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DIFFERENCES OF IDEMPOTENTS IN C^*-ALGEBRAS AND THE QUANTUM HALL EFFECT

A. M. Bikchentaev*

Let φ be a trace on the unital C^*-algebra $\mathcal{A}$ and $\mathfrak{M}_\varphi$ be the ideal of the definition of the trace φ. We obtain a C^* analogue of the quantum Hall effect: if $P, Q \in \mathcal{A}$ are idempotents and $P - Q \in \mathfrak{M}_\varphi$, then $\varphi((P-Q)^{2n+1}) = \varphi(P-Q) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let the isometries $U \in \mathcal{A}$ and $A = A^* \in \mathcal{A}$ be such that $I + A$ is invertible and $U - A \in \mathfrak{M}_\varphi$ with $\varphi(U - A) \in \mathbb{R}$. Then $I - A$, $I - U \in \mathfrak{M}_\varphi$ and $\varphi(I - U) \in \mathbb{R}$. Let $n \in \mathbb{N}$, $\dim \mathcal{H} = 2n + 1$, the symmetry operators $U, V \in \mathcal{B}(\mathcal{H})$, and $W = U - V$. Then the operator W is not a symmetry, and if $V = V^*$, then the operator W is nonunitary.

Keywords: Hilbert space, linear operator, idempotent, symmetry, projection, unitary operator, trace-class operator, C^*-algebra, trace, quantum Hall effect

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1. Introduction

Let P and Q be idempotents in the Hilbert space $\mathcal{H}$. Various properties (invertibility, Fredholm properties, trace-class, positivity, etc.) of the difference $X = P - Q$ were studied in [1]–[6]. Each tripotent ($A = A^3$) is the difference $P - Q$ of some idempotents P and Q with $PQ = QP = 0$ (see Proposition 1 in [7]). Therefore, tripotents inherit some idempotent properties [8]. If X is a trace-class operator, then the traces of all odd powers of X coincide:

$$\operatorname{tr}(P - Q) = \operatorname{tr}((P-Q)^{2n+1}) = \dim\ker(X - I) - \dim\ker(X + I) \in \mathbb{Z}, \tag{1}$$

where I is the identity operator in $\mathcal{H}$. If X is a compact operator, then the right-hand side of (1) yields a natural "regularization" for the trace and shows that it is always an integer [5], [6].

Pairs of idempotents play an important role in the quantum Hall effect [9]. For idempotents P, Q, and R with the trace-class operators $P - Q$ and $Q - R$, from the equality $\operatorname{tr}(P - Q) = \operatorname{tr}(P - R) + \operatorname{tr}(R - Q)$ and (1), we obtain

$$\operatorname{tr}((P-Q)^3) = \operatorname{tr}((P-R)^3) + \operatorname{tr}((R-Q)^3). \tag{2}$$

The physical meaning of the additivity in Eq. (2) comes from the interpretation of $\operatorname{tr}((P-Q)^3)$ as *Hall conductivity*. The additivity of (cubic) Eq. (2) can be considered a variant of Ohm's law for the additivity of conductivity [10].

*Kazan Federal University, Kazan, Russia, e-mail: Airat.Bikchentaev@kpfu.ru.

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Here, we obtain a C^* analogue of the quantum Hall effect (Theorem 1) and prove the realness for the trace of differences of a wide class of symmetries from the unital C^*-algebra (Corollaries 2 and 3). We show that in the complete matrix algebra $\mathbb{M}_n(\mathbb{C})$ with odd n, the difference of the symmetries cannot be a symmetry (Theorem 3).

2. Definitions and notation

We call a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$ a C^*-algebra. For a unital C^*-algebra $\mathcal{A}$, we let $\mathcal{A}^{\mathrm{id}}$, $\mathcal{A}^{\mathrm{sa}}$, $\mathcal{A}^{\mathrm{sym}}$, $\mathcal{A}^{\mathrm{u}}$, $\mathcal{A}^{\mathrm{pr}}$, and $\mathcal{A}^+$ respectively denote its submanifolds of idempotents, Hermitian elements, symmetries, unitary elements, projectors, and positive elements. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{id}}$, then $P^\perp = I - P \in \mathcal{A}^{\mathrm{id}}$. We call the C^*-algebra $\mathcal{A}$ with the predual Banach space $\mathcal{A}_*$ the W^*-algebra: $\mathcal{A} \simeq (\mathcal{A}_*)^*$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$ and $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. Any C^*-algebra can be realized as a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand–Naimark theorem; see Theorem 3.4.1 in [11]).

