A manuscript

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THE FIRST BOUNDARY PROBLEM FOR EQUATIONS OF MIXED TYPE WITH SINGULAR COEFFICIENT

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General description of work

Relevance of the topic. One of the principal and the most important sections of the modern theory of differential equations are partial boundary value problems for the equations mixed type because of its applied and theoretical significance. In connection with this statement, and the study of boundary value problems for such equations is the subject of research by many scientists. This interest is DUE to important practical applications mixed-type equations in the areas of magnetic and hydrodynamic flow with the transition through the speed of sound, transonic gas dynamics, membrane theory of shells with curvature of variable sign, the theory of infinitesimal bendings surfaces et al., as well as theoretical significance obtained results. The set of mathematical models of mass and heat transfer in capillary-porous media, reservoir systems, the movement of low compressibility fluid in the channel surrounded by the porous medium, propagation in inhomogeneous medium of an electromagnetic field, the formation of temperature field, the motion of a viscous and viscoelastic fluid reduced to the study of boundary value problems for equations of mixed type.

In the works of F.Trikomi and S.Gellerstedt was the beginning of the study boundary value problems for equations of mixed type of model is now It is known as the "Tricomi problem "and"Gellerstedt problem ".

Further development of the theory of equations of mixed type received in works of F.I. Frankl, A.V. Bitsadze, K.I. Babenko, S. Agmon, L. Nirenberg, M.N. Protter, A.K. Aziz, M.Schneider, J.R. Cannon, S.S. Morawetz, J.M. Rassias, L. Bers, T.D. Juraev, V.I. Zhegalova, T.Sh.Kalmenova, G.D. Karatoprakliev, I.L. Karol, N.Y. Kapustin, Y.M. Krikunov, A.I. Kozhanov, M.A. Lavrent'ev, M.E. Lerner and O.A. Repin, E.I. Moiseev, A.M. Nakhushev, N.B. Pleschinskii, N.I. Popivanov, S.P. Pulkin, L.S. Pulkina, O.A. Repin, K.B. Sabitov, M.S. Salahitdinov and A.K. Urinov, M.M. Smirnov, A.P. Soldatov, R.S. Khairullin, M.M. Hachev et al.

Special interest to the Dirichlet problem for mixed equations type became evident after the publication of the famous work F.I.Franklya, where it was first noted that the problem tranzvukovoy gas dynamics are reduced to this problem. On the Dirichlet problem for Chaplygin equation, for example, reduces the task of crossing the sound barrier of steady two-dimensional irrotational flows ideal gas nozzles when supersonic wave adjoin the walls of the nozzle close to the minimum section.

A.V.Bitsadze proved incorrect Dirichlet problem for Lavrent'ev equation $u_{xx} + (\text{sgn}y)u_{yy} = 0$. Following the publication of his work was necessary to search for the mixed domains, for which the Dirichlet problem is posed.

In the future, B.V. Shabat was investigated Dirichlet problem for Lavrent'ev-Bitsadze equation in $y > -h$, $h > 0$. Hyperbolic part of this area lies entirely within the characteristic triangle, which is built on the segment the real axis $[0, 1]$.

N.N. Vakhania and J.R. Cannon proved the correctness of the Dirichlet problem for Lavrent'ev-Bitsadze equation in rectangular domains at certain restrictions on the domain of hyperbolicity.

A uniqueness criterion for solutions of the Dirichlet problem for equations mixed type of the first kind in a cylindrical domain set A.M.Nahushev.

Correctness nonlocal Dirichlet problem for Lavrenteva Bitsadze in the D , where $D_{-} = \{0 < -y, x < 1\}$, D_{+} - simply connected region lying in the half-plane $y > 0$, limited segment $[0, 1]$ axes x and simple arc σ with ends at the points $(0, 0)$, $(1, 0)$, proved V.I. Zhegalov.

A.P. Soldatov unique solvability of problem solving Dirichlet type for the Lavrent'ev-Bitsadze equation in a mixed domain Limited at $y > 0$ and $y < 0$, respectively smooth arcs with common endpoints $(0, 0)$ and $(0, 1)$, and the arc in hyperbolic region (with $y < 0$) lies within the characteristic triangle.

R.I. Sokhadze studied the first boundary value problem for an equation of mixed elliptic-hyperbolic type

$$u_{xx} + yu_{yy} + au_y = 0,$$

where $0 < a < 1$ and $a > 1$ - not an integer, in a rectangular domain $D = \{(x, y) \mid 0 < x < 1, -\alpha < y < \beta\}$ with certain conditions on α and β.

O.A. Repin for a mixed-type equation with a singular coefficients of lower derivatives

$$u_{xx} + \mathrm{sgn} y u_{yy} + \frac{2p}{|y|} u_y = 0, \;\; 0 < 2p < 1,$$

in D, limited half-lines $x = 0$ and $x = 1$ with ends points $A(0, 0)$ and $B(1, 0)$, arranged in a half $y > 0$, and characteristics $AC : x + y = 0$; $BC : x - y = 1$, coming out of the points A and B and intersecting at the point $C(1/2, -1/2)$, is studied nonlocal boundary value problem.

In recent years, in connection with the development of the method of spectral analysis in relation to the mixed-type equation Dirichlet problem He studied in rectangular domains.

K.B. Sabitov studied the Dirichlet problem for a degenerate equation mixed type of the first kind

$$\mathrm{sgn} y \cdot |y|^n u_{xx} + x^m u_{yy} - b^2 x^m \mathrm{sgn} y \cdot |y|^m u = 0, \quad n > 0, m = 0, b \geq 0, \qquad (1)$$

the rectangular domain and a half-strip. Uniqueness criterion solutions, built as the sum of the Fourier series, set on the basis of spectral method for solving boundary value problems.

