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Concerning the Theory of τ-Measurable Operators Affiliated to a Semifinite von Neumann Algebra

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Abstract—Let $\mathscr{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathscr{H}$, let τ be an exact normal semifinite trace on $\mathscr{M}$, and let $L_1(\mathscr{M},\tau)$ be the Banach space of τ-integrable operators. The following results are obtained. If $X = X^*$, $Y = Y^*$ are τ-measurable operators and $XY \in L_1(\mathscr{M},\tau)$, then $YX \in L_1(\mathscr{M},\tau)$ and $\tau(XY) = \tau(YX) \in \mathbb{R}$. In particular, if $X, Y \in \mathscr{B}(\mathscr{H})^{\mathrm{sa}}$ and $XY \in \mathfrak{S}_1$, then $YX \in \mathfrak{S}_1$ and $\operatorname{tr}(XY) = \operatorname{tr}(YX) \in \mathbb{R}$. If $X \in L_1(\mathscr{M},\tau)$, then $\tau(X^*) = \overline{\tau(X)}$. Let A be a τ-measurable operator. If the operator A is τ-compact and $V \in \mathscr{M}$ is a contraction, then it follows from $V^*AV = A$ that $VA = AV$. We have $A = A^2$ if and only if $A = |A^*||A|$. This representation is also new for bounded idempotents in $\mathscr{H}$. If $A = A^2 \in L_1(\mathscr{M},\tau)$, then $\tau(A) = \tau(\sqrt{|A|}\,|A^*|\sqrt{|A|}) \in \mathbb{R}^+$. If $A = A^2$ and A (or A^*) is semihyponormal, then A is normal, thus A is a projection. If $A = A^3$ and A is hyponormal or cohyponormal, then A is normal, and thus $A = A^* \in \mathscr{M}$ is the difference of two mutually orthogonal projections $(A + A^2)/2$ and $(A^2 - A)/2$. If $A, A^2 \in L_1(\mathscr{M},\tau)$ and $A = A^3$, then $\tau(A) \in \mathbb{R}$.

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1. INTRODUCTION

Let $\mathscr{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathscr{H}$, let τ be an exact normal semifinite trace on $\mathscr{M}$, and let $L_1(\mathscr{M},\tau)$ be the Banach space of τ-integrable operators. In this paper, we obtain the following results on the algebraic and order properties of the trace τ and the elements of the $*$-algebra $\widetilde{\mathscr{M}}$ of all τ-measurable operators.

If $X, Y \in \widetilde{\mathscr{M}}^{\mathrm{sa}}$ and $XY \in L_1(\mathscr{M},\tau)$, then

$$YX \in L_1(\mathscr{M},\tau) \qquad \text{and} \qquad \tau(XY) = \tau(YX) \in \mathbb{R}$$

(Theorem 3.1). In particular, if $X, Y \in \mathscr{B}(\mathscr{H})^{\mathrm{sa}}$ and $XY \in \mathfrak{S}_1$, then

$$YX \in \mathfrak{S}_1 \qquad \text{and} \qquad \operatorname{tr}(XY) = \operatorname{tr}(YX) \in \mathbb{R}.$$

If $X \in L_1(\mathscr{M},\tau)$, then

$$\tau(X^*) = \overline{\tau(X)}$$

(Theorem 3.3). If the operator A is τ-compact and $V \in \mathscr{M}$ is a contraction, then it follows from $V^*AV = A$ that

$$VA = AV$$

(Theorem 3.4). An example of an unbounded operator $A \in \widetilde{\mathscr{M}}$ with $A = A^2$ is given (Example 4.2).

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