We define the *trace* on the C^*-algebra $\mathcal{A}$ as a map $\varphi\colon \mathcal{A}^+ \to [0, +\infty]$ such that $\varphi(X+Y) = \varphi(X)+\varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ (and $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. For the trace φ, we define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ \colon \varphi(X) < +\infty\}, \qquad \mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}}\mathfrak{M}_\varphi^+, \qquad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ has a well-defined extension by linearity to a functional on $\mathfrak{M}_\varphi$, denoted by the same letter φ. Such an extension allows identifying finite traces (i.e., $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$) with positive functionals on $\mathcal{A}$.

3. Basic results

Lemma 1 (Lemma 1 in [12]). *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$ and the elements $A, B \in \mathcal{A}$ be such that $A - B \in \mathfrak{M}_\varphi$. Then $AB - BA \in \mathfrak{M}_\varphi$ and $\varphi(AB - BA) = 0$.*

Lemma 2 (Theorem 3 in [12]). *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$ and $P, Q \in \mathcal{A}^{\mathrm{id}}$. If $P - Q \in \mathfrak{M}_\varphi$, then $\varphi(P - Q) \in \mathbb{R}$.*

Theorem 1. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$ and $P, Q \in \mathcal{A}^{\mathrm{id}}$. If $P - Q \in \mathfrak{M}_\varphi$, then $\varphi((P - Q)^{2n+1}) = \varphi(P - Q) \in \mathbb{R}$ for all $n \in \mathbb{N}$.*

Proof. Step 1. We show that

$$(P - Q)^{2n+1} = P - Q + \lambda_1(PQP - QPQ) + \cdots + \lambda_n(\underbrace{PQP \cdots QP}_{2n+1} - \underbrace{QPQ \cdots PQ}_{2n+1}) \tag{3}$$

for some $\lambda_k \in \mathbb{Z}$, $k = 1, 2, \ldots, n$. We use induction. For $n = 1$, we have

$$(P - Q)^3 = P - Q - (PQP - QPQ), \tag{4}$$

and equality (3) is satisfied with $\lambda_1 = -1$. We suppose that (3) holds for $k = n$ and show that it holds for $k = n + 1$. We note that $(P - Q)^{2n+3} = (P - Q)(P - Q)^{2n+1}(P - Q)$ and

$$(P - Q)(\underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1})(P - Q) =$$

$$= (\underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1}) - (\underbrace{PQP \cdots QP}_{2m+3} - \underbrace{QPQ \cdots PQ}_{2m+3})$$

for all $m = 1, 2, \ldots, n-1$. We have thus established equality (3).

Step 2. We show that

$$X_m \equiv \underbrace{PQP \cdots QP}_{2m+1} - \underbrace{QPQ \cdots PQ}_{2m+1} \in \mathfrak{M}_\varphi \quad \text{and} \quad \varphi(X_m) = 0 \quad \text{for all } m \in \mathbb{N}.$$

We set $A = PQ$ and $B = \underbrace{QPQ \cdots QP}_{2m}$. Then

$$X_m = PQ \cdot \underbrace{QPQ \cdots QP}_{2m} - \underbrace{QPQ \cdots QP}_{2m} \cdot PQ = AB - BA.$$

