = \widetilde{P} + Z$, where $\widetilde{P} \in \mathcal{M}^{\text{pr}}$ is the range projection of the idempotent P and a nilpotent $Z \in \widetilde{\mathcal{M}}$ with $Z^2 = 0$ and $Z\widetilde{P} = 0, \widetilde{P}Z = Z$.

EXAMPLE 2 [18, Example 1]. Suppose that $0 < p, q < \infty$ and $a_n = 2^{n+1}n^{-q}$, $n \in \mathbb{N}$. Endow the von Neumann algebra $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with a faithful normal finite trace $\tau = \bigoplus_{n=1}^{\infty} 2^{-n} \text{tr}_2$ and put $A = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & a_n \\ 0 & 0 \end{pmatrix}$. We have $A = A^2$ and $A \in L_p(\mathcal{M}, \tau)$ for $pq > 1$ and $A \notin L_p(\mathcal{M}, \tau)$ for $pq \leq 1$.

Lemma 9. The inequality $Z|T|Z^* \leq |TZ^*|$ is fulfilled for all $T \in \widetilde{\mathcal{M}}$ and $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$.

PROOF. Since $t \mapsto \sqrt{t}$ ($t \geq 0$) is operator monotone, Hansen's Theorem of [19] gives $Z|T|Z^* = Z\sqrt{T^*TZ^*} \leq \sqrt{ZT^*TZ^*} = |TZ^*|$. \square

2. The Main Results

Let τ be a faithful normal semifinite trace on a von Neumann algebra \mathcal{M} .

Theorem 1. Let \mathcal{E} be an ideal space on (\mathcal{M}, τ) . The following are equivalent for a normal $X \in \widetilde{\mathcal{M}}$ and $P \in \mathcal{M}^{\text{id}}$:

- (i) $X \in \mathcal{E}$;
- (ii) $XP + P^\perp X \in \mathcal{E}$;
- (iii) $PXP + P^\perp X \in \mathcal{E}$;
- (iv) $XP + P^\perp XP^\perp \in \mathcal{E}$.

PROOF. (ii) \Rightarrow (i): $XP \in \mathcal{E}$ by (1); therefore,

$$P^\perp X \in \mathcal{E}. \quad (3)$$

STEP 1. Let $X \in \widetilde{\mathcal{M}}^+$. Consider the decomposition $P = \widetilde{P} + Z$ described in Lemma 8 with $Z\widetilde{P} = 0$ and $\widetilde{P}Z = Z, Z^2 = 0$. From (1) we obtain $XZ = (X\widetilde{P} + XZ)Z = XP \cdot Z \in \mathcal{E}$ by Lemma 5; thus, $X\widetilde{P} = XP - XZ \in \mathcal{E}$. By (3),

$$ZX = Z(\widetilde{P}^\perp - Z)X = Z \cdot P^\perp X \in \mathcal{E};$$

consequently, $\widetilde{P}^\perp X = P^\perp X + ZX \in \mathcal{E}$. Thus, the operators $\widetilde{P}X\widetilde{P}$ and $\widetilde{P}^\perp X\widetilde{P}^\perp$ lie in \mathcal{E} and $\widetilde{P}X\widetilde{P} + \widetilde{P}^\perp X\widetilde{P}^\perp \in \mathcal{E}$. Now, $X \in \mathcal{E}$ by Lemma 7.

STEP 2. Suppose that $X \in \widetilde{\mathcal{M}}$ is normal. Consider the polar decomposition $X = V|X|$, where $V \in \mathcal{M}$ and $V|X| = |X|V$ (see [20]). We infer that

$$|X|P = V^*V|X|P = V^* \cdot XP \in \mathcal{E} \quad (4)$$

on using (1) and Lemma 6. For $P_1 = P^{*\perp}$ and $X_1 = X^*$, we obtain $|X_1| = |X|$ and

$$X_1P_1 + P_1^\perp X_1 = (XP + P^\perp X)^* \in \mathcal{E}.$$

By the above (see (4)), we have $|X_1|P_1 \in \mathcal{E}$; therefore, $P^\perp|X| = (|X_1|P_1)^* \in \mathcal{E}$. Now, $|X| \in \mathcal{E}$ by step 1. Consequently, $X \in \mathcal{E}$.

