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CUTTING METHODS WITH UPDATING APPROXIMATION SETS FOR SOLVING NONLINEAR PROGRAMMING PROBLEMS

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GENERAL DESCRIPTION OF THE DISSERTATION

Relevance of the dissertation's topic. Optimization problems are solved in practice very often. Therefore, experts in the field of mathematical programming pay attention to the development of conditional minimization methods with convenient numerical implementation.

The class of cutting methods is famous and rather wide. A lot of cutting methods for solving nonlinear programming problem are based on the following fact. In these methods the feasible set of the initial problem or the epigraph of the objective function are successively embedded into some sets with a simpler structure to construct iteration points. Each of the approximation sets is constructed from the previous one by cutting (usually by planes) of some subset which containts the current iteration points.

A main problem that arises in numerical implementation of such methods is as follows. The number of cutting planes which are formed approximation sets indefinitely increases from step to step. Therefore, the complexity of finding iteration points grows with the growth of the step number. This lack of the mentioned methods is significant in case of solving even small dimensional problem with high accuracy. Thus, the dissertation researches connected with improving famous cutting methods and developing new methods of the mentioned class being free from this lack are relevant.

There are some difficults of applying the famous cutting methods approximating either the feasible set or the epigraph of the function to solve practical problems. If the objective function of the initial convex programming problem is nonlinear and the feasible set is also determined by nonlinear functions, then in these methods the complexity of solving auxiliary problems for constructing iteration points is similar to the laboriousness of solving the initial one. Consequently, the research results from one of the dissertation's chapter are relevant for constructing cutting methods which allow to solve a linear programming problem under construction iteration points independently of the type of the initial convex programming problem.

Further, mixed algorithms are used to accelerate the process of solving practice minimization problems very often. During constrcuting of such algorithms it is necessary to investigate their convegence while the methods determined these algorithms are convergence. Thus, it is useful to develop techniques which admit to construct obviously convergent mixed algorithms on the basis of some class of methods. In the dissertation one of such techniques is proposed for constrcuting convergent mixed algorithms in terms of the cutting methods.

At last time specialist in the mathematical programming field pay attention to explore opportunity of using parallel computations in optimization methods. Opportunity of applying parallel computaions is proved in the developed cutting methods.

Purpose of the dissertation consists in constructing cutting-plane methods with updating approximation sets and easily implemented algorithms for solving nonlinear programming problems.

Research tasks of the dissertaion. Propose evaluation criteria of the estimating the quality of approximating sets and in terms of these criteria develop an approach to construct cutting methods with convenient numerical algorithms, approximating the epigraph or the feasible set and with possibility of periodically discarding accumulated cutting planes.

Research methods are based on the theory of nonlinear programming and applying of generalized support vectors, deal with designed in the dissertation techniques of the estimating the quality of approximation sets and developed on these techniques approaches of updating approximation sets.

Scientific novelty. The approach is proposed on the basis of the developed evaluation criteria of the estimating the quality of approximation sets to contruct new cutting-plane methods which can use different procedures of updating embedding sets. The new techique is developed to prove convergence of these methods. Develop the cutting methods using approximation of the epigraph of the objective function or the feasible set simulteniously. On the basis of developed cutting methods the new technique is proposed for constructing mixed convergence algorithms using any relaxation convex programming methods. The ways of using parallel computations is also new for finding iteration points in the methods of the researched class.

Theoretical and practical significance. The proposed approach of obtaining cutting methods with updating approximation sets and the developed technique of proving thier convergence can be used to construct new cutting methods or to improve the famous ones. This technique can be used in case of constructing other criteria of approximation quality which differ from ones proposed in the dissertaion. The proposed approach of constructing mixed algorithms allows to obtain new algorithms on the basis of other classes of optimization methods.

The research results also have practical significance. The proposed principle of updating approximation sets is important for the cutting methods from the practical point. This one guarantees performability of cutting methods, because the problem of infinitely increasing the number of cutting planes is solved. Numerical results has confirmed practical significance of these principles of updating.

The solution time of each tests solved by the proposed methods using procedures of updating is substantially less than the decision time of each examples settled by the same methods without applying procedures of updating. On the basis of the numerical experimets the developed methods have advantages over some famous methods of the same class.