The first boundary problem for two classes of equations of mixed type second type

$$u_{xx} + \mathrm{sgn} y \cdot |y|^m u_{yy} - b^2 u = 0, \quad 0 < m < 2, \quad b = const \geq 0,$$

$$u_{xx} + yu_{yy} + au_y - b^2 u = 0, \quad a = const,$$

in a rectangular domain $D = \{(x,y)|\ 0 < x < 1,\ -\alpha < y < \beta\}$, where a, b, α, β - set real numbers, in this case $\alpha > 0$, $\beta > 0$, in depending on the values of the parameters m and a is studied K.B. Sabitov and A.H. Tregubova (Suleymanova). The criteria for the uniqueness of the solutions the first boundary value problem set based on completeness properties the system of eigenfunctions of one-dimensional spectral problem. Solutions tasks are constructed as a sum of series in eigenfunctions

R.S. Khairullin a criterion for the solution of Dirichlet problem for equation

$$u_{xx} + yu_{yy} + au_y = 0$$

in a rectangular domain with D $a \leq -1/2$, in this case, the line changes are specified other transmission conditions.

K.B. Sabitov and E.V. Vagapova using spectral method expansions have established the first criterion for the uniqueness of the boundary problem for the equation (1) in a rectangular domain for all $n, m > 0$. The solution is presented as the sum of the Fourier - Bessel.

The Dirichlet problem for partial differential equations of higher orders investigated E.A. Utkina. The solution is reduced is equivalent to the task to a system of Fredholm equations whose solvability is established based on the method of a priori estimates.

The purpose and problems of dissertation research. In this the first boundary problem is studied for two classes of equations of mixed type with a singular coefficient:

$$Su \equiv u_{xx} + (\mathrm{sgn}y)u_{yy} + \frac{k}{x}u_x = 0, \tag{2}$$

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + (\mathrm{sgn}y)|y|^m \frac{\partial^2 u}{\partial y^2} + \frac{k}{x}\frac{\partial u}{\partial x} - a^2 u = 0, \tag{3}$$

the rectangular domain $D = \{(x,y)|0 < x < l, -\alpha < y < \beta\}$, where k, $a \geq 0$, $0 < m < 2$, $l > 0$, $\alpha > 0$, $\beta > 0$ - given real numbers (parameters of the problem), depending on the values of these parameters.

The main problems of the study are the production and proof of uniqueness and existence of the first boundary value problem for equations (2) and (3) in a rectangular domain D.

The problem of research is the first boundary value problem for two classes of equations of mixed type with a singular coefficient.

The theoretical and methodological basis of the study questions of existence and uniqueness of solutions of the first boundary problems for equations of mixed type with a singular coefficient up methods of the general theory of partial differential equations derivatives and spectral analysis.

Scientific novelty of research. The results are taken out on defense, they are new.

The main results of the thesis submitted for protection. On defense are made the following results.

1) Found intervals of the parameter k: $k < 1$ $k \neq 0$; $k \geq 1$; $k = 0$, where the Dirichlet problem and Keldysh problem for the equations of mixed type with a singular coefficient and Lavrent'ev-Bitsadze equation in a rectangular domain set correctly. In each of these cases, the only established criteria, decisions built in the form of a series in eigenfunctions of one-dimensional spectral problem with justification convergence of series in the class of regular solutions of the equation (2).

2) Established intervals change parameters k and m: $k < 1$ and $0 < m < 1$; $k < 1$ and $1 \leq m < 2$; $k \geq 1$ and $0 < m < 1$; $k \geq 1$ and $1 \leq m < 2$, where the Dirichlet problem and Keldysh problem for the mixed-type equation with characteristic degeneration and singular coefficient in the rectangular domain set correctly. In each of these cases the criteria set uniqueness theorems of existence of problem solving, which are constructed as a sum of the Fourier - Bessel. Installed sufficient conditions for the convergence of the series in the class of solutions of the equations (3).

Theoretical and practical significance of the research. The work is theoretical. The results and methods study of scientific interest and can be used for the further development of the theory of boundary value problems for partial differential equations of mixed type.

Testing results of the study. Results the thesis discussed at the seminars laboratory of Applied Mathematics and Informatics Department Physical - Mathematical and Technical Sciences Institute of Applied RB Research (Sterlitamak, the head Seminar - Dr., Professor K.B. Sabitov), the Department of Differential Equations of Kazan (Volga) Federal University (Kazan, head of workshop - Dr., Prof. V.I. Zhegalov). The main results were presented at International conference devoted to 100 - anniversary Sobolev "Proc. Functionality space. Approximation Theory "(Novosibirsk, 2008), International conference dedicated to the 70th anniversary of the Moscow State University rector Academician Victor Sadovnichy "Modern problems of mathematics, mechanics and their applications "(Moscow, 2009), International scientific conference "Boundary problems for differential equations and analytic functions "(Kazan, 2014), International Conference "Differential Equations and mathematica modeling " (Ulan-Ude, 2015), Twelfth Kazan International Summer Scientific School-Conference "Theory Functions, its Applications and Adjacent Problems "(Kazan, 2015).

Publications. The main results of the thesis were published in [1] - [9]. In this article [1] - [4] published in journals recommended by the Higher Attestation Commission (HAC) Ministry of Education and Science of the Russian Federation for the publication research results.