The equivalence of (ii)–(iv) is established in Lemma 6. The theorem is proved. \square

Corollary 1. *Let a normal operator $X \in \widetilde{\mathcal{M}}$ and $P \in \mathcal{M}^{\text{id}}$ be such that $XP + P^\perp X \in L_1(\mathcal{M}, \tau)$. Then $X \in L_1(\mathcal{M}, \tau)$ and $\tau(XP + P^\perp X) = \tau(X)$.*

EXAMPLE 3. The normality of $X \in \widetilde{\mathcal{M}}$ is substantial in Theorem 1 and Corollary 1. Endow the von Neumann algebra $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with the faithful normal semifinite trace $\tau = \bigoplus_{n=1}^{\infty} \text{tr}_2$ and put

$$X = \bigoplus_{n=1}^{\infty} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Then $XP = P^\perp X = 0$ but $X \notin L_1(\mathcal{M}, \tau)$.

Theorem 2. *Let \mathcal{E} and \mathcal{F} be ideal spaces on (\mathcal{M}, τ) and take $p \in \{2^k : k \in \mathbb{N}\}$. Then the sets $\mathcal{E} \cap \mathcal{F}$, $\mathcal{E} + \mathcal{F} = \{A + B : A \in \mathcal{E}, B \in \mathcal{F}\}$, $\mathcal{E} \cdot \mathcal{F} = \{X \in \widetilde{\mathcal{M}} : \exists \{A_k\}_{k=1}^n \subset \mathcal{E}, \text{ and } \{B_k\}_{k=1}^n \subset \mathcal{F} \text{ such that } |X| \leq \sum_{k=1}^n |A_k B_k|\}$ as well as $\mathcal{E}_p = \{A \in \widetilde{\mathcal{M}} : |A|^p \in \mathcal{E}\}$ are ideal spaces on (\mathcal{M}, τ) too. Moreover, $\mathcal{E} \cdot \mathcal{F} = \mathcal{F} \cdot \mathcal{E}$, $(\mathcal{E} \cap \mathcal{F})^{\text{sa}} = \mathcal{E}^{\text{sa}} \cap \mathcal{F}^{\text{sa}}$, $(\mathcal{E} \cap \mathcal{F})^+ = \mathcal{E}^+ \cap \mathcal{F}^+$, and $(\mathcal{E} + \mathcal{F})^{\text{sa}} = \mathcal{E}^{\text{sa}} + \mathcal{F}^{\text{sa}}$, $(\mathcal{E} + \mathcal{F})^+ = \mathcal{E}^+ + \mathcal{F}^+$.*

PROOF. If $A, B \in \widetilde{\mathcal{M}}$ and $c > 0$ then

$$|A + B|^2 \leq (1 + c)|A|^2 + \left(1 + \frac{1}{c}\right)|B|^2,$$

where equality holds if and only if $B = cA$. This follows since $(\sqrt{c}A - \frac{1}{\sqrt{c}}B)^*(\sqrt{c}A - \frac{1}{\sqrt{c}}B) \geq 0$. Therefore, $A + B \in \mathcal{E}_2$ for all $A, B \in \mathcal{E}_2$. Let $A = U|A|$ be the polar decomposition of $A \in \mathcal{E}_2$. Then $|A^*|^2 = U|A|^2U^* \in \mathcal{E}_2$ by Lemma 5; thus, $A^* \in \mathcal{E}_2$. Let $A \in \mathcal{E}_2$, $B \in \widetilde{\mathcal{M}}$, and $|B| \leq |A|$. Owing to Lemma 2, there is $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$ with $|B| = Z|A|Z^*$. Hence, $Z^*Z \leq I$ and, by Lemmas 1 and 5, we infer

$$|B|^2 = Z|A|Z^*Z|A|Z^* \leq Z|A|^2Z^* \in \mathcal{E}.$$

Thus, $|B|^2 \in \mathcal{E}$ and \mathcal{E}_2 are ideal spaces on (\mathcal{M}, τ) . Note also that

$$\mathcal{E}_4 = (\mathcal{E}_2)_2, \quad \mathcal{E}_8 = (\mathcal{E}_4)_2, \dots, \quad \mathcal{E}_{2^k} = (\mathcal{E}_{2^{k-1}})_2$$

for all $k \in \mathbb{N}$.