Approbation of the dissertation. Research results are discussed and reported at the final scientific conference of KFU in 2011–2014, the All-Russian conference «Statistika. Modelirovanie. Optimizacija (in Russian)» (Chelyabinsk, November 28 – December 3, 2011), the 14th All-Russian conference «Matematicheskoe programmirovanie i prilozhenija (in Russian)» (Ekaterinburg, February 28 – March 4, 2011), the final scientific and educational conference of Kazan university students in 2011 - 2012, the 5th All-Russian conference «Problemy optimizacii i jekonomicheskie prilozhenija (in Russian)» (Omsk, July 2 – 6, 2012), the republican N.I. Lobachevsky award competitions of scientific works of students and post-graduates (Kazan, Regional Youth Public Organization «Student League of the Republic of Tatarstan», April 9 – 12, 2013, April 9 – 12, 2014), the 37th, 38th All-Russian youth conferences «Informacionnye tehnologii i sistemy (in Russian)» (Kaliningrad, September 1 – 6, 2013, Nizhny Novgorod, September 1 – 5, 2014), the 16th All-Russian conference «Matematicheskie metody raspoznavanija obrazov (in Russian)» (Kazan, October 6 – 10, 2013), the International conference «Discrete Optimization and Operations Research (DOOR–2013)» (Novosibirsk, June 24 – 28, 2013), the 17th International conference «Problemy teoreticheskoj kibernetiki (in Russian)» (Kazan, June 16 – 20, 2014), the 15th, 16th Baikal International Triannual School-Seminars «Methods of Optimization and Their Applications» (Baikal, June 23 – 29, 2011, June 30 – July 6, 2014), the 9th All-Russian and the 10th International conferences «Mesh Methods for Boundary-Value Problems and Applications» (Kazan, September 17 – 22, 2013, September 24 – 29, 2014).

Publications. The main results of the dissertaion have been published at 21 works, where 4 of these are in the journals from the VAK list, and one of these works is in the journal from SCOPUS.

Structure and volume of the dissertation. The dissertaion contains 147 pages and consists of the Introduction, 3 chapters, the Conclusions and the Reference included 163 titles.

Main results. In the dissertaion the following results are obtained:

1) approach is proposed for constructing cutting methods to solve nonlinear programming problems which allows to include different procedures of updating approximation sets into these methods to periodically dropping accumulated cutting planes. The mentioned approach is based on the developed in the work evaluation

criteria of estimating the quality of approximation sets in the neighborhood of iteration points;

2) two cutting methods with approximation of the feasible set, four cutting methods using approximation of the epigraph of the objective function and the projection method of the point onto the convex set are developed on the basis of the approach for solving conditional minimization problems given in the form by different conditions of defining the objective function and the feasible set. There are opportunities of updating approximation sets in these methods by dropping arbitrary number of any constructed cutting planes;

3) for solving convex programming problems two cutting methods are constructed which use approximation of the feasible set and the epigraph of the objective function simulteneously for obtaining iteration points. Methods are characterized by the following fact. During approximation of the epigraph of the objective function and the constraint set by polyhedral sets linear programming problem is solved at each iteration. The principle of updating approximation sets is included into one of these methods. This principle is based on the simulteneous estimation of the approximation quality of the epigraph and the approximation quality of the feasible set;

4) for most cutting cutting methods in view of some additional condtions defined for the objective function and constraint functions accuracy evaluations of the solution and rates of convergence are obtained for the sequence of approximation;

5) the technique of constructing mixed conditional minimization algorithms is proposed on basis of developed methods and any relaxation convex programming methods. Convergence of such algorithms is guaranteed by convergence of the developed cutting methods. These algorithms also allow to update approximation sets;

6) it is represented that parallel computations can be used to construct iteration points in the proposed methods.

All of the main research results are submitted for protection.

REVIEW OF THE DISSERTATION

The introduction substantiates relevance of the performed researches, demonstrates scientific novelty of the obtained results, there are considerations of developed approaches from which to construct cutting methods, and the content of the dissertation is briefly discussed.

The first part deals with the developed cutting methods which are based on the approximation of the feasible set to solve nonlinear programming problems and represents a method for finding projection of the point on the convex set.