The contribution of the author in the development of selected problems. Thesis It is an independent research of the author.

The structure and scope of work. The thesis consists of an introduction, two chapters and bibliography containing 106 titles. Overall dissertation volume - 109 pages.

The author expresses his deep appreciation and gratitude to the scientific the head of the K.B. Sabitov for the proposed research topics, valuable advice, constant attention and support.

Summary of the thesis

In the introduction it provides an overview of the literature, formulated setting goals, the main results.

In the first chapter, consisting of three sections, for mixed-type equation with a singular coefficient (2) in rectangular domain D, depending on the parameter k is studied The first boundary problem. Spectral analysis method is set criteria of uniqueness and existence theorems of solutions tasks for each period of the parameter k.

Consider the equation (2) in a rectangular domain D and we put the following problems:

Problem 1.1 (Dirichlet problem). Let $k < 1$, $k \neq 0$. It is required to determine in D the function $u(x, y)$, which is satisfies the following conditions:

$$u(x,y) \in C(\overline{D}) \cap C^1(D) \cap C^2(D^+ \cup D^-), \tag{4}$$

$$Su(x,y) \equiv 0, \quad (x,y) \in D^+ \cup D^-, \tag{5}$$

$$u(x,\beta) = \varphi(x), \quad u(x,-\alpha) = \psi(x), \quad 0 \leq x \leq l, \tag{6}$$

$$u(l,y) = 0, \quad -\alpha \leq y \leq \beta, \tag{7}$$

$$u(0,y) = 0, \quad -\alpha \leq y \leq \beta, \tag{8}$$

where $\varphi(x)$, $\psi(x)$ - sufficiently smooth functions, $\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0$, $D^+ = D \cap \{y > 0\}$, $D^- = D \cap \{y < 0\}$.

In view of the results of M.V. Keldysh and S.P. Pulkina in the class of bounded in D of solutions equation for $k \geq 1$ interval $x = 0$ D the boundary exempt from the boundary condition. In this connection, it is proposed the next task with incomplete boundary data, ie, task E (on terminology M.V. Keldysh)

Problem 1.2 (Keldysh problem). Let $k \geq 1$. Required define in D the function $u(x, y)$, which satisfies conditions (4) - (7).

Problem 1.3 (Dirichlet problem). Let $k = 0$. Required define in D the function $u(x, y)$, which satisfies conditions (4) - (8).

It should be noted that S.P.Pulkin for the equation (2) at $k \geq 1$ Tricomi studied the problem in G, which is limited piecewise smooth curve Γ, located in the first quarter, meet at the points $A(1,0)$ and $B(0,1)$ and characteristics $OC: x+y=0$ and $BC: x-y=1$, $O(0,0)$, $C(1/2,-1/2)$, where the uniqueness established on the basis of the principle of extremum, and Theorem existence of a solution of this problem is proved by the integral equations. The case $0<k<1$ investigated in V.F.Volkodavova and students. K.B. Sabitov single method studied the problem T for the equation (2) for all $k>0$. The paper K.B.Sabitov and R.R.Ilyasov proposed a new method for constructing the solution of problem Tricomi problem for the equation (2) in the area of G as a series in special functions for parameter values $0<k<1$ when the curve $\Gamma \equiv \Gamma_0 = \{(x,y)|x^2+y^2=1, x>0, y>0\}$.

The Dirichlet problem for the Lavrent'ev - Bitsadze, i.e. for equation (2) when $k=0$ in a rectangular domain D if $l=1$ was first studied in J.R.Cannon. Fourier method solution is constructed as the sum of orthogonal series. On condition where $\varphi(x), \psi(x) \in C^4[0,1]$ $\psi(0)=\psi''(0)=\psi(1)=\psi''(1)==\varphi(0)=\varphi''(0)=\varphi(1)=\varphi''(1)=0$ and the number of α can be set to $\alpha = p/j$, $j=1,2,3$, $p\in\mathbb{N}$ and $\alpha=p/q$, $(p,q)=1$, $np=mq+r$, $n\in\mathbb{N}$, $M,r\in\mathbb{N}_0=\mathbb{N}\cup 0$, $0\le r<q$, $\min\limits_{0\le r<q}|\frac{r}{q}-\frac{3}{4}|\ge\delta_q>0$, $N>\mathbb{N}_q=const>0$ is marked convergence of the series.

In §1.1 investigate the Dirichlet problem for the equation (2) at $k<1$ and $k\neq 0$, in §1.2 Keldysh studied the problem for equation (2) with $k\geq 1$, and §1.3 is devoted to the study of the first boundary problem for the equation (2) when $k=0$.

We first present results on the Dirichlet problem for equation (2) at $k<1$ and $k\neq 0$. Based on the method of spectral analysis solution of the problem (4) - (8) is constructed in the form of sums of the Fourier - Bessel

$$u(x,y)=\sum_{n=1}^{+\infty}u_n(y)X_n(x), \tag{9}$$

where

$$u_n(y)=\begin{cases}\triangle^{-1}(n)(\varphi_n(\cos\lambda_n\alpha sh\lambda_n y+\sin\lambda_n\alpha ch\lambda_n y)+\psi_n sh\lambda_n(\beta-y)), y>0,\\ \triangle^{-1}(n)(\varphi_n\sin\lambda_n(y+\alpha)+\psi_n(sh\lambda_n\beta\cos\lambda_n y-ch\lambda_n\beta\sin\lambda_n y)), y<0,\end{cases} \tag{10}$$

$$X_n(x)=x^{\frac{1-k}{2}}J_{\frac{1-k}{2}}(\lambda_n x), \tag{11}$$

$$\lambda_n=\frac{\mu_n}{l},\quad n=1,2,3,...,$$

$$u_n(\beta)=\int\limits_0^l\varphi(x)x^kX_n(x)dx=\varphi_n,\quad u_n(-\alpha)=\int\limits_0^l\psi(x)x^kX_n(x)dx=\psi_n,$$

