ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2017, Vol. 38, No. 1, pp. 16–23. © Pleiades Publishing, Ltd., 2017.

Modules Close to SSP- and SIP-Modules

Adel N. Abyzov*, **Tran Hoai Ngoc Nhan****, and **Truong Cong Quynh*****

(Submitted by M. M. Arslanov)

Department of Algebra and Mathematical Logic, Kazan (Volga Region) Federal Univesrity, 18 Kremlyovskaya str., Kazan, 420008 Russia,
Department of IT and Mathematics Teacher Training, Dong Thap University, Vietnam,
Department of Mathematics, Danang University, 459 Ton Duc Thang, Danang city, Vietnam
Received June 22, 2016

Abstract—In this paper, we investigate some properties of SIP, SSP and CS-Rickart modules. We give equivalent conditions for SIP and SSP modules; establish connections between the class of semisimple artinian rings and the class of SIP rings. It shows that R is a semisimple artinian ring if and only if R_R is SIP and every right R-module has a SIP-cover. We also prove that R is a semiregular ring and $J(R) = Z(R_R)$ if only if every finitely generated projective module is a CS-Rickart module which is also a $C2$ module.

DOI: 10.1134/S1995080217010024

Keywords and phrases: *SSP-module, SIP-module, SIP-CS, CS-Rickarts*.

1. INTRODUCTION AND NOTATION

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. The Jacobson radical ideal in R is denoted by $J(R)$. The notations $N \leq M$, $N \leq_e M$, $N \trianglelefteq M$, or $N \subset_d M$ mean that N is a submodule, an essential submodule, a fully invariant submodule, and a direct summand of M, respectively. We refer to [6, 9, 18], and [23] for all the undefined notions in this paper.

Recall that a module M is called a *SIP module* (respectively, *SSP module*) if the intersection (or the sum) of any two direct summands of M is also a direct summand of M (see [12, 14, 22]). It is known that every Rickart right R-module M (i.e., every endomorphism of M has the kernel a direct summand) has the SIP (see [16, Proposition 2.16]) and every d-Rickart right R-module M (i.e., every endomorphism of M has the image a direct summand) has the SSP ([17, Proposition 2.11]).

A module M is called an *SIP-CS module* if the intersection of any two direct summands of M is essential in a direct summand of M. It is known that every CS-Rickart module has the CS-SIP (see [2, Proposition 1.(4)]).

In this paper, we provide some characterizations of SIP, SSP, SIP-CS and CS-Rickart modules.

*E-mail: Adel.Abyzov@ksu.ru
**E-mail: tranhoaingocnhan@gmail.com
***E-mail: tcquynh@dce.udn.vn

2. SIP MODULES AND SSP MODULES

Let $f : A \to B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows: $\langle f \rangle = \{a + f(a) \mid a \in A\}$. The following result is obvious and we can omit its proof.

Lemma 2.1. *Let $M = X \oplus Y$ and $f : A \to Y$, a homomorphism with $A \leq X$. Then*

(1) $A \oplus Y = \langle f \rangle \oplus Y$;

(2) $\mathrm{Ker}(f) = X \cap \langle f \rangle$.

We next study some properties of SIP and SSP modules via homomorphisms:

Proposition 2.2. *The following conditions are equivalent for a module M:*

(1) M is SSP;

(2) For any split monomorphism $f : A \to M$ with A a direct summand of M, $A + \mathrm{Im}(f)$ is a direct summand of M;

(3) For any split epimorphism $f : M \to M/A$ with A a direct summand of M, $A + \mathrm{Ker}(f)$ is a direct summand of M.

Proof. (1) ⇒ (2), (3) are obvious.

(2) ⇒ (1). Assume that $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ an R-homomorphism. Call $T = \langle f \rangle$ a submodule of M and hence $M = T \oplus A_2$. We consider the homomorphism $\psi : A_1 \to M$ given by $\psi(x) = x + f(x)$. It is easily to see that ψ is a split monomorphism. By (2), $A_1 + \psi(A_1) = A_1 + T$ is a direct summand of M. Furthermore, $A_1 + T = A_1 \oplus \mathrm{Im}(f)$, which implies $\mathrm{Im}(f)$ is a direct summand of A_2.