In § 1.1 propose a cutting method to solve the problem

$$\min\{f(x) : x \in D\}, \tag{1}$$

where $f(x)$ is a continuous function defined in an n-dimensional Euclidean space $\mathrm{R_n}$, $D = \{x \in \mathrm{R_n} : f_j(x) \le 0,\ j \in J\}$, $f_j(x)$, $j \in J = \{1, \ldots, m\}$, are convex functions in $\mathrm{R_n}$. Suppose that for each $j \in J$ the set $D_j = \{x \in \mathrm{R_n} : f_j(x) \le 0\}$ has a nonempty interior int D_j.

Define $F(x) = \max\{f_j(x) : j \in J\}$, $D_\varepsilon = \{x \in \mathrm{R_n} : F(x) \le \varepsilon\}$, where $\varepsilon \ge 0$, $K = \{0, 1, \ldots\}$, $W(x, D_j) = \{a \in \mathrm{R_n} : \langle a, z - x\rangle \le 0\ \forall z \in D_j, ||a|| = 1\}$.

The proposed method for solving problem (1) constructs sequences of approximations y_i, $i \in K$, x_k, $k \in K$, by the following rule.

Method 1.1. Construct a convex closed set $M_0 \subset \mathrm{R_n}$ containing a point $x^* = \arg\min\{f(x) : x \in D\}$. Select points $v^j \in \operatorname{int} D_j$ for each $j \in J$. Determine a number $\varepsilon_0 \ge 0$. Put $k = 0$, $i = 0$.

1. Find a point $y_i \in M_i \bigcap \{x \in \mathrm{R_n} : f(x) \le f(x^*)\}$, set $J_i = \{j \in J : y_i \notin D_j\}$. If $J_i = \emptyset$, then y_i is a decision of the problem.

2. For each $j \in J_i$ find a point $z_i^j \notin \operatorname{int} D_j$ in some ways which belongs to the interval (v^j, y_i). Choose some subset $H_i \subset J_i$.

3. For each $j \in H_i$ select a finited set $A_i^j \subset W(z_i^j, D_j)$ and put $M_{i+1} = Q_i \bigcap\limits_{j \in H_i} \{x \in \mathrm{R_n} : \langle a, x - z_i^j\rangle \le 0\ \forall a \in A_i^j\}$. The set Q_i is constructed as follows. If $y_i \notin D_{\varepsilon_k}$, then $Q_i = M_i$. Otherwise, put $i_k = i$, $x_k = y_{i_k}$ and select the convex closed set $Q_i = Q_{i_k}$ such that $x^* \in Q_i$.

4. Define a number $\varepsilon_{k+1} \ge 0$, the value of k is increased by one. Further, go to Step 1 and increment the number i.

The approximation quality for the feasible set is determined with the sets M_i in neighborhood of points y_i by the condition $y_i \in D_{\varepsilon_k}$. At some step $i = i_k$, where the mentioned condition is defined for y_i, the approximation quality is considered to be sufficient to fix the point x_k, and the approximation set M_{i_k+1} can be updated due to the arbitrary choice of the set $Q_i = Q_{i_k}$. On iterations on which the condition $y_i \notin D_{\varepsilon_k}$ is fulfilled for the points y_i the approximation quality is considered to be unsatisfactory for updating, i. e. cutting planes will be accumulated to the iteration $i = i_k$.

There are several procedures for updating embedding sets M_i in method 1.1. Let $y_i \in D_{\varepsilon_k}$. Put $Q_i = M_{r_i}$, where $0 \le r_i \le i = i_k$. Then the inclusion $x^* \in Q_i$ is defined for all $r_i = 0, \ldots, i$, and the set Q_{i_k} can be formed on the basis of any sets $M_0, \ldots, M_{i_k}$ constructed for the step i_k. In particular, it is possibly to determine $Q_i = M_0$, thus in case of $i = i_k$, $k \in K$, the approximation set M_{i_k+1} is fully updated.

In case of choosing positive numbers ε_k it is proved there exists a number $i = i_k$ such that the inclusion $y_i \in D_{\varepsilon_k}$ is fulfilled. Consequently, there are some possibilities of updating embedding sets.

Theorem 1. Let a sequence $\{x_k\}$, $k \in K$, be constructed by the proposed method, and let numbers ε_k, $k \in K$, be such that $\varepsilon_k > 0$ for all $k \in K$ and $\varepsilon_k \to 0$ as $k \to \infty$. Then any limit point of this sequence belongs to the solution set of problem (1), and if inequalities $f(x_{k+1}) \geq f(x_k)$ are fulfilled for all $k \in K$, then the whole sequence $\{x_k\}$, $k \in K$, converges to the solution set.