μ_n - n roots of equation $J_{\frac{1-k}{2}}(\mu_n)=0$, $\mu_n=\lambda_n l$, $J_\nu(z)$ - Bessel function of order ν, provided that for all $n\in\mathbb{N}$

$$\triangle(n)=ch\lambda_n\beta\sin\lambda_n\alpha+sh\lambda_n\beta\cos\lambda_n\alpha\neq 0. \tag{12}$$

Suppose condition (12) is violated when some l, α, β, k and $n = s \in \mathbb{N}$, i.e $\triangle(s) = 0$, then homogeneous problem (4) - (8) (where $\varphi(x) = \psi(x) \equiv 0$) has a nonzero solution

$$u_s(x,y) = \begin{cases} \widetilde{d}_s(sh\lambda_s ych\lambda_s\beta - sh\lambda_s\beta ch\lambda_s y)X_s(x), & y > 0, \\ \widetilde{d}_s(ch\lambda_s\beta \sin\lambda_s y - sh\lambda_s\beta\cos\lambda_s y)X_s(x), & y < 0, \end{cases} \tag{13}$$

where $\widetilde{d}_s$ - arbitrary constant not equal to zero.

The expression $\triangle(n)$ can be represented as

$$\triangle(n) = \sqrt{ch2\lambda_n\beta}\sin(\mu_n\widetilde{\alpha} + \theta_n), \tag{14}$$

where $\widetilde{\alpha} = \frac{\alpha}{l}$, $\theta_n = \arcsin\frac{ch\lambda_n\beta}{\sqrt{ch2\lambda_n\beta}}$. From (14) shows that the expression of $\triangle(n) = 0$ relatively $\widetilde{\alpha}$ only when

$$\widetilde{\alpha} = \frac{1}{\mu_n}(\pi z - \theta_n), \quad z = 1, 2, ... \ . \tag{15}$$

The following assertion.

Theorem 0.1.1. *If the solution of the problem (4) - (8) exists, it is unique if and only if, when the conditions (12) for all $n \in \mathbb{N}$.*

The proof of the uniqueness of the solution of the problem is carried out on the basis of completeness of the system (11) in the space $L_2[0,l]$ with weight x^k.

Since the expression of $\triangle(n)$, which is included in the denominators formula (10), has a countable set of zeros (15) and $\widetilde{\alpha}$, β, k - any number of periods job, it may be small enough for large n, i.e. a so-called problem of "small denominators". For, to justify the existence of the solution of this problem, you must show the existence of the numbers $\widetilde{\alpha}$, β and k, such that expression $\triangle(n)$ separated from zero to corresponding asymptotic behavior for large n.

We have the following

Lemma 0.1.1. *If $\widetilde{\alpha} = p/q$, $p, q \in \mathbb{N}$, $(p,q) = 1$ and $k \neq \frac{1}{p}(4r + q - 4qd)$, $r = 1, ..., q-1$, $d \in \mathbb{N}$, then there exist constants $C_0 > 0$ and $n_0 \in \mathbb{N}$, such that for all $n > n_0$ satisfies evaluation*

$$|\triangle(n)| \geq C_0 e^{\lambda_n\beta}. \tag{16}$$

If specified in lemma 0.1.1 of numbers $\widetilde{\alpha}$ $\triangle(s) = 0$ for some $n = s = m_1, m_2, ..., m_h$, where $1 \leq m_1 < m_2 < ... < m_h \leq n_0$, m_i, $i = \overline{1,h}$ and h - natural numbers, then the solvability of the problem (4) - (8) are necessary and sufficient conditions are the following:

$$\psi_s ch\lambda_s\beta - \varphi_s\cos\lambda_s\alpha = 0, \quad s = m_1, m_2, ..., m_h. \tag{17}$$

Solution (4) - (8) in this case is determined as the sum of a number of

$$u(x,y)=\left(\sum_{n=1}^{m_1-1}+...+\sum_{n=m_{h-1}+1}^{m_h-1}+\sum_{n=m_h+1}^{+\infty}\right)u_n(y)X_n(x)+\sum_s u_s(x,y), \qquad (18)$$

here in the last sum of s takes values $m_1, m_2, ..., m_h$, the function $u_s(x,y)$ is determined by the formula (13). If the lower limit is greater than the upper, then the final amount of (18) should be considered to be zero.

Theorem 0.1.2. *Suppose that the function $\varphi(x)$, $\psi(x) \in C^4[0,l]$ and $\varphi(0) = \psi(0) == \varphi'(0) = \psi'(0) = \varphi''(0) = \psi''(0) = 0$, $\varphi(l) = \psi(l) = \varphi'(l) = \psi'(l) = \varphi''(l) == \psi''(l) = 0$, and the estimate (16) when $n > n_0$. Then if $\triangle(n) \neq 0$ for all $n = \overline{1, n_0}$, then there exists a unique solution of (4) - (8) and the decision by a number of (9); if $\triangle(n) = 0$ for some $n = m_1, m_2, ..., m_h \leq n_0$, then the problem (4) - (8) is solvable only if the following conditions (17) and the solution in this case is determined by a number (18).*

