0000-0003-3278-7795

ABOUT OUTER AND INNER DIGITIZATION OF MATHEMATICS TEACHING

Aslanbek Naziev[1]

[1]*Ryazan State University named after S.A. Esenin, Ryazan 390048, RF*

[1]*Ryazan Institute for Educatiom Development, Ryazan 390023, RF*

[1]State Social Humanitarian University, Kolomna, Mosc. Reg., RF
[1]a.naziev@365.rsu.edu.ru

Abstract. We draw attention to the difference between digitization inner (internal) and outer (external). So popular nowadays, the video recording of ordinary lectures, transmitted via digital channels, is not true digitization of education. It is, so to say, “analogous digitization”, superimposed externally on the education process. Such a “digitization” is natural to call external (or, better, facade). Another thing is internal digitization, coming from the essence of subjects, carried out in organic unity with them. Only such a digitization of the education process we consider genuine. An excellent tool for implementing just such a digitization is, created by Donald Knut and Leslie Lamport — a desktop publishing system for a comfortable set up of scientific texts. In our article, we give after a brief reference on for those who are not familiar with it and, after that, show in detail on some our products how allows the true digitization of the mathematics teaching.

***Keywords**: Mathematics, Teaching, Digitization, LaTeX, TikZ, tkz-euclide.*

1. LaTeX brief overview

LaTeX [1] is a desktop publishing system made by Donald Knut [2] and Leslie Lamport [3]; see photos below. It consists of two layers, the “severe” basic layer TeX, produced by Knut, and the friendly interface to TeX, added by Lamport. The result was LaTeX, desktop publishing system remarkably suited for comfortable producing complex scientific texts (mathematical, physical, technical, chemical, and even musical) of the highest typographical quality. But the ideal means of integration of ICT and mathematics it becomes due to its numerous extensions appeared in the last 15–20 years and continuously appearing up today.

One of these extensions is the package Ti*k*Z and its extension tkz-euclide. Features of these packages are described in detail by their authors Till Tantau [5] (the first package) and Allan Mattes (Alain Matthes in French) [6] (the second package). From these descriptions, we know that the packages (especially the second) allow producing and describing plane geometric drawings in the TEX notation in much the same way as in the ordinary course of geometry: tkzGetPoint (take into account a point), tkzDrawPoint (draw a point), tkzLabelPoint (denote the point), tkzDrawLine (draw the line), etc.

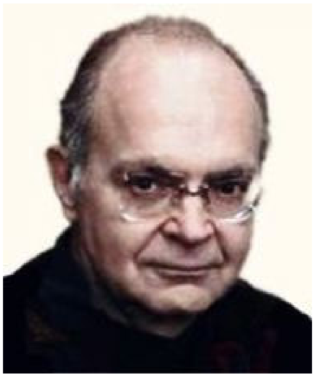


Fig. 1. Donald Knut

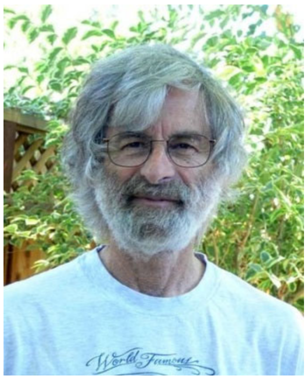


Fig. 2. Leslie Lamport

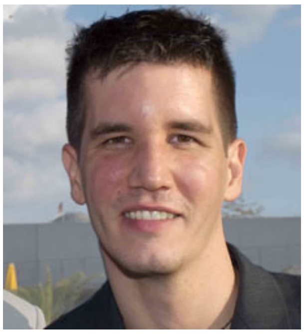


Fig. 3. Till Tantau

Fig. 4. Allan Mattes

2. Internal digitization with an example

Now, we shall try to show what we consider the internal digitization of mathematics teaching and how using LATEX makes it possible. The digitization of something, we consider internal if it proceeds in close unity with the subject's content. Let us clarify this thought with an example.

Suppose we want to make a dynamic model of the tangent line sliding on the waves of the sinusoid. What have we to know to do it? We must know that the sine function is differentiable at every point of its domain, and the equation of the tangent line to the graph of a differentiable function at the arbitrary point. Further, we have to know the derivative of the sine function. At last, we need to know something from $\LaTeX$, namely, that it has the drawing tool mfpic and the package multido, which permits easy repeating calculations by the same formula in many points.