(3) ⇒ (1). Suppose that $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ an R-homomorphism. Let $T = \langle f \rangle$ be a submodule of M. Then $M = T \oplus A_2$. Call the homomorphism $\psi : M \to M/T$ given by $\psi(a_1 + a_2) = a_2 + T$ for all $a_1 \in A_1, a_2 \in A_2$. Clearly, ψ is a split epimorphism and $\mathrm{Ker}(\psi) = A_1$. By (3), $A_1 + T$ is a direct summand of M. On the other hand, $A_1 + T = A_1 \oplus \mathrm{Im}(f)$, which implies $\mathrm{Im}(f)$ is a direct summand of A_2. □

Corollary 2.3. The following conditions are equivalent for a module M:

(1) M is SSP;

(2) For any two direct summands A_1 and A_2 with $A_1 \simeq A_2$, then $A_1 + A_2$ is a direct summand of M.

Similarly with SIP, we also have some characterizations of SIP-modules:

Proposition 2.4. *The following conditions are equivalent for a module M:*

1. *M is* SIP;
2. *For any split monomorphism $f : A \to M$ with A a direct summand of M, $A \cap f(A)$ is a direct summand of M;*
3. *For any split epimorphism $f : M \to M/A$ with A a direct summand of M, $A \cap \mathrm{Ker}(f)$ is a direct summand of M.*

Corollary 2.5. The following conditions are equivalent for a module M:

1. M is SIP;
2. For any two direct summands A_1 and A_2 with $A_1 \simeq A_2$, then $A_1 \cap A_2$ is a direct summand of M.

Proposition 2.6. *Let R be a ring, M an R-R-bimodule and $T = R \propto M$ the corresponding trivial extension. The following conditions are equivalent:* (1) T has the SSP;

(2) (a) R *has the* SSP;

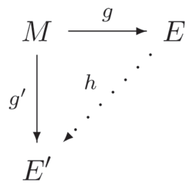
(b) *For every regular x of R with $x = xyx$, we have $xM(1 - xy) = 0$.*

Proof. (1) $\Rightarrow$ (2). By [12, Proposition 4.5].

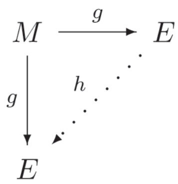
(2) $\Rightarrow$ (1). Assume that $x = xyx$. For any $m \in M$, call $z = xm(1 - xy)$. It follows that $z = (xy)z(1 - xy)$. Note that xy is idempotent of R. By [12, Proposition 4.5], $z = (xy)z(1 - xy) = 0$. □

Let R be a ring and Ω, a class of right R-modules which is closed under isomorphisms and summands. According to Enochs in [10], we study the notion of Ω-envelope and the notion of Ω-cover:

An R-homomorphism $g : M \to E$ is called an *Ω-envelope of a right R-module M*; if $E \in \Omega$ such that any diagram:

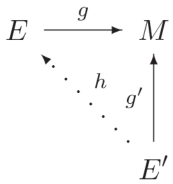


with $E' \in \Omega$, can be completed, and the diagram:

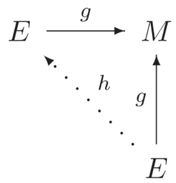


can be completed only by an automorphism h.

An R-homomorphism $g : E \to M$ is called *an Ω-cover of a right R-module M*; if $E \in \Omega$ such that any diagram:



with $E' \in \Omega$, can be completed, and the diagram:



can be completed only by an automorphism h.

A right R-module M is called a *$C3$-module* if whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. Dually, M is called a *$D3$-module* if whenever M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Proposition 2.7. *The following conditions are equivalent for a ring R*:

(1) *R is a semisimple artinian ring*;

(2) *Every right R-module has a $D3$-cover*;

(3) *Every* 2*-generated right R-module has a* D3*-cover*;

(4) *Every right R-module has a* D3*-envelope*;