In the case of **1.3** problem solution is constructed as the sum of orthogonal series:

$$u(x,y)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}u_n(y)\sin\lambda_n x, \qquad (19)$$

where

$$u_n(y)=\begin{cases}\triangle^{-1}(n)(\varphi_n(\cos\lambda_n\alpha sh\lambda_n y+\sin\lambda_n\alpha ch\lambda_n y)+\psi_n sh\lambda_n(\beta-y)), & y>0,\\ \triangle^{-1}(n)(\varphi_n\sin\lambda_n(y+\alpha)+\psi_n(sh\lambda_n\beta\cos\lambda_n y-ch\lambda_n\beta\sin\lambda_n y)), & y<0,\end{cases} \qquad (20)$$

$\lambda_n = \pi n/l$, provided that, for all $n \in \mathbb{N}$

$$\triangle(n)=ch\lambda_n\beta\sin\lambda_n\alpha+sh\lambda_n\beta\cos\lambda_n\alpha\neq 0. \qquad (21)$$

For example, when some l, α, β and $n = s \in \mathbb{N}$: $\triangle(s) = 0$. In this case, the homogeneous Problem 1.3 (where $\varphi(x) = \psi(x) \equiv 0$) has a nontrivial solution

$$u_s(x,y)=\begin{cases}\widetilde{d}_s(sh\lambda_s y ch\lambda_s\beta-sh\lambda_s\beta ch\lambda_s y)\sin\lambda_s x, & y>0,\\ \widetilde{d}_s(ch\lambda_s\beta\sin\lambda_s y-sh\lambda_s\beta\cos\lambda_s y)\sin\lambda_s x, & y<0,\end{cases}$$

where $\widetilde{d}_s$ - arbitrary constant not equal to zero.

The question naturally arises about the zeros of the expression $\triangle(n)$. For that transform it to the form:

$$\triangle(n)=\sqrt{\mathrm{sh}^2\lambda_n\beta+\mathrm{ch}^2\lambda_n\beta}\sin(\lambda_n\alpha+\theta_n)=\sqrt{\mathrm{ch}\,2\lambda_n\beta}\sin(\pi n\widetilde{\alpha}+\theta_n), \qquad (22)$$

where $\widetilde{\alpha} = \alpha/l$, $\theta_n = \arcsin\frac{\mathrm{sh}\,\lambda_n\beta}{\sqrt{\mathrm{ch}\,2\lambda_n\beta}}$. From (22) implies that $\triangle(n) = 0$ only when

$$\widetilde{\alpha}=(\pi z-\theta_n)/\pi n, n, z\in\mathbb{N},$$

ie with such $\widetilde{\alpha}$ the theorem is violated the uniqueness of the solution of the problem (4) - (8) when $k = 0$.

We have the following

Theorem 0.1.3. *If a solution of 1.3 exists, it is unique if and only if the following conditions are satisfied (21) for all* $n \in \mathbb{N}$.

From the formula (20) shows that the denominator is expression of $\triangle(n)$. Therefore, in order to justify 1.3 The existence of a solution, in addition to conditions (21), you must show the existence of the numbers $\widetilde{\alpha}$ and β, such that expression of $\triangle(n)$ bounded away from zero

The validity of the following statements.

Lemma 0.1.2. *If* $\widetilde{\alpha} \in \mathbb{N}$ *, then for any* $\beta > 0$ *there exists a constant* $C_0 = C_0(\beta) > 0$*, such that for all* $n \in \mathbb{N}$ *satisfies the estimate*

$$|\triangle(n)| \geq C_0 e^{\lambda_n \beta} > 0.$$

Lemma 0.1.3. *If* $\widetilde{\alpha}$ *- any fractional number, i.e.* $\widetilde{\alpha} = \frac{p}{q}$, $(p, q) = 1$, $\frac{p}{q} \notin \mathbb{N}$, $(q, 4) = 1$, *then there constant* $\beta_0 = \beta_0(\widetilde{\alpha}) > 0$ *and* $C_0 = C_0(\widetilde{\alpha}, \beta) > 0$*, such that for all* $\beta > \beta_0$ *and* $n \in \mathbb{N}$ *satisfies the estimate*

$$|\triangle(n)| \geq C_0 e^{\lambda_n \beta}.$$

Lemma 0.1.4. *If* $\widetilde{\alpha} > 0$ *is any irrational algebraic number of degree* $m \geq 2$*, then there are constants* $\beta_0 > 0$ *and* $C_0 > 0$ *such that for all* $\beta > \beta_0$ *and* $n \in \mathbb{N}$ *the estimates*

$$|\triangle(n)| \geq C_0 e^{\lambda_n \beta} \frac{1}{n^{1+\varepsilon}}, \;\; m > 2,$$
$$|\triangle(n)| \geq C_0 e^{\lambda_n \beta} \frac{1}{n}, \;\; m = 2,$$

where $\varepsilon > 0$ *- sufficiently small number.*

Theorem 0.1.4. *Suppose that the function* $\varphi(x)$, $\psi(x) \in C^3[0, l]$ *and* $\varphi(0) = = \psi(0) = \varphi''(0) = \psi''(0) = 0$, $\varphi(l) = \psi(l) = \varphi''(l) = \psi''(l) = 0$ *and the conditions of lemmas 0.1.2 and 0.1.3. Then there is a unique solution to the problem and it is 1.3 the decision by a number of (19).*