Assume that $A \in \mathcal{E} + \mathcal{F}$, $B \in \widetilde{\mathcal{M}}$, and $|B| \leq |A|$. Then $A = A_1 + A_2$ with $A_1 \in \mathcal{E}$, $A_2 \in \mathcal{F}$, and there are partial isometries $V, W \in \mathcal{M}$ such that $|B| \leq |A_1 + A_2| \leq V|A_1|V^* + W|A_2|W^*$ [21, Theorem 2.2]. Consequently, $|B| = ZV|A_1|V^*Z^* + ZW|A_2|W^*Z^*$ for some $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$ by Lemma 2. If $B = U|B|$ is the polar decomposition then $B = U|B| = UZV|A_1|V^*Z^* + UZW|A_2|W^*Z^*$, where $UZV|A_1|V^*Z^* \in \mathcal{E}$ and $UZW|A_2|W^*Z^* \in \mathcal{F}$ by Lemma 5. Thus, $B \in \mathcal{E} + \mathcal{F}$.

Let $X = U|X|$ be the polar decomposition of $X \in \mathcal{E} \cdot \mathcal{F}$. Then, by Lemmas 1 and 9, we have

$$|X^*| = U|X|U^* \leq \sum_{k=1}^n U|A_k B_k|U^* \leq \sum_{k=1}^n |A_k \cdot B_k U^*|,$$

where $\{B_k U^*\}_{k=1}^n \subset \mathcal{F}$ by Lemma 5. Thus, $X^* \in \mathcal{E} \cdot \mathcal{F}$.

Suppose that $X, Y \in \mathcal{E} \cdot \mathcal{F}$ and $|Y| \leq \sum_{j=1}^l |C_j D_j|$ with some $\{C_j\}_{j=1}^l \subset \mathcal{E}$ and $\{D_j\}_{j=1}^l \subset \mathcal{F}$. Then there are partial isometries $V, W \in \mathcal{M}$ with $|X + Y| \leq V|X|V^* + W|Y|W^*$ [21, Theorem 2.2]. Lemmas 1 and 9 yield

$$|X + Y| \leq \sum_{k=1}^n V|A_k B_k|V^* + \sum_{j=1}^l W|C_j D_j|W^* \leq \sum_{k=1}^n |A_k \cdot B_k V^*| + \sum_{j=1}^l |C_j \cdot D_j W^*|,$$

where $\{B_k V^*\}_{k=1}^n, \{D_j W^*\}_{j=1}^l \subset \mathcal{F}$ by Lemma 5. Hence, $X + Y \in \mathcal{E} \cdot \mathcal{F}$.

For checking that $\mathcal{E} \cdot \mathcal{F} = \mathcal{F} \cdot \mathcal{E}$, it suffices to show that $\mathcal{E} \cdot \mathcal{F} \subset \mathcal{F} \cdot \mathcal{E}$. Suppose that $X \in \mathcal{E} \cdot \mathcal{F}$ and $|X| \leq \sum_{k=1}^n |A_k B_k|$ with some $\{A_k\}_{k=1}^n \subset \mathcal{E}$ and $\{B_k\}_{k=1}^n \subset \mathcal{F}$. Let $B_k^* A_k^* = U_k |B_k^* A_k^*|$ be the polar decomposition of $B_k^* A_k^* = (A_k B_k)^*$, $k = 1, 2, \dots, n$. Then, by Lemma 9,

$$|A_k B_k| = |(B_k^* A_k^*)^*| = U_k |B_k^* A_k^*| U_k^* \leq |B_k^* \cdot A_k^* U_k^*|, \quad k = 1, 2, \dots, n.$$

Thus, $|X| \leq \sum_{k=1}^n |B_k^* \cdot A_k^* U_k^*|$. Since $\{A_k^* U_k^*\}_{k=1}^n \subset \mathcal{E}$, by Lemma 5 we obtain $X \in \mathcal{F} \cdot \mathcal{E}$.

If $X \in (\mathcal{E} + \mathcal{F})^{\text{sa}}$ then $X = X^* = A + B$ with $A \in \mathcal{E}$ and $B \in \mathcal{F}$. Then $X = 2^{-1}(A + A^*) + 2^{-1}(B + B^*)$, where $2^{-1}(A + A^*) \in \mathcal{E}^{\text{sa}}$ and $2^{-1}(B + B^*) \in \mathcal{F}^{\text{sa}}$.