(5) *Every* 2*-generated right R-module has a* D3*-envelope*.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (3), $M = R_R \oplus S$ has a D3-cover, say $\alpha : C \to M$ where C is a D3-module. Let $\iota_1 : S \to M$ and $\iota_2 : R_R \to M$ be the inclusion maps for all $i = 1, 2$. Note that S and R_R are D3-modules, and there are homomorphisms $\beta_1 : S \to C, \beta_2 : R_R \to C$ such that $\alpha\beta_i = \iota_i$. Clearly, $id_M = \iota_1 \oplus \iota_2 = \alpha(\beta_1 \oplus \beta_2)$. This shows that M is isomorphic to a direct summand of C, which implies that M is a D3-module. We deduce that Ker(φ) is a direct summand of R_R by [4, Proposition 4]. It follows that S is a projective module. Thus R is semisimple.

(1) $\Rightarrow$ (4) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (1). Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (5), $M = R_R \oplus S$ has a D3-envelope, named $\iota : M \to E$ where E is a D3-module. Since S and R are D3-modules, there exist $f_1 : E \to S,\ f_2 : E \to R$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to S$ and $\pi_2 : M \to R$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = id_M$, and hence ι is a split monomorphism. Thus $N \oplus E(N)$ is isomorphic to a direct summand of E. This gives that $S \oplus R$ is also a D3-module. We deduce that Ker(φ) is a direct summand of R_R. So S is a projective module. Thus R is semisimple. □

Corollary 2.8. The following conditions are equivalent for a ring R:

(1) R is a semisimple artinian ring;

(2) R_R is SIP and every right R-module has a SIP-cover;

(3) R_R is SIP and every 2-generated right R-module has a SIP-cover;

(4) R_R is SIP and every right R-module has a SIP-envelope;

(5) R_R is SIP and every 2-generated right R-module has a SIP-envelope.

A ring R is called a right *V-ring* if every simple right R-module is injective.

Proposition 2.9. *The following conditions are equivalent for a ring R*:

(1) *R is a right* V-*ring*;

(2) *Every finitely cogenerated right R-module has a* C3-*envelope*;

(3) *Every finitely cogenerated right R-module has a* C3-*cover*.

Proof. (1) $\Rightarrow$ (2), (3) are obvious.

(2) $\Rightarrow$ (1) Let N be an arbitrary simple module. Assume that $\iota : M = N \oplus E(N) \to E$ is the C3-envelope, where E is a C3-module. Since N and $E(N)$ are C3-modules, there exist $f_1 : E \to N,\ f_2 : E \to E(N)$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to N_i$ and $\pi_2 : M \to E(N)$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = id_M$, and so the monomorphism ι splits. Thus $N \oplus E(N)$ is isomorphic to a direct summand of E. It follows that $N \oplus E(N)$ is also a C3-module. Therefore N is a direct summand of $E(N)$. This gives N is injective. Thus R is a right V-ring.

(3) $\Rightarrow$ (1) The proof is similar to the proof (3) $\Rightarrow$ (1) of Proposition 2.7. □

Similarly, we also get the following result for injectivity of semisimple modules:

Proposition 2.10. *The following conditions are equivalent for a ring R*:

(1) *R is a right Noetherian right* V-*ring*;

(2) *Every right R-module with essential socle has a* C3-*envelope*;

(3) *Every right R-module with essential socle has a* C3-*cover*.

3. SIP-CS MODULES

A module M is called *relatively CS-Rickart to* N (or N*-CS-Rickart*) if for every $\varphi \in \mathrm{End}_R(M, N)$, $\mathrm{Ker}\varphi$ is an essential submodule of a direct summand of M. A module M is called *relatively d-CS-Rickart to* N (or N*-d-CS-Rickart*) if for every $\varphi \in \mathrm{End}_R(N, M)$, $\mathrm{Im}\varphi$ lies above a direct summand of M. A module M is called *CS-Rickart (d-CS-Rickart)* if M is M-CS-Rickart (resp., M-d-CS-Rickart). M is called a *SIP-CS module* if A_i is essential in a direct summand of M for all $i \in \mathcal{I}$, $\mathcal{I}$ is a finite index set, then $\bigcap_{i\in\mathcal{I}} A_i$ is essential in a direct summand of M. M is called a *lifting SSP module* if A_i lies above a direct summand of M for all $i \in \mathcal{I}$, $\mathcal{I}$ is a finite index set, then $\sum_{i\in\mathcal{I}} A_i$ lies above a direct summand of M. The class of CS-Rickart (d-CS-Rickart, SIP-CS, lifting SSP) modules is studied by the authors in [1, 2].