Theorem 0.1.5. *If the function* $\varphi(x)$, $\psi(x) \in C^4[0, l]$ *and* $\varphi(0) = \psi(0) = \varphi''(0) = \psi''(0) = 0$, $\varphi(l) = \psi(l) = \varphi''(l) = \psi''(l) = 0$ *and the number of* $\widetilde{\alpha}$ *is an irrational algebraic number degree 2, then there exists a unique solution of the problem and it is 1.3 the decision by a number of (19).*

Theorem 0.1.6. *If the function $\varphi(x), \psi(x) \in C^{4+\delta}[0,l]$, $\varepsilon \leq \delta \leq 1$ and $\varphi(0) = = \psi(0) = \varphi''(0) = \psi''(0) = 0$, $\varphi(l) = \psi(l) = \varphi''(l) = \psi''(l) = 0$ and the number $\widetilde{\alpha}$ is irrational algebraic number of degree $m > 2$, then there exists the only solution to the problem 1.3 and the decision by a number of (19).*

Theorem 0.1.7. *Suppose that the conditions of theorem 0.1.4, and $\triangle(n) \neq 0$ for all $n \in \mathbb{N}$. Then the solution Problem 1.3 we have the estimates*

$$\|u(x,y)\|_{L_2[0,l]} \leq N_{01}(\|\varphi(x)\|_{L_2[0,l]} + \|\psi(x)\|_{L_2[0,l]}),$$
$$\|u(x,y)\|_{C(\overline{D})} \leq N_{02}(\|\varphi'(x)\|_{C[0,l]} + \|\psi'(x)\|_{C[0,l]}),$$

where the constants N_{01} and N_{02} are independent of $\varphi(x)$ and $\psi(x)$.

Chapter 2, consisting of four sections, devoted study the first boundary value problem for a mixed-type equation with characteristic degeneration and singular coefficient (3) depending on the parameter k and m in a rectangular domain D.

Problem 2.1 (Dirichlet problem). Let $k < 1$, $k \neq 0$ and $0 < m < 1$. It is required to determine in D the function $u(x,y)$, which is satisfies the following conditions:

$$u(x,y) \in C^2(D^+ \cup D^-) \cap C^1(D) \cap C(\overline{D}), \tag{23}$$

$$Lu(x,y) \equiv 0, \quad (x,y) \in D^+ \cup D^-, \tag{24}$$

$$u(x,\beta) = \varphi(x), \quad u(x,-\alpha) = \psi(x), \quad 0 \leq x \leq l, \tag{25}$$

$$u(l,y) = 0, \quad -\alpha \leq y \leq \beta, \tag{26}$$

$$u(0,y) = 0, \quad -\alpha \leq y \leq \beta, \tag{27}$$

where $\varphi(x)$, $\psi(x)$ - sufficiently smooth functions, $\varphi(0) = \psi(0) = \varphi(l) = \psi(l) = 0$, $D^+ = D \cap \{y > 0\}, D^- = D \cap \{y < 0\}$.

Problem 2.2 (Dirichlet problem). Let $k < 1$ and $1 \leq m < 2$. It is required to determine in D the function $u(x,y)$, which is satisfies the following conditions:

$$u(x,y) \in C^2(D^+ \cup D^-) \cap C(\overline{D}), \tag{28}$$

$$\lim_{y \to 0+0} y^{m-1} u_y(x,y) = - \lim_{y \to 0-0} (-y)^{m-1} u_y(x,y), 0 < x < l, 1 < m < 2, \tag{29}$$

$$\lim_{y \to 0+0} \frac{u_y(x,y)}{\ln y} = - \lim_{y \to 0-0} \frac{u_y(x,y)}{\ln(-y)}, \quad 0 < x < l, \quad m = 1. \tag{30}$$

$$Lu(x,y) \equiv 0, \quad (x,y) \in D^+ \cup D^-, \tag{31}$$

$$u(x,\beta) = \varphi(x), \quad u(x,-\alpha) = \psi(x), \quad 0 \leq x \leq l, \tag{32}$$

$$u(l,y) = 0, \quad -\alpha \leq y \leq \beta, \tag{33}$$

$$u(0,y) = 0, \quad -\alpha \leq y \leq \beta. \tag{34}$$

Problem 2.3 (problem Keldysh). Let $k \geq 1$ and $0 < m < 1$. It is required to determine in D the function $u(x,y)$, which is satisfies the conditions of (23) - (26).

Problem 2.4 (problem Keldysh). Let $k \geq 1$ and $1 \leq m < 2$. It is required to determine in D the function $u(x,y)$, which is satisfies the conditions of (28) - (33)

Here, for example, we found for **2.3** problem for the equation (3) at $k \geq 1$ and $0 < m < 1$, the solution of which is represented as the sum of a number of Fourier - Bessel

$$u(x,y) = \sum_{n=1}^{+\infty} u_n(y) X_n(x), \tag{35}$$

where

$$X_n(x) = x^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(\lambda_n x),$$

$$\lambda_n = \frac{\mu_n}{l}, \quad n = 1, 2, 3, ...,$$

$J_\nu(t)$ - function Bessel of the first kind, μ_n - roots of $J_{\frac{k-1}{2}}(\mu_n) = 0$,

$$u_n(y) = \begin{cases} \frac{1}{E(n)\sqrt{\alpha\beta}} \left(\varphi_n \sqrt{\alpha y} E_n(\alpha, y) + \psi_n \sqrt{\beta y} M_n(y, \beta)\right) & y > 0, \\ \frac{1}{E(n)\sqrt{\alpha\beta}} \left(\varphi_n \sqrt{-\alpha y} N_n(\alpha, -y) + \psi_n \sqrt{-\beta y} E_n(-y, \beta)\right), & y < 0, \end{cases}$$