Further,

$$A - B = PQ - \underbrace{QPQ \cdots QP}_{2m} =$$

$$= (P^2Q - QPQ) + (QP \cdot Q^2 - QPQP) + \cdots + (\underbrace{QPQ \cdots PQ^2}_{2m-1} - \underbrace{QPQ \cdots QP}_{2m}) =$$

$$= (P - Q)PQ + QPQ(Q - P) + \cdots + \underbrace{QPQ \cdots PQ}_{2m-1}(Q - P)$$

belongs to $\mathfrak{M}_\varphi$, and by Lemma 1, the element $AB - BA$ also belongs to $\mathfrak{M}_\varphi$ and $\varphi(AB - BA) = 0$. From (3), we obtain

$$\varphi((P-Q)^{2n+1}) = \varphi(P-Q) + \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \cdots + \lambda_n \cdot 0 = \varphi(P-Q)$$

for all $n \in \mathbb{N}$. We now note that $\varphi(P - Q) \in \mathbb{R}$ by Lemma 2. The theorem is proved.

Corollary 1. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$. If $P, Q, R \in \mathcal{A}^{\mathrm{id}}$ and $P - Q$, $Q - R \in \mathfrak{M}_\varphi$, then*

$$\varphi((P-R)^{2n+1}) = \varphi((P-Q)^{2n+1}) + \varphi((Q-R)^{2n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Corollary 2. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$. If $U, V, W \in \mathcal{A}^{\mathrm{sym}}$ and $U - V$, $V - W \in \mathfrak{M}_\varphi$, then $\varphi(U - V) \in \mathbb{R}$ and*

$$\varphi((U-W)^{2n+1}) = \varphi((U-V)^{2n+1}) + \varphi((V-W)^{2n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Proof. The formula $U = 2P - I$ ($P \in \mathcal{A}^{\mathrm{id}}$) establishes a bijection between the manifolds $\mathcal{A}^{\mathrm{id}}$ and $\mathcal{A}^{\mathrm{sym}}$. Let $V = 2Q - I$ and $W = 2R - I$ with $Q, R \in \mathcal{A}^{\mathrm{id}}$. Then

$$\varphi(U - V) = \varphi(2P - 2Q) = 2\varphi(P - Q) \in \mathbb{R}$$

by Lemma 2. By Corollary 1, we have

$$\varphi((U-W)^{2n+1}) = 2^{2n+1}\varphi((P-R)^{2n+1}) =$$

$$= 2^{2n+1}(\varphi((P-Q)^{2n+1}) + \varphi((Q-R)^{2n+1})) =$$

$$= \varphi((U-V)^{2n+1}) + \varphi((V-W)^{2n+1})$$

for each $n \in \mathbb{N} \cup \{0\}$. The statement is proved.

Remark 1. The condition

$$\varphi((P-Q)^3) = \varphi(P-Q) \quad \text{for all } P, Q \in \mathcal{A}^{\text{pr}}$$

characterizes the traces in the class of all positive functionals on the W^*-algebra $\mathcal{A}$. From (4), we have $\varphi(PQP) = \varphi(QPQ)$ for all $P, Q \in \mathcal{A}^{\text{pr}}$, and we can apply Theorem 1 in [13] (see [14]–[16] and the references therein for other characterizations of the trace using the commutation relations for projectors).

Theorem 2. *Let $\mathcal{J}$ be an ideal in a unital $*$-algebra $\mathcal{A}$, let an isometry $U \in \mathcal{A}$ (i.e., $U^*U = I$), and let the operator $A \in \mathcal{A}^{\text{sa}}$ be such that $I + A$ is invertible in $\mathcal{A}$. Then the following conditions are equivalent*:

1. *$U - A \in \mathcal{J}$ and*
2. *$I - A, I - U \in \mathcal{J}$.*

Proof. $1 \Rightarrow 2$. We have $U^* - A = (U - A)^* \in \mathcal{J}$ and

$$-U^*A + AU = U^*(U - A) - (U^* - A)U \in \mathcal{J}.$$

Consequently,

$$I - A^2 = (U^* - A)(U + A) - U^*A + AU \in \mathcal{J}, \qquad I - A = (I - A^2)(I + A)^{-1} \in \mathcal{J}.$$

Therefore, $I - U = I - A - (U - A) \in \mathcal{J}$.

$2 \Rightarrow 1$. We have $U - A = (I - A) - (I - U) \in \mathcal{J}$. The theorem is proved.