If $f(x)$ is a convex function, $f_j(x)$, $j \in J$, are strongly convex functions with strong convexity constants μ_j respectively, the set D satisfies Slater condition, and the absolute minimum point of the function $f(X)$ does not belong to the set int D in case of this one exists, then evaluation of the accuracy of the solution $||x_k - x^*|| \leq \delta_k$, where $\delta_k = \sqrt{\varepsilon_k/\mu}$, is obtained. If the function $f(x)$ also satisfies Slater condition, then the evaluation $|f(x_k) - f(x^*)| \leq L\delta_k$ is determined for each $k \in K$. Moreover, if $\varepsilon_k = 1/k^p$, where $p > 0$, then for the sequence $\{x_k\}$, $k \in K$, rates of convergence $||x_k - x^*|| \leq 1/(\mu k)^{p/2}$, $|f(x_k) - f(x^*)| \leq L/(\mu k)^{p/2}$ are obtained on the basis of evaluations given above.

Further, a cutting method is proposed in § 1.2 for solving problem (1) given by the constraint set with the following form:

$$D = D' \bigcap D'', \tag{2}$$

where D', D'' are convex closed sets in $\mathrm{R_n}$, in addition, the second one may have an empty interior.

The proposed cutting method differs from Method 1.1 as follows. This method use approximation of the set D' containing the feasible set rather than the set D. The method is also characterized by ways of constructing approximation sets. Namely, the embedding set M_{i+1} is given on the basis of hyperplanes which cut the point y_i from M_i. These hyperplanes are constructed by subgradients of the functions defined the set D'. Moreover, in this method there are greater opportunities to select points of the main sequence. In view of these opportunities mixed algorithms can be constructed in terms of the method developed in § 1.2, and convergence of such mixed algorithms is provided by convergence of the proposed method. Opportunity of periodically updating approximation sets is typical of the method from § 1.2. These updates occurs in such iterations when approximation quality of the set is considered to be satisfactory.

Considering additional assumptions for the objective function and functions defined the set D' and in case of $D'' = \mathrm{R_n}$ evaluation of the accuracy of the solution and the rate of convergence are obtainded. These estimates are similar to those given in § 1.1.

In § 1.3 the cutting method is proposed for finding projection of the point $y \in \mathrm{R_n}$ onto the convex set D given by form (2). At each iteration polyhedral set approximating the region D' is constructed by some way, and the next approximation is obtained by projection of the point y onto the intersection of this constructed set with the set D''. If the set D'' is determined by some system of the linear inequalities or equalities, then there is no difficulties to obtain approximations, because the projection problem is solved by the famous algorithms in a finite number of steps.

In § 1.4 for solving problem (1) another cutting method using approximation of the feasible set is constructed. The methos differs from the methods from §§ 1.1, 1.2 in opportunity of using use parallel computions for finding appromixations. In this method the process of obtaining the main iteration points is as follows. Firstly, several polyhedral sets are constructed which approximates the constraint region of the initial problem. Further, some auxiliary points are obtained as a approximate minimum of the objective function in each these sets. At last, the next main iteration point is constructed on the basis of the auxiliary points as record one. Note that these auxiliary points can be found in parallel by different minimization algorithms.

In the second chapter cutting methods are developed for solving convex programming problem. The proposed methods use embedding process not of the feasible set, but of the epigraph of the objective function of the initial problem. These methods are similar to the methods from chapter 1 in possibility of updating embedding sets on account of discarding cutting planes. All methods from this chapter allow to obtain the proximity of the objective function value at the current iteration point to the optimal value. Two evaluation criteria of the estimating the quality of approximation sets are proposed in the neighborhood of iteration points. These criteria allow to include some procedures of updating embedding sets into the developed methods. The first criterion is obtained on the basis of the assessment proximity of the current iteration values to the optimal value, the second one is based on the assessment proximity of the iteration points to the epigraph of the objective function. Convergence of the proposed methods are proved, their implementations are discussed, approaches of periodically dropping arbitrary number of any cutting planes which are obtained in the solution process are proposed, evaluations of the accuracy of the solution and rates of convergence are represented.