Lemma 3.1. *The following implications hold for a module* $M = M_1 \oplus \ldots \oplus M_n$:

(1) *if* M *is relatively* CS-Rickart *to* N *then* M_i *relatively* CS-Rickart *to* N;

(2) *if* M *is relatively* d-CS-Rickart *to* N *then* M_i *relatively* d-CS-Rickart *to* N.

Proof. We only need to prove for the case $n = 2$, $i = 1$.

(1) Assume that $M = M_1 \oplus M_2$ is relatively CS-Rickart to N. There exists $\varphi : M \to N$ such that $\varphi = \psi \oplus 0|_{M_2}$ for each $\psi : M_1 \to N$. By assumption, there exists a direct summand D of M such that $\mathrm{Ker}(\varphi) \leq_e D$. Since $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\psi) \oplus M_2$ and $D = (D \cap M_1) \oplus M_2$, it follows that $\mathrm{Ker}(\psi) \oplus M_2 \leq_e (D \cap M_1) \oplus M_2$. Therefore $\mathrm{Ker}(\psi) \leq_e D \cap M_1$. Since D is a direct summand of M, $D \cap M_1$ is a direct summand of M_1. Hence M_1 is relatively CS-Rickart to N.

(2) Assume that $M = M_1 \oplus M_2$ is relatively d-CS-Rickart to N. There exists $\varphi : M \to N$ such that $\varphi = \psi \oplus 0|_{M_2}$ for each $\psi : M_1 \to N$. By assumption, $\mathrm{Im}\varphi = \mathrm{Im}\psi$ lies above a direct summand of N. Thus, M_1 is relatively d-CS-Rickart to N. □

Proposition 3.2. *The following implications hold for a module* M:

(1) *if* M *is a* SIP-CS *module with* $C2$ *condition and* $M = M_1 \oplus M_2$ *then* M_1 *relatively* CS-Rickart *to* M_2;

(2) *if* M *is a lifting* SSP *module with* $D2$ *condition and* $M = M_1 \oplus M_2$ *then* M_1 *relatively* d -CS-Rickart to M_2.

Proof. Let $f : M_1 \to M_2$ be an R-homomorphism. Then $M = \langle f \rangle \oplus M_2$.

(1) We have that $\mathrm{Ker}(f) = \langle f \rangle \cap M_1 \leq_e eM$ for some $e^2 = e \in S$, by M is SIP-CS. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the canonical projection and hence $eM \cap M_2 = 0$, implies that $\pi_1(eM) \cong eM$. Since M is a $C2$ module, $\pi_1(eM)$ is a direct summand of M. Then, since $\mathrm{Ker}(f) \leq_e eM$, $\mathrm{Ker}(f) = \pi_1(\mathrm{Ker}(f)) \leq_e \pi_1(eM)$. Hence, M_1 is relatively CS-Rickart to M_2.

(2) We have that $\mathrm{Im}(f) \oplus M_1 = \langle f \rangle + M_1$ lies above eM for some $e^2 = e \in S$, by M is lifting SSP. Since $\langle f \rangle + M_1 + M_2 = M$, $eM + M_2 = M$ by [8, 3.2.(1)]. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the canonical projection. Then $\pi_{1|eM}$ splits by $D2$ condition. It follows that $eM = (eM \cap M_2) \oplus N$ by $\mathrm{Ker}(\pi_{1|eM}) = eM \cap M_2$. We have $N \oplus M_2 = N + (eM \cap M_2 + M_2) = eM + M_2 = M$ and obtain that $N \oplus \mathrm{Im}(f) = M_1 \oplus \mathrm{Im}(f) \supset eM$.