If $X \in (\mathcal{E} + \mathcal{F})^+$ then $X = |X| = A + B$ with $A \in \mathcal{E}$ and $B \in \mathcal{F}$. By [21, Theorem 2.2], $X = |A + B| \leq U|A|U^* + V|B|V^*$ with some partial isometries $U, V \in \mathcal{M}$. By Lemma 2, there exists $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$ such that $X = ZU|A|U^*Z^* + ZV|B|V^*Z^*$. Now, $ZU|A|U^*Z^* \in \mathcal{E}^+$ and $ZV|B|V^*Z^* \in \mathcal{F}^+$ by Lemma 5. The theorem is proved. \square

Theorem 3. *The structure of ideal spaces is modular: if \mathcal{E} , \mathcal{F} , and \mathcal{G} are ideal spaces on (\mathcal{M}, τ) and $\mathcal{E} \subset \mathcal{G}$ then $(\mathcal{E} + \mathcal{F}) \cap \mathcal{G} = \mathcal{E} + (\mathcal{F} \cap \mathcal{G})$; moreover, $(\mathcal{E} + \mathcal{F}) \cdot \mathcal{G} = \mathcal{E} \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G}$.*

PROOF. Prove the inclusion \supset . If $A \in \mathcal{E}$ and $B \in \mathcal{F} \cap \mathcal{G}$ then $A + B \in \mathcal{G}$ and $A + B \in \mathcal{E} + \mathcal{F}$.

Prove the inclusion \subset . If $X \in (\mathcal{E} + \mathcal{F}) \cap \mathcal{G}$ then $X = A + B$ with some $A \in \mathcal{E}$ and $B \in \mathcal{F}$. Therefore, $B = X - A \in \mathcal{G}$ (since $\mathcal{E} \subset \mathcal{G}$); i.e., $B \in \mathcal{F} \cap \mathcal{G}$.

Since $\mathcal{E} \cdot \mathcal{G}, \mathcal{F} \cdot \mathcal{G} \subset (\mathcal{E} + \mathcal{F}) \cdot \mathcal{G}$ and $(\mathcal{E} + \mathcal{F}) \cdot \mathcal{G}$ is a subspace of $\widetilde{\mathcal{M}}$, we have $\mathcal{E} \cdot \mathcal{G} + \mathcal{F} \cdot \mathcal{G} \subset (\mathcal{E} + \mathcal{F}) \cdot \mathcal{G}$. Let $X \in (\mathcal{E} + \mathcal{F}) \cdot \mathcal{G}$, i.e., $|X| \leq \sum_{k=1}^n |(A_k + B_k)C_k|$ with some $\{A_k\}_{k=1}^n \subset \mathcal{E}$, $\{B_k\}_{k=1}^n \subset \mathcal{F}$ and $\{C_k\}_{k=1}^n \subset \mathcal{G}$. By [21, Theorem 2.2], there are partial isometries $V_k, W_k \in \mathcal{M}$ such that

$$|(A_k + B_k)C_k| \leq V_k |A_k C_k| V_k^* + W_k |B_k C_k| W_k^*, \quad k = 1, 2, \dots, n.$$

Then $|X| \leq \sum_{k=1}^n V_k |A_k C_k| V_k^* + W_k |B_k C_k| W_k^*$ and, by Lemma 2, there exists $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq 1$ such that

$$|X| = \sum_{k=1}^n ZV_k |A_k C_k| V_k^* Z^* + ZW_k |B_k C_k| W_k^* Z^*.$$

Let $X = U|X|$ be the polar decomposition of X . Then $X = A + B$, where $A = \sum_{k=1}^n UZV_k |A_k C_k| V_k^* Z^* \in \mathcal{E} \cdot \mathcal{G}$ and $B = \sum_{k=1}^n UZW_k |B_k C_k| W_k^* Z^* \in \mathcal{F} \cdot \mathcal{G}$ by Lemma 5. The theorem is proved. \square

REMARK 1. We have $\mathcal{E} + \mathcal{F} = \widetilde{\mathcal{M}}$ for $\mathcal{E} = \mathcal{M}$ and $\mathcal{F} = \widetilde{\mathcal{M}}_0$ [22]. If $0 < p < \infty$ and $\mathcal{E} = L_p(\mathcal{M}, \tau)$ then $\mathcal{E}_2 = L_{2p}(\mathcal{M}, \tau)$. In [23], for operators $X, Y \in \widetilde{\mathcal{M}}$, sufficient conditions were established that $XY, YX \in L_1(\mathcal{M}, \tau)$. For such operators, $\tau([X, Y]) = 0$ by Lemma 3.