More concretely, we take the equation of the tangent line to the graph of a differentiable function f at the arbitrary point x_0,

$$y = f(x_0) + f'(x_0) \cdot (x - x_0),$$

replace in it "x_0" by "t", "f" by "$\sin$", get

$$y = \sin(t) + \sin'(t) \cdot (x - t),$$

remember that $\sin' = \cos$, and obtain the equation of the tangent line to the function $\sin$,

$$y = \sin(t) + \cos(t) \cdot (x - t).$$

This equation tells that our tangent line consists of all the points of the form

$$(x,\ (\sin(t) - \cos(t) \cdot t) + \cos(t) \cdot x)$$

with $x \in \mathbb{R}$. Of course, we cannot draw the whole line; we only can draw some of its intervals. We choose the interval of length 2 centered at the point $(t,\ \sin t)$. Simple calculations show that this interval has ends

$$\left(t \pm \frac{1}{\sqrt{1+\cos^2 t}},\ \sin t \pm \frac{\cos t}{\sqrt{1+\cos^2 t}}\right).$$

Varying t, we obtain the family of intervals sliding on the sinusoid. Again, the whole sinusoid we cannot draw, we only can draw some of its portions. For the picture below, we have chosen the portion over the segment $[-5,\ 5]$. Once again, it is impossible to draw such intervals for all $t \in [-5,\ 5]$. We only can draw them for some more or less big *finite* set of points. We take points of the form $t = i \cdot \pi/16$ with the whole i from –26 to 26 and apply the package multido. With some more details, by placing a page with the picture inside the command \multido{…} and putting after picture (but inside \multido{…}!) the command \newpage for the transition to a new page, we obtain a series of replacing one other pages with the same sine waves but varying segments of tangents.

Here is one slide of the whole model consisting of 53 similar slides replacing one other.

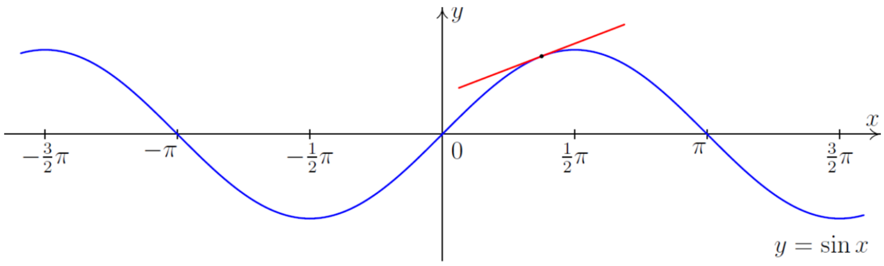


Fig. 5. Sliding tangent line to the graph of sin

And here is the whole code.

```
\documentclass[12pt]{amsart}
\usepackage[metapost]{mfpic}
\usepackage{multido}
\usepackage{color}
\usepackage[pdftex]{web}
\margins{0pt}{0pt}{0pt}{0pt}
\screensize{1.68in}{5.76in}
\pagestyle{empty}
\parindent=0pt
\begin{document}
\thispagestyle{empty}
\opengraphsfile{tangraphsin}
\multido{\i=-26+1}{53}{
\begin{mfpic}[40]{-5.2}{5.2}{-1.5}{1.5}
\axes
\xmarks{-3/2pi,-2pi,-pi,-pi/2,0,pi/2,pi,3/2pi}
\tlabel[cl](.1,1.41){$y$}\tlabel[bc](5.1,.1){$x$}
\tlabel[tc](4.5,-1.2){$y = \sin x$}
\pen{.8pt} \drawcolor{blue}\function{-5,5,.2}{sin x}
\drawcolor{red}
\lines{(\i*pi/16-(1+(cos\i*pi/16)**2)**(-1/2), sin(\i*pi/16)-cos(\i*pi/16)*((1+(cos\i*pi/16)**2)**(-1/2))),
((\i*pi/16+(1+(cos\i*pi/16)**2)**(-/2)),sin(\i*pi/16)+cos(\i*pi/16)*((1+(cos\i*pi/16)**2)**(-1/2)))}
\point[2pt]{((\i*pi)/16,sin((\i*pi)/16))}
\tlabel[tc](-4.71,-.1){$-\frac{3}{2}\pi$}\tlabel[tr](-3.14,-.1){$-\pi$}
\tlabel[tc](-1.57,-.1){$-\frac{1}{2}\pi$}\tlabel[tl](.1,-.1){$0$}
\tlabel[tc](1.57,-.1){$\frac{1}{2}\pi$}\tlabel[tr](3.14,-.1){$\pi$}
\tlabel[tc](4.71,-.1){$\frac{3}{2}\pi$}
\end{mfpic}
\newpage
 }
\end{document}
```