By modular law, $eM = N \oplus eM \cap \mathrm{Im}(f)$. As $\dfrac{\mathrm{Im}(f) \oplus M_1}{eM} \ll \dfrac{M}{eM}$, we have that $\dfrac{N \oplus \mathrm{Im}(f)}{N \oplus eM \cap \mathrm{Im}(f)} \ll \dfrac{N \oplus M_2}{N \oplus eM \cap \mathrm{Im}(f)}$. This is equivalent to $\dfrac{\mathrm{Im}(f)}{eM \cap \mathrm{Im}(f)} \ll \dfrac{M_2}{eM \cap \mathrm{Im}(f)}$, which implies that $\mathrm{Im}(f)$ lies above the direct summand $eM \cap \mathrm{Im}(f)$ of M. □

Corollary 3.3. The following implications hold for a module $M = M_1 \oplus \ldots \oplus M_n$:

(1) if M is a SIP-CS module with $C2$ condition then M_i is relatively CS-Rickart to M_j for every $i \neq j$;

(2) if M is a lifting SSP with $D2$ condition then M_i is relatively d-CS-Rickart to M_j for every $i \neq j$.

Proof. If M is a SIP-CS module with $C2$ condition (respectively, lifting SSP with $D2$ condition), then by Proposition 3.2, $\bigoplus_{i\neq j} M_i$ is relatively CS-Rickart to M_j (respectively, relatively d-CS-Rickart to M_j). By Lemma 3.1, M_i is relatively CS-Rickart to M_j (respectively, relatively d-CS-Rickart to M_j) for every $i \neq j$. □

Corollary 3.4. The following implications hold for a module M:

(1) if $M \oplus M$ is a SIP-CS module with $C2$ condition then M is a CS-Rickart module;

(2) if $M \oplus M$ is a lifting SSP with $D2$ condition then M is a d-CS-Rickart module.

Proof. Follow from Corollary 3.3. □

The singular submodule $Z(M)$ of a right R-module M is defined as $Z(M) = \{m \in M : ann^r_R(m)$ is an essential right ideal of $R\}$ where $ann^r_R(m)$ denotes the right annihilator of m in R. The singular submodule of R_R is called the (right) singular ideal of the ring R and is denoted by $Z(R_R)$. It is well known that $Z(R_R)$ is indeed an ideal of R.

Next we give a necessary and sufficient condition for a ring over which every finitely generated projective module to be a SIP-CS-module which is also a $C2$ module.

Theorem 3.5. *The following conditions are equivalent for a ring R:*

(1) *R is a semiregular ring and $J(R) = Z(R_R)$;*

(2) *Every finitely generated projective module is a* CS-Rickart *module which is also a $C2$ module;*

(3) *Every finitely generated projective module is a* SIP-CS *module which is also a $C2$ module;*

(4) *Every finitely generated projective module is a* SIP-CS *module which is also a $C3$ module.*

Proof. (1) ⇒ (2). Follows from [2, Theorem 2].

(2) ⇒ (3). Follows from [2, Proposition 1].

(3) ⇒ (2). Let P be a finitely generated projective module. By the hypothesis, P is a SIP-CS module which is also a $C2$ module. Then $P \oplus P$ is a SIP-CS module which is also a $C2$ module. Since Proposition 3.2, P is relatively CS-Rickart to P, it means that P is a CS-Rickart module.

(3) ⇔ (2). Follows from [1, Corollary 3.5]. □

Lemma 3.6. *The following conditions are equivalent for a module M:*

(1) *M is a* SIP-CS *module;*

(2) *Intersection of every pair of direct summands of M is essential in a direct summand of* M.

Proof. It is obvious. □

Proposition 3.7. *Assume that M is a* SIP-CS *module. Then for any decomposition $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ is a homomorphism, then* Ker(f) *is essential in a direct summand of M.*

Proof. Assume that $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ an R-homomorphism. Call $T = \langle f \rangle$ a submodule of M. So $M = T \oplus M_2$ and Ker$(f) = T \cap M_1$. On the other hand, by the hypothesis, M is a SIP-CS and hence Ker(f) is essential in a direct summand of M by Lemma 3.6. □

Corollary 3.8. Let M be a module and N, a nonsingular module. If $M \oplus N$ is a SIP-CS module, then every homomorphism from M to N has the kernel a direct summand of M.