where

$$M_n(y, \beta) = I_{\frac{1}{2q}}(p_n \beta^q) K_{\frac{1}{2q}}(p_n y^q) - I_{\frac{1}{2q}}(p_n y^q) K_{\frac{1}{2q}}(p_n \beta^q),$$

$$N_n(\alpha, -y) = \overline{Y}_{\frac{1}{2q}}(p_n(-y)^q) J_{\frac{1}{2q}}(p_n \alpha^q) - \overline{Y}_{\frac{1}{2q}}(p_n \alpha^q) J_{\frac{1}{2q}}(p_n(-y)^q),$$

$$E_n(\alpha, y) = \overline{Y}_{\frac{1}{2q}}(p_n \alpha^q) I_{\frac{1}{2q}}(p_n y^q) + J_{\frac{1}{2q}}(p_n \alpha^q) K_{\frac{1}{2q}}(p_n y^q),$$

$$E_n(-y, \beta) = I_{\frac{1}{2q}}(p_n \beta^q) \overline{Y}_{\frac{1}{2q}}(p_n(-y)^q) + J_{\frac{1}{2q}}(p_n(-y)^q) K_{\frac{1}{2q}}(p_n \beta^q),$$

$$q p_n = \sqrt{a^2 + \lambda_n^2}, \quad q = (2-m)/2,$$

provided that, for all $n \in \mathbb{N}$

$$E(n) = J_{\frac{1}{2q}}(p_n \alpha^q) K_{\frac{1}{2q}}(p_n \beta^q) + I_{\frac{1}{2q}}(p_n \beta^q) \overline{Y}_{\frac{1}{2q}}(p_n \alpha^q) \neq 0. \tag{36}$$

For example, the condition is violated (36) for some $\alpha, \beta, l, k, m, a$ and $n = s \in \in \mathbb{N}$, i.e. $E(s) = 0$. In this case the problem (23) - (26) with $\varphi(x) = \psi(x) \equiv 0$ has a nontrivial solution

$$u_s(x,y) = \begin{cases} \frac{E_s(\alpha,y)\sqrt{y}}{J_{\frac{1}{2q}}(p_s \alpha^q)} X_s(x), & y > 0, \\ \frac{E_s(-y,\beta)\sqrt{-y}}{I_{\frac{1}{2q}}(p_s \beta^q)} X_s(x), & y < 0. \end{cases}$$

The expression $E(n)$ represented as follows:

$$E(n) = I_{\frac{1}{2q}}(p_n \beta^q) \gamma(n),$$

where

$$\gamma(n)=J_{\frac{1}{2q}}(\widetilde{\mu_n}\widetilde{\alpha_q})\frac{K_{\frac{1}{2q}}(p_n\beta^q)}{I_{\frac{1}{2q}}(p_n\beta^q)}+\overline{Y}_{\frac{1}{2q}}(\widetilde{\mu_n}\widetilde{\alpha_q})=\gamma_1(n)+\gamma_2(n),$$

$$\widetilde{\alpha_q}=\alpha_q/l,\quad \alpha_q=\alpha^q/q,\quad \widetilde{\mu_n}=\mu_n\widetilde{p_n},\quad \widetilde{p_n}=\sqrt{1+(al/\mu_n)^2}.$$

The existence of zeros $\gamma(n)$ with respect to $\widetilde{\alpha_q}$ follows from the fact that the functions $J_{\frac{1}{2q}}(\widetilde{\mu_n}t)$ and $\overline{Y}_{\frac{1}{2q}}(\widetilde{\mu_n}t)$, $t=\widetilde{\alpha_q}$, are linearly independent solutions Bessel equation

$$y^{''}(t)+\frac{1}{t}y^{'}(t)+\left[\widetilde{\mu_n}^2-\left(\frac{1}{2qt}\right)^2\right]y(t)=0. \tag{37}$$

It follows that the function $J_{\frac{1}{2q}}(\widetilde{\mu_n}t)$ and $\gamma(n)$ are also linearly independent solutions equation (37). According to the general theory of linear differential equations zeros of two linearly independent solutions Bessel equation strictly alternate, i.e, the interval between any successive zeros of any of those decisions is contained in exactly one zero otherwise. Function $J_{\frac{1}{2q}}(\widetilde{\mu_n}t)$ has a countable set positive zeros. Then the function $\gamma(n)$ is also countable with respect to the set of positive zeros $t=\widetilde{\alpha_q}$.

The following assertion.

Theorem 0.2.1. *If the solution of the problem (23) - (26) exists, it is unique if and only if, when the conditions (36) for all* $n\in\mathbb{N}$.

Since $\widetilde{\alpha_q}$, β and k - any number of job gaps, the expression $E(n)$ for sufficiently large n may be sufficiently small, i.e. there is "little problem denominators. "To solve this problem, you must show there are numbers $\widetilde{\alpha_q}$, β and k, such the expression $E(n)$ bounded away from zero for sufficiently large n.