Proposition 1. *Let $P = P^2 \in \widetilde{\mathcal{M}}$ and let $P = \tilde{P} + Z$ be the decomposition described in Lemma 8. The following are equivalent:*

- (i) $P \in \mathcal{M}_0$;
- (ii) $\tilde{P} \in \mathfrak{M}_\tau^+$ and $Z \in \widetilde{\mathcal{M}}_0$.

PROOF. (i) \Rightarrow (ii): We have $Z = \tilde{P}Z = PZ \in \widetilde{\mathcal{M}}_0$ and

$$\tilde{P} = P - Z \in \widetilde{\mathcal{M}}_0. \quad (5)$$

Since $\tilde{P} \in \mathcal{M}^{\text{pr}}$, we have $\mu_t(\tilde{P}) \in \{0, 1\}$ for all $t > 0$. Now, $\tilde{P} \in \mathfrak{M}_\tau^+$ by (5), and the proposition is proved. \square

Theorem 4. *Let operators $X, Y \in \widetilde{\mathcal{M}}$ and $P \in \mathcal{M}^{\text{id}}$ be such that $XP - PY \in L_1(\mathcal{M}, \tau)$. Then $\tau(XP - PY) = \tau(PXP - PYP)$ and for $X = Y$ we get $\tau([X, P]) = 0$.*

PROOF. Owing to Lemma 5,

$$PXP - PYP = \frac{1}{2}(S_P(XP - PY)S_P + (XP - PY)) \in L_1(\mathcal{M}, \tau).$$

By Lemma 7,

$$\tau(XP - PY) = \tau(S_P(XP - PY)S_P) = \tau(2(PXP - PYP) - (XP - PY));$$

therefore, $\tau(XP - PY) = \tau(PXP - PYP) = \tau(P(X - Y)P)$, and the theorem is proved. \square

Theorem 4 and Lemma 1 imply

Corollary 2. *Let $X, Y \in \widetilde{\mathcal{M}}^{\text{sa}}$ with $Y \leq X$ and $P \in \mathcal{M}^{\text{pr}}$ be such that $XP - PY \in L_1(\mathcal{M}, \tau)$. Then $\tau(XP - PY) \geq 0$.*

Proposition 2. *Suppose that \mathcal{E} is an ideal space on (\mathcal{M}, τ) , $P, Q \in \mathcal{M}^{\text{id}}$, and $P = \widetilde{P} + Z$, $Q = \widetilde{Q} + T$ are the decomposition described in Lemma 8. The following are equivalent:*

- (i) $P - Q \in \mathcal{E}$;
- (ii) $\widetilde{P} - \widetilde{Q}, Z - T \in \mathcal{E}$;
- (iii) $Q^\perp P, QP^\perp \in \mathcal{E}$.

PROOF. (i) \Rightarrow (ii): We have

$$\widetilde{P} - \widetilde{Q}\widetilde{P} + T\widetilde{P} = (P - Q)\widetilde{P} \in \mathcal{E} \quad (6)$$

and $T\widetilde{P} = \widetilde{Q}(\widetilde{P} - \widetilde{Q}\widetilde{P} + T\widetilde{P}) \in \mathcal{E}$. Now (6) yields $\widetilde{P} - \widetilde{Q}\widetilde{P} \in \mathcal{E}$. Similarly, $\widetilde{Q} - \widetilde{P}\widetilde{Q} \in \mathcal{E}$. Therefore, $\widetilde{P} - \widetilde{Q} = (\widetilde{P} - \widetilde{Q}\widetilde{P})^* - (\widetilde{Q} - \widetilde{P}\widetilde{Q}) \in \mathcal{E}$.