The first paragraph of the second chapter treats the method for solving a minimization problem given in the form by the convex discrete maximum function $f(x)$ defined in $\mathrm{R_n}$ and the convex closed set $D \subset \mathrm{R_n}$. The developed method is as follows. At the ith iteration ($i \geq 0$) find a point (y_i, γ_i) as a solution of the following problem:

$$\min\{\gamma : x \in G_i, \quad (x, \gamma) \in M_i, \quad \gamma \geq \bar{\gamma}_i\}, \tag{3}$$

where M_i is a set approximating the epigraph of the objective function, G_i is some subset of the region D, $\bar{\gamma}_i \leq f^* = \min\{f(x) : x \in D\}$. If the difference $f(y_i) - \gamma_i$ is quite small, then the approximation quality of the epigraph is considered to be satisfactory with the set M_i. In this case y_i is fixed as the point of the main approximation sequence, and the set M_i is updated. Otherwise, the next approximation set M_i is constructed by cutting off the point (y_i, γ_i) from the set M_i.

Note that there are a lot of ways of choosing approximation sets M_0, G_i and the number $\bar{\gamma}_i$. Also due to the type of the objective function the method has significant opportunities for constructing cutting planes formed approximation sets.

In paragraph 2.2 a cutting method is proposed for solving problem (1) with the convex function $f(x)$ defined on convex set D.

Let $\mathrm{epi}\,(f, G) = \{(x, \gamma) \in \mathrm{R_{n+1}} : x \in G, \gamma \geq f(x)\}$, where $G \subset \mathrm{R_n}$, $W(u, Q) = \{a \in \mathrm{R_{n+1}} : \langle a, z - u\rangle \leq 0 \ \forall z \in Q, ||a|| = 1\}$, where $Q \subset \mathrm{R_{n+1}}$.

The proposed method constructs sequences $\{y_i\}$, $i \in K$, $\{x_k\}$, $k \in K$, and works as follows.

Method 2.2. Choose points $v^j \in \mathrm{int\,epi}\,(f, \mathrm{R_n})$ for all $j \in J$, convex bounded closed set $G_0 \subset D$ containing a point $x^* = \arg\min\{f(x) : x \in D\}$, convex closed set $M_0 \subset \mathrm{R_{n+1}}$ such that $(x^*, f(x^*)) \in M_0$. Define $\varepsilon_0 > 0$, $\bar{\gamma}_0 \leq f(x^*)$ and set $i = 0$, $k = 0$.

1. Find a solution (y_i, γ_i) of problem (3), where $y_i \in \mathrm{R_n}$, $\gamma_i \in \mathrm{R_1}$. If $f(y_i) = \gamma_i$, then the point y_i is a solution of the initial problem.

2. If $f(y_i) - \gamma_i > \varepsilon_k$, then set $Q_i = M_i$, $u_i = y_i$, and Step 4 is performed. Otherwise, go to Step 3.

3. Choose a convex closed set $Q_i \subset \mathrm{R_{n+1}}$ such that the inclusion $(x^*, f(x^*)) \in Q_i$ is fulfilled. Determine a point $x_k \in \mathrm{R_n}$ in accordance with conditions $x_k \in G_i$, $f(x_k) \leq f(y_i)$. Set $i_k = i$, $\sigma_k = \gamma_{i_k}$, $u_i = u_{i_k} = x_k$, define $\varepsilon_{k+1} > 0$, the value of k is increased by one.

4. For each $j \in J$ find a point $z_i^j \notin \mathrm{int\,epi}\,(f, \mathrm{R_n})$ belonging to the interval $(v^j, (u_i, \gamma_i))$ in some ways.

5. For each $j \in J$ choose a finite set $A_i^j \subset W(z_i^j, \mathrm{epi}\ (f, \mathrm{R_n}))$ and determine

$$M_{i+1} = Q_i \bigcap T_i,$$

where $T_i = \bigcap\limits_{j \in J} \{(x, \gamma) \in \mathrm{R_{n+1}} : \langle a, (x, \gamma) - z_i^j\rangle \leq 0 \ \forall a \in A_i^j\}$.

6. Select a convex closed set $G_{i+1} \subset G_0$ containing the point x^*. Define a number $\bar{\gamma}_{i+1}$ satisfying the condition $\bar{\gamma}_0 \leq \bar{\gamma}_{i+1} \leq f^*$. The value of i is increased by one and go to Step 1.