Proof. Let $f : M \to N$ be a non-zero homomorphism. By Proposition 3.7, Ker(f) is essential in a direct summand of $M \oplus N$. Assume that A is a direct summand of $M \oplus N$ such that Ker$(f) \leq_e A$. Call $\pi_M : M \oplus N \to M$ the canonical projection and $h = (f \circ \pi_M)|_A : A \to N$. Therefore Ker$(h) =$ Ker$(f) \oplus (N \cap A)$. We have that Ker$(f) \leq_e A$ and obtain that Ker$(f) \oplus (N \cap A) \leq_e A$. It follows that Ker$(f) \oplus (N \cap A) = A$. Thus Ker$(f)$ is a direct summand of M. □

Corollary 3.9. Let M be an indecomposable module and N be a nonsingular module. If $M \oplus N$ is a SIP-CS module, then every nonzero homomorphism from M to N is a monomorphism.

Proposition 3.10. *Let M be a nonsingular right R-module. If $(R \oplus M)_R$ is a* SIP-CS *module, then every cyclic submodule of M is projective.*

Proof. Let m be a non-zero arbitrary element of M. Call the homomorphism $\varphi : R_R \to M$ given by $\varphi(x) = mx$. As $(R \oplus M)_R$ is a SIP-CS module, Ker(φ) is a direct summand of R_R by Corollary 3.8. It follows that Im(φ) is isomorphic to a direct summand of R_R. Thus mR is a projective module. □

A ring R is called right *(semi)hereditary* if every (finitely generated) right ideal of R is projective.

Theorem 3.11. *The following statements are equivalent for a right nonsingular ring R:*

(1) *R is right hereditary;*

(2) *Every projective right R-module is a* SIP-CS *module.*

Proof. (1) ⇒ (2) is obvious.

(2) $\Rightarrow$ (1) Let I be a right ideal of R. We will show that I is a projective module. Call an epimorphism $\varphi : F \to N$ for some free right R-module F. Let ι be the inclusion map from I to R_R. Consider the homomorphism $\iota \circ \varphi : F \to R_R$. By (2), $F \oplus R_R$ is a SIP-CS module. We have from Corollary 3.8, $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\iota \circ \varphi)$ is a direct summand F. This gives that $F = \mathrm{Ker}(\varphi) \oplus B$ for some submodule B of F. Thus, I is projective. □

The author Warfied proved that if R is right serial, then R is right nonsingular if and only if R is right semihereditary.

The same argument of the proof of Theorem 3.11, we also have the following result of semihereditary rings:

Theorem 3.12. *The following statements are equivalent for a right nonsingular ring R:*

(1) *R is right semihereditary;*

(2) *Every finitely generated projective right R-module is a* SIP-CS *module;*

(3) *Every finitely generated free right R-module is a* SIP-CS *module.*

Let M be a right R-module and $S = \mathrm{End}(M)$. We denote

$$\Delta(S) = \{f \in S \mid \mathrm{Ker}(f) \leq_e M\}.$$

An R-module is called a *self-generator* if it generates all its submodules.

Theorem 3.13. *The following conditions are equivalent for a self-generator module M with $S = \mathrm{End}(M)$:*

(1) *S is a semiregular ring with $J(S) = \Delta(S)$;*

(2) *M is a* CS-Rickart *and* C2 *module.*

Proof. (1) $\Rightarrow$ (2) Assume that S is a semiregular ring with $J(S) = \Delta(S)$. As M is a self-generator, $J(S) = \Delta(S) \leq Z(S_S)$. We deduce that S is right C2. This gives that M is a C2-module by [20, Theorem 7.14(1)]. Let $\alpha : M \to M$ be an endomorphism of M. As S is a semiregular ring, there exists $\beta \in S$ such that $\beta = \beta\alpha\beta$ and $\alpha - \alpha\beta\alpha \in J(S)$ by [19, Theorem 2.9]Ni. Call $e = 1 - \beta\alpha$. Then $e^2 = e \in S$. As $\alpha - \alpha\beta\alpha \in \Delta(S)$, $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \leq_e M$ and hence $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \cap e(P) \leq_e e(M)$. It is easily to check that $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \cap e(M) = \mathrm{Ker}(\alpha)$. We deduce that $\mathrm{Ker}(\alpha) \leq_e e(M)$.