3. An example of external digitization in comparison with the internal one

3.1 An example of external digitization leading to a mistake

Now, let us give an example of an external digitalization. The next picture shows the cover of a Russian school textbook, one of the main textbooks recommended for massive use in teaching space geometry. The book was published near 2000 by the main Russian educational publisher, "Prosvescschenie." By that time, all books over the world were for several decades published digitally, and this one, of course, too.

Let us take a close look at the drawing on the book's cover.

Fig. 6. The cover of a textbook

We see on it a sphere with an equator and tangent lines to the sphere, drawn through the points of the equator and passing through the same point, say T, outside the sphere. Can this be? No! The tangent lines to the sphere are perpendicular to the radii drawn to the points of tangency. All these radii are also perpendicular to the straight line passing through the center of the sphere and T (let us call this line the axis). It turns out that each tangent line and the axis are perpendicular to the radius drawn to the tangent point. This means that these lines cannot intersect! (Otherwise, a triangle will exist with two right angles.) It turns out that a school textbook that has near 20 editions begins with a mistake on the cover!

Moreover, in the text of the book, one can find 21 more drawings, erroneous as well. And the book was printed, let us repeat it, *digitally*. Conclusion: digitalization in no way prevents mistakes.

If to think it over, the conclusion we have come to is completely natural and even obvious. Indeed, digitalization is an extremely powerful tool that can deal with everything, even with mistakes! Therefore, there is no reason to hope that digitalization will lead only to success. Thomas Huxley once said that mathematics may be compared to a mill of exquisite workmanship, which grinds given stuff to any degree of fineness; but, what one gets out depends upon what he put in. If he puts in the grain, the outputs will be meal; but if he puts in quinoa, the outputs will be quinoa dust.

With digitization, the situation is completely similar. One can digitize everything, but what he gets out depends upon what he puts in (whether somebody thought otherwise?). That is why we speak about the necessity of distinguishing the digitization external and internal. External digitization merely superimposed on the ready-made stuff after its creation. In contrast, the inner digitization proceeds from the essence of the stuff and participates in the process of its creation.

In our first example, we have already shown how this is possible and how one can accomplish it. Now we show this for the second example. Namely, we will show how one could avoid the noted errors and obtain the correct solution making digitization in the close connection with the properties of objects under consideration and looking at the root of the problem.

Note that it is not at all difficult to detect the fallacy of the given drawing. For this, it is enough to consider mentally the cross-section of the combination of a sphere and a conical cap on it with a plane passing through the center of the sphere and the top of the cap. The section of the sphere will turn out to be a large circle of the sphere; the section of the cap will be the angle formed by two generators of the cone tangent to the sphere. It is quite clear that these generators cannot touch the circle at diametrically opposite points, because then they would both be perpendicular to the same diameter and therefore parallel.

Of course, all that we said is well known to the authors of the textbook. The reason for their error lies not in ignorance, but in the approach to the image processing. The usual approach to the building of digital images has this order of actions; first image, then digitalization, with math somewhere on the side. Contrary to this, it is necessary, we claim, to carry out digitalization in organic unity with a mathematical consideration of the problem. Now we intend to demonstrate this for the second example.

3.2 Some preliminary problems with solutions

In order to ease the way to the required solution, we first consider a simpler problem, namely, the problem of the correct drawing of a cone. On the Internet and in many textbooks, as well as in pupil's note-books, one can find a lot of erroneous drawings, similar to adduced below. Among them, there are even made with LaTeX, because LaTeX alone also does not protect from errors, only in close interaction with mathematical problem-solving.

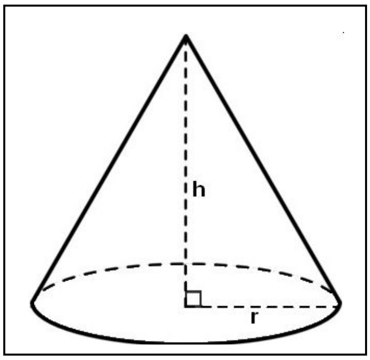


Fig. 7. Erroneous cone drawing

What is incorrect in this picture? The base of a cone must be represented by an ellipse. Contour generators of the conic surface must be tangential to this ellipse. But the tangent lines to the ellipse at the ends of any diameter are parallel. So, contour generators of the conic surface (which both go through the point representing the vertex of the cone) cannot go through the ends of a diameter.