The approximation quality for the epigraph of the function is defined with sets M_i in the neighborhood of points x_i by the value $f(y_i) - \gamma_i$. On iterations when y_i satisfies the inequality $f(y_i) - \gamma_i > \varepsilon_k$, approximation quality is considered to be insufficient to fix the point x_k. On such iterations the sets Q_i are determined by the equality $Q_i = M_i$, and cutting planes are accumulated during constructing the set M_{i+1}. At some step $i = i_k$ on which $f(y_i) - \gamma_i \leq \varepsilon_k$ approximation quality becomes satisfactory, the point x_k is fixed according to the condition $f(x_k) \leq f(y_{i_k})$, and due to the virtually arbitrary choice of the set $Q_i = Q_{i_k}$ there is some opportunity to update the approximation set M_{i_k+1}.

Various techniques of updating approximation sets can be developed on the basis of criteria for assessing the approximation quality for epigraph. Suppose that the inequality $f(y_i) - \gamma_i \leq \varepsilon_k$ is fulfilled for the point (y_i, γ_i). In this case according to Step 3 of Method 2.2 the set $Q_i = Q_{i_k}$ is determined in terms of the inclusion $(x^*, f(x^*)) \in Q_i$. Put, for example, $Q_i = \mathrm{R}_{\mathrm{n}+1}$ or $Q_i = M_0$. Then it follows that $M_{i+1} = T_i$ or $M_{i+1} = M_0 \bigcap T_i$ respectively, and all cutting planes obtained to the current iteration are not used in construction of the set M_{i+1}. If $f(y_i) - \gamma_i \leq \varepsilon_k$ is fulfilled, then the set $Q_i = Q_{i_k}$ can be given by $Q_i = M_{r_i}$, where $0 < r_i \leq i = i_k$. Consequently, during forming the set M_{i+1} there is possibility of discarding only some part of cutting planes accumulated to the step $i = i_k$.

It is proved that for all $k \in K$ there exists a number $i = i_k$ such that the inequality $f(y_i) - \gamma_i \leq \varepsilon_k$ is fulfilled, consequently, there is opportunity to update an approximation set.

Theorem 2. Let a sequence $\{(x_k, \sigma_k)\}$, $k \in K$, be constructed by the method under the assumption that $\varepsilon_k \to 0$, $k \to \infty$. Then $\lim\limits_{k \in K} f(x_k) = f(x^*)$, $\lim\limits_{k \in K} \sigma_k = f(x^*)$.

Selection conditions of x_k defined at Step 3 allow to apply a lot of ways for constructing these points x_k. Firstly, these conditions permit to use various famous relaxation methods of conditional minimization for finding the points x_k. Secondly, there are some opportunities of developing varied mixed algorithms on the basis of the Method 2.2. For example, considering y_{i_k} as the starting point, the famous gradient or subgradient methods can be used within certain steps to minimize the function $f(x)$ on the set G_{i_k}. Thus, a new algorithm has been constructed in terms of the proposed cutting method, and, moreover, convergence of such an algorithm is proved in accordance with Theorem 2.

The generality of the conditions for obtaining x_k also allow to apply parallel computations in the Method 2.2. Namely, some auxiliary points $w_k^1, \dots, w_k^l$, $l \geq 1$, can be constructed such that $w_k^j \in G_{i_k}$, $f(w_k^j) < f(y_{i_k})$, $j = 1, \dots, l$ in parallel at the iteration $i = i_k$ by any algorithms, and it is possible to obtain the sought-for point x_k by $f(x_k) = \min\{f(w_k^1), \dots, f(w_k^l)\}$.

In § 2.3 the improved level-set method is proposed for solving problem (1), where $f(x)$ is a convex function, D is a convex bounded closed set. The proposed method is largely different from the famous one in opportunity of periodically dropping cutting planes accumulated in the solution process. As a result this feature makes the developed modification attractive from the practival viewpoint. The proposed method also differs from the famous method in more general way of constructing iteration points. In particular, there is opportunities of using parallel computations in constructing approximations.

In § 2.4 another cutting method is developed on basis of way represented in § 2.2 of constructing approximation sets for solving problem (1) with convex objective function. For obtaining iteration points approximation sets M_i do not embed the epigraph the objective function $f(x)$, but the epigraph of some auxiliary function $g(x)$. This method is greatly different from the previous methods in a new evaliation criterion of estimating the approximation quality of the set M_i, consequently, in the criterion of dropping cutting planes too. Namely, if the current iteration point from M_i is quite closeness to the epigraph of the function $g(x)$, then the approximation quality is considered to be satisfactory, and update of the set M_i occurs.