The equivalence (i) \Leftrightarrow (iii) follows from the equalities

$$P - Q = Q^\perp P - QP^\perp, \quad Q^\perp P = Q^\perp(P - Q), \quad QP^\perp = -(P - Q)P^\perp$$

and Lemma 4. The proposition is proved. \square

Proposition 3. *Let \mathcal{E} be an ideal space on (\mathcal{M}, τ) . If $P, Q \in \mathcal{M}^{\text{id}}$ and $\{P - Q, P + Q - I\} \cap \mathcal{E} \neq \emptyset$ then $[P, Q] \in \mathcal{E}$.*

PROOF. The proposition follows from Lemma 5 and

$$(P + Q - I)(P - Q) = QP - PQ = (Q - P)(P + Q - I). \quad \square$$

Proposition 4. *Suppose that \mathcal{E} is an ideal space on (\mathcal{M}, τ) , $X \in \widetilde{\mathcal{M}}$, $P \in \mathcal{M}^{\text{id}}$, and $P = \widetilde{P} + Z$ is the decomposition described in Lemma 8. The following are equivalent:*

- (i) $P^\perp XP \in \mathcal{E} \Leftrightarrow \widetilde{P}^\perp X\widetilde{P}, ZX\widetilde{P}, ZXZ, \widetilde{P}^\perp XZ \in \mathcal{E}$;
- (ii) $P^\perp XP = 0 \Leftrightarrow \widetilde{P}^\perp X\widetilde{P} = ZX\widetilde{P} = ZXZ = \widetilde{P}^\perp XZ = 0$.

PROOF. (i) \Rightarrow (ii): By Lemma 5,

$$\widetilde{P}^\perp X\widetilde{P} - ZX\widetilde{P} = (P^\perp XP)\widetilde{P} \in \mathcal{E}. \quad (7)$$

Therefore, $ZX\widetilde{P} = Z(\widetilde{P}^\perp X\widetilde{P} - ZX\widetilde{P}) \in \mathcal{E}$ and (7) gives $\widetilde{P}^\perp X\widetilde{P} \in \mathcal{E}$. Now,

$$V = P^\perp XP - \widetilde{P}^\perp X\widetilde{P} = \widetilde{P}^\perp XZ - ZX\widetilde{P} - ZXZ \in \mathcal{E}$$

and $ZXZ = ZV \in \mathcal{E}$. Consequently, $\widetilde{P}^\perp XZ - ZX\widetilde{P} = V - VZ \in \mathcal{E}$ and $Z\widetilde{P} = -(V - VZ)\widetilde{P} \in \mathcal{E}$. The rest is obvious. \square

Corollary 3. *Let $\mathcal{E} = L_1(\mathcal{M}, \tau)$ under the conditions of Proposition 4. Then $\tau(ZXZ) = 0$ and $\tau(\widetilde{P}^\perp XZ) = \tau(ZXP)$.*

PROOF. The operators $\widetilde{P} \cdot ZXZ = ZXZ$ and $ZXZ \cdot \widetilde{P} = 0$ lie in $L_1(\mathcal{M}, \tau)$; therefore, $\tau(ZXZ) = \tau(\widetilde{P} \cdot ZXZ) = \tau(ZXZ \cdot \widetilde{P}) = \tau(0) = 0$ by Lemma 3. Similarly, $\tau(P^\perp XP) = \tau(\widetilde{P}^\perp X\widetilde{P}) = 0$, and the proposition ensues from the equality $\tau(P^\perp XP) = \tau(\widetilde{P}^\perp X\widetilde{P}) - \tau(ZXZ) + \tau(\widetilde{P}^\perp XZ) - \tau(ZX\widetilde{P})$. \square

REMARK 2. If $\mathcal{E} = \widetilde{\mathcal{M}}_0$ then Propositions 2–4 are carried over (with similar proofs) to unbounded idempotents $P, Q \in \widetilde{\mathcal{M}}$.

Theorem 5. Suppose that $\lambda > 0$, $X \in \widetilde{\mathcal{M}}$, and $Y = X^*X + XX^*$. Then there exist unitary operators $S_\lambda, S \in \mathcal{M}^{\text{sa}}$ such that

$$|\operatorname{Re} X| \leq \frac{1}{4\lambda}(X^*X + S_\lambda X^*X S_\lambda) + \frac{\lambda}{2}I, \quad (8)$$

$$|\operatorname{Re} X^2| \leq \frac{1}{4}(Y + SY S). \quad (9)$$

PROOF. The inequality $(X \pm \lambda I)^*(X \pm \lambda I) \geq 0$ yields $\mp\lambda(X + X^*) \leq X^*X + \lambda^2 I$. Therefore,

$$-\left(\frac{1}{\lambda}X^*X + \lambda I\right) \leq X^* + X \leq \frac{1}{\lambda}X^*X + \lambda I$$

and by [24, Theorem 1] there exists a unitary operator $S_\lambda \in \mathcal{M}^{\text{sa}}$ such that

$$2|X + X^*| \leq \frac{1}{\lambda}X^*X + \lambda I + S_\lambda \left(\frac{1}{\lambda}X^*X + \lambda I\right) S_\lambda = \frac{1}{\lambda}(X^*X + S_\lambda X^*X S_\lambda) + 2\lambda I.$$