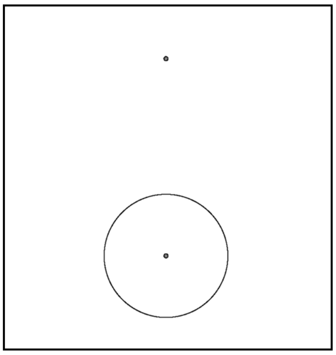
Well, let us try to build a correct drawing of a cone, simply a cone, without any additional requirements. For solving this problem, it is enough to use the simplest constructions of tkz-euclide combined with some information from mathematics. First, remember that the image of a cone's base is an ellipse and that the contour generators of the cone are tangent to the ellipse. This remark can lead to one solution of the problem: set up parameters of an ellipse, set up cone's vertex, calculate the coordinates of the point of tangency for two tangent lines to the ellipse through the vertex and reflect all this on the drawing.

However, we can proceed more efficiently and intelligible. Remember that

- every ellipse is a compressed or stretched circle;
- one can without any problems draw the tangent lines to a *circle* through the given point;
- the computer can easily compress or stretch given figures.

These considerations lead to the following sequence of actions:

- draw a circle with its center and select a point for the vertex of the cone;
- draw two tangent lines to the circle through the point chosen for the vertex of the cone;
-



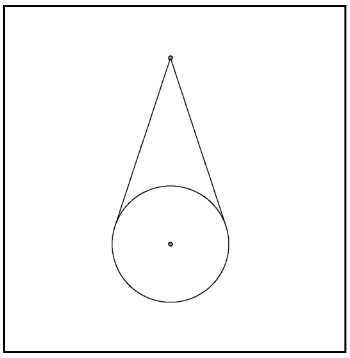


Fig. 8, 9. Steps in drawing a cone

- stretch the obtained picture about the cone's axis (i. e. the line through the center and the vertex of the cone).

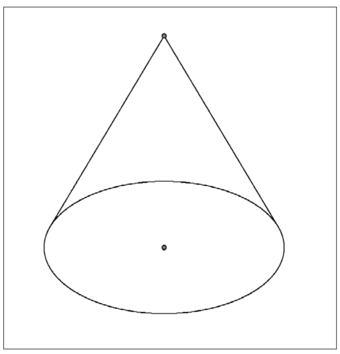


Fig. 10**.** Step in drawing a cone

That is it! However, the drawing obtained this way has one demerit. Namely, the whole ellipse drew continuous, while one arc between the tangency points is invisible; therefore, one has to represent it dashed. For removing this lack, it is useful to know that tkz-euclide allows considering elements of construction without representing them on the picture. With these reflections, we obtain the required result in the following sequence of actions:

- get a circle with its center; draw the center but not the circle; choose and draw a point for the vertex of the cone;
- draw two tangent lines to the circle through the point chosen for the vertex;
- one arc between tangency points draw continuous, the other draw dashed;

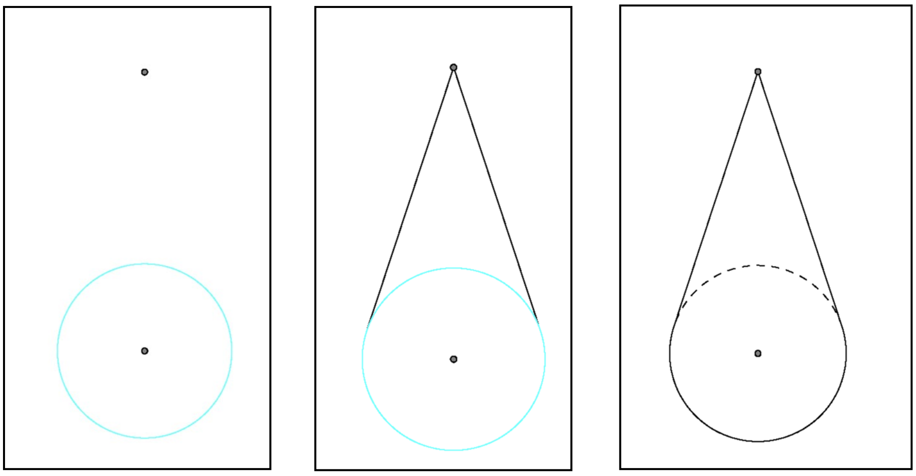


Fig. 11, 12, 13. New steps in drawing a cone

- stretch the obtained picture about the line through the center and the vertex.