The method from § 2.4 allows to find a ε-solution of problem (1). If after finding ε-solution the process of constructing approximations is not stopped, then any limit point of the iteration sequence will belong to the solution set. Note that the proposed method as the previous ones allows to contruct mixed algorithms by using other famous conditional minimization methods. Moreover, parallel computations can be also used in this method for constructing iteration points.

In chapters 1, 2 the developed cutting methods use approximation of either the feasible set or the epigraph of the function to construct iteration points. In the third chapter the new cutting methods are proposed for solving convex programming problem. In these methods iteration points are constructed by applying simulteneous approximation of the feasible set and the epigraph of the objective function. In case of using polyhedral sets for approximating the mentioned sets the auxiliary linear programing problem is solved to construct an iteration point at each step. Discuss implementations of the proposed methods and prove their convergence.

In § 3.1 a general cutting method is proposed for solving problem (1), where $f(x)$

is a convex function. The method constructs a sequence of approximations by the following rule. At the ith iteration find a solution (y_i, γ_i) of the problem

$$\min\{\gamma : (x, \gamma) \in M_i, \quad x \in G_i, \quad \gamma \geq \alpha\},$$

where $M_i \subset \mathrm{R_{n+1}}$ is a set approximated the epigraph of the objective function, $G_i \subset \mathrm{R_n}$ is a set approximated the constraint region in the neighborhood of the solution of the initial problem, and α is a lower bound of the value f^*. The component y_i of the found solution is considered to be a current iteration point. Further, the following approximation sets M_{i+1}, G_{i+1} are constructed. The first one is formed by cutting off the point (y_i, γ_i) from the set M_i, and in case of $y_i \notin D$ the second one is determined on the basis of cutting the approximation y_i from G_i. In the proposed method cutting planes are constructed in terms of generalized-support elements for the epigraph of the objective function or for the feasible set. In this dissertation's part various approaches of constructing cutting planes are proposed, and there are discussions of various ways of determining initial approximating sets.

Paragraph 3.2 deals with a cutting method based on the idea of simultaneous approximation for the epigraph of the objective function and the constraint region. The proposed method allows to update approximating sets and this one is developed for solving convex programming problem (1) with the feasible set D defined in form (2), where $D' = \{x \in \mathrm{R_n} : F(x) \leq 0\}$, and $F(x)$ is a convex function in $\mathrm{R_n}$.

Let $D'_\varepsilon = \{x \in \mathrm{R_n} : F(x) \leq \varepsilon\}$, $f^* = \min\{f(x) : x \in D\}$, $X^*(\varepsilon) = \{x \in D : f(x) \leq f^* + \varepsilon\}$, where $\varepsilon \geq 0$, $x^* = \arg\min\{f(x) : x \in D\}$.

Method 3.2. Choose convex closed sets $M_0 \subset \mathrm{R_n}$, $G_0 \subset \mathrm{R_{n+1}}$ such that $x^* \in M_0$, $\mathrm{epi}\,(f, \mathrm{R_n}) \subset G_0$, and inequalities $\gamma \geq \bar{\gamma} > -\infty$ are fullfilled for any points $(x, \gamma) \in G_0$, where $x \in M_0$. Define numbers $\varepsilon_0 > 0$, $\delta_0 > 0$, and put $i = 0$, $k = 0$.

1. Find a point (y_i, γ_i), where $y_i \in \mathrm{R_n}$, $\gamma_i \in \mathrm{R_1}$, as the decision of the problem

$$\min\{\gamma : x \in M_i \cap D'', \quad (x, \gamma) \in G_i\}.$$

If $y_i \in D'$ and $f(y_i) = \gamma_i$, then y_i is a decision of the initial problem.

2. If $y_i \in D'$ and $f(y_i) - \gamma_i \leq \delta_k$, then $y_i \in X^*(\delta_k)$. Further, the method is stopped or go to Step 3.