(2) $\Rightarrow$ (1) By [21, Theorem 3.2]. □

Corollary 3.14. The following conditions are equivalent for a self-generator module M with $S = \mathrm{End}(M^{(\mathbb{N})})$:

(1) S is a semiregular ring with $J(S) = \Delta(S)$;

(2) $M^{(\mathbb{N})}$ is a CS-Rickart and C2 module;

(3) $M^{(\mathbb{N})}$ is a SIP-CS and C2 module.

*Proof.*By Proposition 3.2 and Theorem 3.13. □

ACKNOWLEDGMENTS

The third author was supported by the Russian Government Program of Competitive Growth of Kazan Federal University. The author would like to thank the members of Department of Algebra and Mathematical Logic, Kazan (Volga Region) Federal University for their hospitality.

REFERENCES

1. A. N. Abyzov and A. A. Tuganbaev, “Modules in which sums or intersections of two direct summands are direct summands,” Fundam. Prikl. Mat. **19**, 3–11 (2014).
2. A. N. Abyzov and T. H. N. Nhan, “CS-Rickart modules,” Lobachevskii J. Math. **35**, 317–326 (2014).
3. M. Alkan and A. Harmanci, “On summand sum and summand intersection property of modules,” Turk. J. Math. **26**, 131–147 (2002).
4. I. Amin, Y. Ibrahim, and M. F. Yousif, “D3-modules,” Commun. Algebra **42** (2), 578–592 (2014).
5. I. Amin, Y. Ibrahim, and M. F. Yousif, “C3-modules,” Algebra Colloq. **22**, 655-670 (2015).
6. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules* (Springer, New York, 1974).
7. V. Camillo, Y. Ibrahim, M. Yousif, and Y. Zhou, “Simple-direct-injective modules,” J. Algebra **420**, 39–53 (2014).
8. J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules. Supplements and Projectivity in Module Theory*, Frontiers in Mathematics (Birkhäuser, Basel, Boston, Berlin, 2006).
9. N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules*, Vol. 313 of *Pitman Research Notes in Mathematics* (Longman Sci. and Tech., Harlow, New York, 1994).
10. E. E. Enochs, “Injective and flat covers, envelopes and resolvents,” Israel J. Math. **39**, 189–209 (1981).
11. C. Faith, *Algebra II. Ring Theory* (Springer, New York, 1967).
12. J. L. Garcia, “Properties of direct summands of modules,” Commun. Algebra **17**, 73–92 (1989).
13. A. Hamdouni, A. Harmanci, and A. Ç. Özcan, “Characterization of modules and rings by the summand intersection property and the summand sum property,” JP J. Algebra, Number Theory Appl. **5**, 469–490 (2005).
14. J. Hausen, “Modules with the summand intersection property,” Commun. Algebra **17**, 135–148 (1989).
15. Y. Ibrahim, M. Tamer Kossan, T. C. Quynh, and M. Yousif, “Simple-direct-projective modules,” Comm. Algebra **44**, 5163–5178 (2016).
16. G. Lee, S. T. Rizvi, and C. S. Roman, “Rickart modules,” Commun. Algebra **38**, 4005–4027 (2010).
17. G. Lee, S. T. Rizvi, and C. S. Roman, “Direct sums of Rickart modules,” J. Algebra **353**, 62–78 (2012).
18. S. H. Mohammed and B. J. Müller, *Continous and Discrete Modules,* Vol. 147 of *London Math. Soc. Lecture Notes Series* (Cambridge Univ. Press, Cambridge, 1990).
19. W. K. Nicholson, “Semiregular modules and rings,” Canad. J. Math. **28**, 1105–1120 (1976).
20. W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings* (Cambridge Univ. Press, Cambridge, 2003).
21. T. C. Quynh, M. T. Kosan, and L. V. Thuyet, “On (semi)regular morphisms,” Commun. Algebra **41**, 2933–2947 (2013).
22. G. V. Wilson, “Modules with the summand intersection property,” Commun. Algebra **14**, 21–38 (1986).
23. R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach, London, 1991).