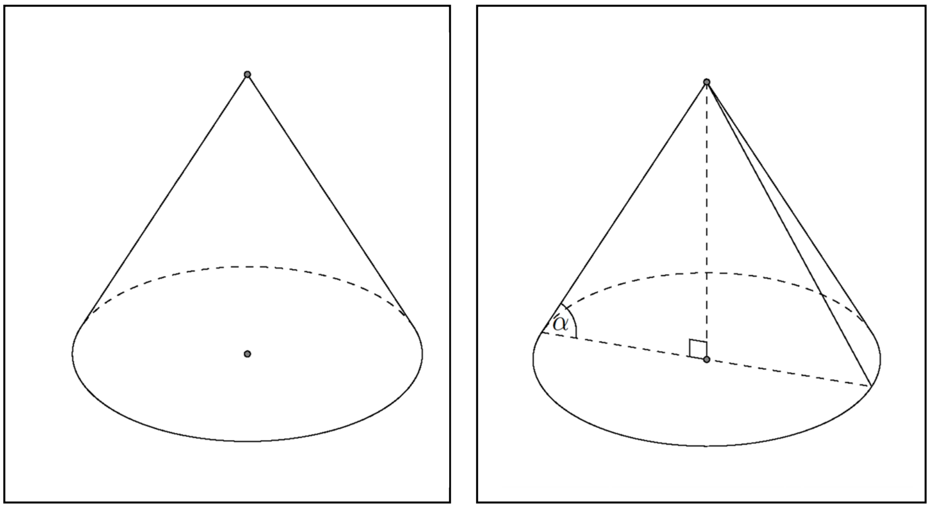


Fig. 14, 15. Correct drawings of a cone

Doing so, one obtains a correct drawing of a cone! In the above series of pictures, we represent same parts that we get but no draw in pale-blue. Further, one can:

- get without drawing the line through one point of tangency and the center;
- get and draw the point of its intersection with the circle;
- draw dashed diameter connecting this point and the point of tangency;
- draw the segment connecting this point with the vertex.

After these actions, one obtains on the above picture the drawing of the cross-section of the cone with the plane through the mentioned point of tangency, the center of the ellipse, and the vertex of the cone.

3.3 Correct drawing of a sphere with a conical cap

Now we, at last, demonstrate the obtaining of the correct picture to that drawing which tried to produce but so unsuccessfully produced the authors of the cited textbook on its cover. We first show the final drawing (in two variants). The contour of the sphere and its equator we draw green while the conical cap and its line of tangency with the sphere, orange. In these terms, the error of the textbook's authors one can describe as follows: they drew the combination so as the tangency line of the cap with the sphere is equator (green circle) while it is another line (orange parallel).

Turn to the description of the way leading to the correct drawing. The reader will note the close interaction and mutual interlacing of mathematical content with the instruments LaTeX. Also, he will see how these peculiarities force to act correctly.

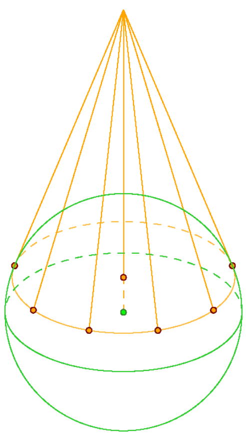


Fig. 16. Sphere with a conical cap

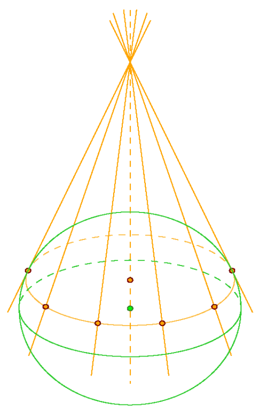


Fig. 17. Sphere with a conical cap, "tails" added

To shorten the description, we present the working drawing in its final stage and solidified form. On the left, we place the correct image, on the right, the auxiliary constructions leading to this image. The dynamics of the construction we reflect through descriptions. We place point notations directly to the right, even though many of them fall on drawn lines. We again use different colors.