3. Sets M_{i+1}, G_{i+1} are constructed by several approaches which depend on fulfilment of the conditions for the point (y_i, γ_i).

a) Let the (y_i, γ_i) is found such that $y_i \notin D'_{\varepsilon_k}$. Then the next approximation sets M_{i+1}, G_{i+1} are determined by the following way. The set M_{i+1} is given by cutting the point y_i from M_i, and G_{i+1} is defined in accodance with cutting the point (y_i, γ_i) from G_i.

b) Let the point (y_i, γ_i) is obtained such that $y_i \in D'_{\varepsilon_k}$, but at the same time the inequality $f(y_i) - \gamma_i > \delta_k$ is fulfilled. Then, firstly, put $M_{i+1} = M_i$, in case of $y_i \in D'$, or the set M_{i+1} is constructed according to Step 3a if $y_i \in D'_{\varepsilon_k} \setminus D'$. Secondly, construct the set G_{i+1} as well as in Step 3a.

c) Let conditions $y_i \in D'_{\varepsilon_k}$, $f(y_i) - \gamma_i \le \delta_k$ are fulfilled for the point (y_i, γ_i) simultaneously. Firstly, if $y_i \in D'$, then put $M_{i+1} = M_i$. Otherwise, in a certain way the convex closed set M_{i+1} is constructed under conditions $M_{i+1} \subseteq M_0$, $x^* \in M_{i+1}$ and $y_i \notin M_{i+1}$. Secondly, the convex closed set S_i is formed according to clauses $S_i \subset G_0$, $\mathrm{epi}\,(f, \mathrm{R_n}) \subset S_i$, and in some way choose the set $G_{i+1} \subset S_i$. Thirdly, set $i_k = i$, $x_k = y_{i_k}$, $\sigma_k = \gamma_{i_k}$, define $\varepsilon_{k+1} > 0$, $\delta_{k+1} > 0$, and the value of k is increased by one.

4. Go to Step 1 and increment the number i.

Note that in Method 3.2 the process of updating approximation sets occurs in such iterations, when the approximation quality of the feasible set and the approximation quality of the epigraph of the function are considered to be satisfactory simultaneously.

Procedures of updating approximation sets as a result of dropping accumulated cutting planes can be caused by choosing sets Q_i, S_i. Let conditions $f(y_i) - \gamma_i \le \delta_k$, $y_i \in D'_{\varepsilon_k}$ are simultaneously determined for the point (y_i, γ_i) subject to some number $i = i_k$. Put, for example, $Q_i = Q_{i_k} = M_r$, $S_i = S_{i_k} = G_l$, where $0 \le r \le i_k$, $0 \le l \le i_k$. Since $M_r \subset M_0$, $G_l \subset G_0$ for all $r = 0, \dots, i_k$, $l = 0, \dots, i_k$, in view of $x^* \in M_r$, $\mathrm{epi}\,(f, \mathrm{R_n}) \subset G_l$ for all $r = 0, \dots, i_k$, $l = 0, \dots, i_k$ it follows that the set Q_{i_k} can be determined according to any sets $M_0, \dots, M_{i_k}$ formed to the i_kth step and the set S_{i_k} can be chosen from any sets $G_0, \dots, G_{i_k}$. In particular, if the sets Q_{i_k}, S_{i_k} is defined by $Q_{i_k} = M_0$, $S_{i_k} = G_0$, then the process of discarding all cutting planes accumulated to i_kth iteration occurs, and sets M_{i+1}, G_{i+1} are constructed in terms of sets M_0, G_0 by cutting off points y_i, (y_i, γ_i) from M_0, G_0 respectively.

Subgradients of the function $F(x)$ is used in cutting planes to construct the following approximation sets M_{i+1}, and the sets G_{i+1} is formed on the basis of subgradients of the function $f(x)$.

In paragraphs 1.5, 2.5 and 3.3 results of the numerical experiments are reported for investigate methods developed in §§ 1.1, 1.2, 2.1, 2.2, 3.3. These methods are made by the programming language C++ in Microsoft Visual Studio 2008. Methods from §§ 1.1, 1.2, 3.3 are tested on tasks, where the feasible set is defined by nonlinear functions. Other methods are investigated with tasks, where the constraint set is a polyhedron and the objective function is nonlinear. A lot of tasks with different number of varibles (to 50) and constraints (to 50) have been solved by the proposed methods. Each task has been solved by the proposed methods with either using procedures of dropping cutting planes or not applying procedures of updating approximation sets. From numerical experiments it follows that significant acceleration of solving all tests occurs in case of using some update procedures in each investigated methods.

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