Lemma 0.2.1. *If* $\widetilde{\alpha_q}=p/t,\ p,t\in\mathbb{N},\ (p,t)=1$ *and the condition* $k\neq\frac{1}{p}(4td+3t-4r)-2,\ r,\ d\in\mathbb{N}_0$, *then there are constant* $C_0>0$ *and* $n_0\in\mathbb{N}$, *depending on* n,m,a *and* α,β, *such that for all* $N>n_0$ *satisfies the estimate*

$$|\sqrt{n}\gamma(n)|\geq C_0>0. \tag{38}$$

If specified in lemma 0.2.1 of numbers $\widetilde{\alpha_q}$ $E(s)=0$ for some $n=s=r_1,r_2,...r_h$, where $1\leq r_1<r_2<...<r_h\leq n_0$, r_h and h - natural numbers that are necessary and sufficient for the solvability of the problem Keldysh the following conditions:

$$\varphi_s\sqrt{\alpha}J_{\frac{1}{2q}}(p_s\alpha^q)+\psi_s\sqrt{\beta}I_{\frac{1}{2q}}(p_s\beta^q)=0,\quad s=r_1,r_2,...r_h. \tag{39}$$

Solution (23) - (26) in this case, It is defined as the sum of the series

$$u(x,y)=\left(\sum_{n=1}^{r_1-1}+...+\sum_{n=r_{h-1}+1}^{r_h-1}+\sum_{n=r_h+1}^{+\infty}\right)u_n(y)X_n(x)+\sum_s u_s(x,y), \tag{40}$$

here in the last sum of s takes values $r_1, r_2, ...r_h$, the function $u_s(x,y)$ is determined by the following formula:

$$u_s(x,y) = \begin{cases} \left[\frac{\varphi_s\sqrt{y} I_{\frac{1}{2q}}(p_s y^q)}{\sqrt{\beta} I_{\frac{1}{2q}}(p_s \beta^q)} + \frac{C_s E_s(\alpha, y)}{J_{\frac{1}{2q}}(p_s \alpha^q)}\right] X_s(x), & y > 0, \\ \left[\frac{\psi_s\sqrt{-y} J_{\frac{1}{2q}}(p_s (-y)^q)}{\sqrt{\alpha} J_{\frac{1}{2q}}(p_s \alpha^q)} + \frac{C_s E_s(-y, \beta)}{I_{\frac{1}{2q}}(p_s \beta^q)}\right] X_s(x), & y < 0, \end{cases}$$

where C_s - arbitrary constants, the final amount of (40) should be considered to be zero, if the lower limit greater than the upper.

Thus proved

Theorem 0.2.2. *Let the function $\varphi(x), \psi(x) \in C^4[0,l]$ and $\varphi'(0) = \psi'(0) = = \varphi''(0) = \psi''(0) = 0$, $\varphi(l) = \psi(l) = \varphi''(l) = \psi''(l) = 0$, and the estimate (38) when $n > n_0$. Then, if $E(n) \neq 0$ for all $n = \overline{1, n_0}$, then there exists a unique solution of the problem (23) - (26), and the decision by a number of (35); if $E(n) = 0$ for some $n = r_1, r_2, ... r_h \leq n_0$, then the problem (23) - (26) is solvable only if the following conditions are satisfied (39) and the solution in this case is determined by a number (40).*

Similar results were obtained for other tasks 1.2, 2.1, 2.2, 2.4, namely the established criteria of uniqueness, solutions constructed as the sum of the series proved the convergence of the series in the class regular solutions.

Publications on the subject of the dissertation

Articles in peer reviewed scientific journals, included in the WAC list

1. Safina, R.M. A criterion of uniqueness of a solution to the Dirichlet problem with the axial symmetry for the three-dimensional mixed type equation with the Bessel operator / R. M. Safina // Russian Mathematics [Izvestiya Vuz.Mathematika], 2014, Volume 58, Issue 6, pp 69 – 73.

2. Safina, R.M. The Dirichlet problem for Pulkin's equation in rectangular domain, / R. M. Safina // Vestnik of Samara state university, Natural science series, 2014, no. 10, p. 91 – 101.

3. Safina, R.M. The Keldysh problem for Pulkin's equation in rectangular domain, / R. M. Safina // Vestnik of Samara state university, Natural science series, 2015, no. 3(125), p. 53 – 63.

4. Safina, R.M. The Keldysh problem for mixed-type equation of the second kind with a Bessel operator / R. M. Safina // Differential Equations , 2015, no. 10. pp 1354 - 1366.

Publications in other publications

5. Safina, R. M. On the formulation of the Tricomi problem for the equation mixed type with a Bessel operator / R. M. Safina // International Conference dedicated to 100 - anniversary of the Sobolev "Differential equations. Functional

spaces. Theory approximations "(Novosibirsk, 5 - 12 October 2008): Abstracts of papers /. Ying - Mathematics SB RAS. Novosibirsk. "2008 — S. 202.

6. Safina, R. M. Solution N for a B - elliptic equation by Green function / R. M. Safina // Proceedings of the international conference dedicated to the 70th anniversary of the Moscow State University Rector Academician VA Sadovnichy. — M: Publishing house"University Book — 2009. — C. 205/ – 206.

7. Safina, R. M. On the Dirichlet problem for the equation in Pulkina rectangular area / R. M. Safina // Proceedings of the Mathematical Center named NI Lobachevsky: Proceedings of the International Scientific Conference "Boundary problems for differential equations and analytic functions - 2014". — Kazan.. Publishing House of Kazan Univ.— 2014 — T.49 — C. 298 - 301.

8. Safina, R. M. The Dirichlet problem for a mixed-type equation the second kind with a singular coefficient / R. M. Safina // Proceedings of the Twelfth International Kazan Summer Research School - Conference "Theory of Functions, its Applications and Adjacent Problems". - 2015. — T.51 — C. 249 – 251.

9. Safina, R. M. Keldysh problem for a mixed-type equation with Singular Coefficient in the rectangular area / R. M. Safina // Abstracts of the International Conference "Differential Equations and Mathematical Modeling". - Ulan-Ude. — 2015. — C. 270 - 271.