We begin with an image of a sphere of the arbitrary radius R with the center O and an equator, both lying in the plane parallel to the plane taken as "horizontal." The picture of the sphere is a circle of radius R ("big circle"); we draw it green. The image of the equator is an ellipse with the major axis $2R$ and minor axis, say, $2r$; we present it green also. To produce this image, we take an invisible circle with the center O and radius r having continuous lower half and dashed upper half. We place these data into the "camera" for the stretching along horizontal with the coefficient $k = R : r$. For definiteness, we take $k = 2$. After LaTeX-ing, we get an ellipse of needed dimensions, with the continuous lower half, and dashed upper half.

The "camera" name is *scope*; it has this format:

```
\begin{scope}[xscale=k]
…
\end{scope}.
```

It acts as follows:

```
\begin{scope}[xscale=k]
\tkzDrawArc[thick,color=LimeGreen](O,A)(B)
\tkzDrawArc[thick,dashed,color=LimeGreen](O,B)(A)
\end{scope}.
```

The result is green ellipse on the left-hand side of the picture.

Further, we need to build the image of the cap's tangency curve with the sphere (orange ellipse on the left). For this, we make on the right side an auxiliary drawing. All the elements on this drawing have their natural dimensions and placed so that corresponding elements on the left side appear to be projections of these right-hand elements. In particular, the black circle on the right-hand side represents in natural size the contour meridian of the sphere.

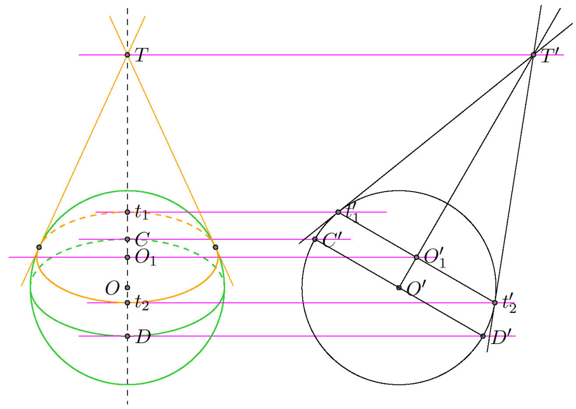


Fig. 18: Building s sphere with a conical cap

Look at the green ellipse on the left side. It represents an equator of the sphere. The segment $[C, D]$ represents the diameter of the equator in compressed form. Represent on the right-hand drawing this diameter in natural size in such a way that the compressed diameter on the left appears to be the projection of the "natural" right-hand one. For this, draw through C and D two horizontal lines (in magenta) and select a segment equal to the diameter with the ends on these lines; let it be $[C', D']$. The midpoint O' of the segment $[C', D']$ represents the sphere's center O

The segment $[O, T]$ on the left represents in compressed form the segment between O and the vertex of the cap. To get this compressed segment, we drew on the right side the mediator to the $[C', D']$ and took on it the point T' being away from O' to the natural distance between O and the vertex of the cap. The point T appears to be the projection of T' onto the axis along the horizontal.

After making all described, draw (on the left side) the image of the tangency curve of the cap with the sphere. For this, draw (on the right side) two tangent lines to the black circle with the points t_1' and t_2' of tangency. The segment $[t_1, t_2]$ is the diameter of the tangency circle, analogous to the diameter $[C, D]$ of the equator. On the left side, it is represented in compressed form, on the right side, in the natural size. How to obtain the required ellipse we already know. Get (without drawing) the circle of the radius O_1t_1, get the tangent lines to this circle, get the tangency points of these lines with the circle, and draw continuous tangent lines, the continuous lower arc of the circle between tangency points, with the dashed upper arc. Place these data into the stretching "camera", and after stretching, obtain the required ellipse with tangent lines and tangency points. To distinguish these parts of the drawing, draw them in a different color, say, orange.

Surely, the reader already guessed how to get more tangent lines from the top of the cap to the tangency line; one needs to get more points on the circle with the center O_1 of the radius O_1t_1, draw segments from T to these points, place these data into the stretching "camera", and after stretching you will obtain what you need.

Such is the process of the correct image generating for the combination of figures from the cover of the cited textbook. Hope, the reader noted impressive close interaction and mutual interlacing of mathematical content and LaTeX-nical instruments in the above constructions. It is this kind of interaction that we consider *internal digitization*.

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