Thus, (8) holds

Since $Z = i(X - X^*)$ is selfadjoint, $Z^2 \geq 0$; therefore, $Y \geq 2\operatorname{Re} X^2$. Since $(X + X^*)^2 \geq 0$, we have $2\operatorname{Re} X^2 \geq -Y$. Thus, $-Y \leq 2\operatorname{Re} X^2 \leq Y$, and (9) follows from [24, Theorem 1]. Note that if $X = I$ then (8) for $\lambda = 1$ and (9) become equalities. The theorem is proved. \square

Since $\Im X = \operatorname{Re}(iX)$, $(iX)^*(iX) = X^*X$, $(iX)(iX)^* = XX^*$, and $\Im X^2 = \operatorname{Re}(iX^2) = \operatorname{Re}((zX)^2)$ with $z = e^{\frac{\pi}{4}i}$, $(zX)^*(zX) = X^*X$, and $(zX)(zX)^* = XX^*$ for all $X \in \widetilde{\mathcal{M}}$, we have

Corollary 4. Suppose that $\lambda > 0$, $X \in \widetilde{\mathcal{M}}$, and $Y = X^*X + XX^*$. Then there exist unitary operators $T_\lambda, T \in \mathcal{M}^{\text{sa}}$ such that

$$|\Im X| \leq \frac{1}{4\lambda}(X^*X + T_\lambda X^*X T_\lambda) + \frac{\lambda}{2}I, \quad (10)$$

$$|\Im X^2| \leq \frac{1}{4}(Y + TY T). \quad (11)$$

Corollary 5 (cf. [25, Corollary 3.4]). Let $X = X^2 \in \widetilde{\mathcal{M}}$ and $Y = X^*X + XX^*$. Then there exist unitary operators $U, V \in \mathcal{M}^{\text{sa}}$ such that $4|\operatorname{Re} X| \leq Y + UYU$ and $4|\Im X| \leq Y + VYV$.

Corollary 6. Suppose that $p \geq 1$ and $X \in L_{2p}(\mathcal{M}, \tau)$. Then

- (i) $\operatorname{Re} X^2, \Im X^2 \in L_p(\mathcal{M}, \tau)$ and $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \leq \|X^*X\|_p$;
- (ii) if $\tau(I) = 1$ then $\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \leq \sqrt{\|X^*X\|_p}$.

PROOF. (i) By (9), (11), the triangle inequality for $\|\cdot\|_p$, the equality $\|XX^*\|_p = \|X^*X\|_p$, and the unitary invariance of the norm $\|\cdot\|_p$, we obtain $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \leq \|X^*X\|_p$. If $X \in \mathcal{M}^{\text{pr}}$ then this inequality turns into equality.

(ii) If $\tau(I) = 1$ then, by (8), (10), the triangle inequality for $\|\cdot\|_p$, and the unitary equivalence of the norm $\|\cdot\|_p$, we come to the relation

$$\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \leq \frac{1}{2} \left(\frac{1}{\lambda} \|X^*X\|_p + \lambda \right).$$

Note that $\min_{\lambda>0} \left(\frac{1}{\lambda} \|X^*X\|_p + \lambda \right)$ is attained at the point $\lambda_0 = \sqrt{\|X^*X\|_p}$. The corollary is proved. \square

Lemma 5 and Theorem 5 give

Corollary 7. Suppose that \mathcal{E} is an ideal space on (\mathcal{M}, τ) and $X \in \widetilde{\mathcal{M}}$. If $X \in \mathcal{E}_2$ then $\operatorname{Re} X^2 \in \mathcal{E}$. In particular, if $0 < p < +\infty$ and $X \in L_{2p}(\mathcal{M}, \tau)$ then $\operatorname{Re} X^2 \in L_p(\mathcal{M}, \tau)$ and $\|\operatorname{Re} X^2\|_p \leq 2^{\max\{0, 2-2p\}} \|X\|_{2p}$.

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