Corollary 3. *Let φ be a trace on a unital C^*-algebra $\mathcal{A}$, let an isometry $U \in \mathcal{A}$, and let $A \in \mathcal{A}^{\text{sa}}$ be such that the operator $I + A$ is invertible in $\mathcal{A}$ and $U - A \in \mathfrak{M}_\varphi$ with $\varphi(U - A) \in \mathbb{R}$. Then $I - A$, $I - U \in \mathfrak{M}_\varphi$, and $\varphi(I - U) \in \mathbb{R}$.*

Lemma 3. *Let $n \in \mathbb{N}$ and $\dim \mathcal{H} = 2n + 1$. If $P, Q, R \in \mathcal{B}(\mathcal{H})^{\text{id}}$, then $P + Q + R \neq 3I/2$.*

Proof. We assume that $P + Q + R = 3I/2$. For the canonical trace, we then have $\operatorname{tr}(P) \in \mathbb{N} \cup \{0\}$ and

$$\frac{3}{2}(2n + 1) = \operatorname{tr}(P + Q + R) = \operatorname{tr}(P) + \operatorname{tr}(Q) + \operatorname{tr}(R) \in \mathbb{N},$$

i.e., we obtain a contradiction. The lemma is proved.

Theorem 3. *Let $n \in \mathbb{N}$, $\dim \mathcal{H} = 2n + 1$, the operators $U, V \in \mathcal{B}(\mathcal{H})^{\text{sym}}$, and $W \equiv U - V$. Then*

1. *the operator $W \notin \mathcal{B}(\mathcal{H})^{\text{sym}}$, and*
2. *if $V = V^*$, then $W \notin \mathcal{B}(\mathcal{H})^{\text{u}}$.*

Proof. 1. We assume that $W \in \mathcal{B}(\mathcal{H})^{\text{sym}}$. Then the operators

$$P = \frac{U + I}{2}, \qquad Q = \frac{V + I}{2}, \qquad R = \frac{W + I}{2}$$

are in $\mathcal{B}(\mathcal{H})^{\text{id}}$, and the equality $U - V = W$ is equivalent to the equality $2P + 2Q^\perp + 2R^\perp = 3I$, which contradicts Lemma 3.

2. We now assume that $W \in \mathcal{B}(\mathcal{H})^{\mathrm{u}}$ and let

$$U = 2P - I \quad \text{with } P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}} \quad \text{and} \quad V = 2Q - I \quad \text{with } Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}.$$

Because

$$\frac{W + 2Q}{2} = P = P^2 = \frac{W^2 + 4Q + 2WQ + 2QW}{4},$$

we have $2W = W^2 + 2WQ + 2QW$. Multiplying both sides of this equality by the operator W^* from the left, we obtain

$$2I = W + 2Q + 2W^*QW. \tag{5}$$

Therefore, $2I - W \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ and $W = W^*$. Consequently, $W = 2R - I$ with $R = (W + I)/2 \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. From (5), we obtain

$$3I = 2R + 2Q + 2W^*QW,$$

which contradicts Lemma 3. The theorem is proved.

Corollary 4. *Let $n \in \mathbb{N}$, $\dim \mathcal{H} = 2n + 1$, and the operators $U, V \in \mathcal{B}(\mathcal{H})^{\mathrm{u}} \cap \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$. Then the operator $U - V \notin \mathcal{B}(\mathcal{H})^{\mathrm{u}}$.*

Example 1. The condition $\dim \mathcal{H} = 2n + 1$ is important in Theorem 2 and Corollary 4. We set $x = 1/2 + \sqrt{3}/16$ and consider the operators $U = 2P - I$ and $V = 2Q - I$ in $\mathbb{M}_2(\mathbb{C})$, where

$$P = \begin{pmatrix} x & 1/4 \\ 1/4 & 1 - x \end{pmatrix}, \qquad Q = \begin{pmatrix} x & -1/4 \\ -1/4 & 1 - x \end{pmatrix}.$$

Then $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{pr}}$, and the operator $U - V$ is unitary.

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