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FORMAL MATRIX RINGS AND THEIR ISOMORPHISMS

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Abstract

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General points of work

Relevancy. This dissertation is about isomorphism an automorphism problem for formal matrix rings. This work mainly concern formal matrix rings over ring and upper triangular matrix rings.

Homological classification of rings is one of the classical problems of algebra. It was questioned how ring properties affects structure category of modules over ring. This problem was studied by K. Morita [1], who in 1958 described equivalent rings in terms of Morita context. It is ordered tuple $(R, S, M, N, \varphi, \psi)$, where R, S – rings, M, N – bimodules, which are connected by bimodule homomorhisms φ и ψ.

Given Morita context one can define matrix ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ with usual operations of matrix addiction and multiplication. Those rings are called Morita context rings or formal matrix rings of order 2. Similarly it is easy to define formal matrix rings of arbitrary order n. If ring R contains nontrivial idempotent e, ring R can be viewed as formal matrix ring $\begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. Thus endomorpism ring of decomposable module is formal matrix ring. So it is necessary to study formal matrix rings. Ring-theoretic properties of formal matrix rings were studied recently [2] [3] [4] [5] [6].

Upper triangular formal matrix rings are used for providing examples of rings with asymmetric properties (e.g. left noetherian, but not right noetherian). While studying of upper triangular formal matrix question arise: given upper triangular

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matrix ring can one found its diagonal rings [7]? In original paper and in [8] explicit form of isomorphism of upper triangular Morita context rings was found: at first with restriction of diagonal rings not having nontrivial idempotents, and then with restriction with diagonal rings be semicentral reduced. In paper [9] explicit form of isomorphism of Morita context rings with zero trace ideals was found. In paper [10] explicit form of isomorphism of upper triangular formal matrix rings was found, but it has iterative form: for a series of rings there are series of isomorphisms. Problem of describing explicit form of isomorpism is still open.

Natural special case of formal matrix rings is Morita context rings over ring R, that is formal matrix rings of the form $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$. Such rings were introduced and studied by P.A. Krylov in paper [11]. They were marked as $K_s(R)$, where s – central element of R, which define multiplication in $K_s(R)$. For those rings isomorphism problem was studied and solved: under which restriction on s and t rings $K_s(R)$ and $K_t(R)$ are isomorphic? Under some assumption on ring R, this isomorphism is equivalent to s and t differ by invertible element, up to automorphism of R. This paper started series of publications. In [12] this criteria lowered assumptions on R, and in paper [13] new class of formal matrix rings over R of order n was introduced, with similar isomorphism condition. For formal matrix rings over ring notions of determinant, characteristic polynomial werr introduced, Hamilton-Caley theorem was proved. Isomorphism problem was prooved even for greater class of formal

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[8] Anh P.N., van Wyk L. Automorphism group of generalized triangular matrix rings// Linear Algebra and its Appl. – 2011. – V. 434. – P. 1018—1026.

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[10] Anh P.N., van Wyk L. Isomorphisms between strongly triangular matrix rings // Linear Algebra and its Appl. – 2013. – V. 438. – P. 4374—4381.

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[12] Tang G., Li C., Zhou Y. Study of Morita contexts // Comm. in Algebra. – 2014. – V. 42. – № 4. – P. 1668—1681.

[13] Tang G., Zhou Y. A class of formal matrix rings // Linear Algebra and its Appl. – 2013. – V. 438. – № 12. – P. 4672—4688.

matrix rings [14]. Grothendieck and Whitehead groups were studied. Big collection of research on formal matrix rings can be found in [15].

Since formal matrix rings extends matrix rings, it is natural to study the same problems. It is fairly known that Skolem-Noether theorem implies that all automorphisms of matrix ring over field are inner. However that doesn't hold even for arbitrary commutative ring. Outer automorphism group was introduced as measure of difference between group of all automorphisms and group of inner automorphisms. In paper [16] it was shown that given matrix algebra of finite order there is a fixed power, in which any automorphism is inner.

It is hard to describe automorphism group of arbitrary matrix algebra, but it is easy to see that isomorphism $M_2(R) \cong M_2(S)$ over commutative rings R and S is equivalent to isomorphisms of rings R and S. That doesn't hold true for noncommutative case. It was studied in papers [17] [18], [19], where examples were provided. Next step was studying incidence algebras, which are a special case of formal matrix rings.

Incidence algebras [20] $I(X, R)$ of locally-finite partially ordered set X over commutative ring R was introduced by Gian-Carlo Rota Жан-Карло Рота [21] as natural tool for combinatorics. Mobius function was defined, Mobius invertion formula and Inclusion–exclusion principle were proved. Soon it became clear that incidence algebras are interested as standalone object. They contains R.P. Stanley introduced

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and solved [22] [23] isomorphism problem for incidence algebras of posets over field. It was stated that $I(X, F) \cong I(Y, F)$ implies isomorphism of orders X and Y. W. R. Belding [24] extended this theorem on qosets, under assumptions that at least one of the qosets is finite. It was followed by series of papers [25] [26] [27] [28] [29]. Review of all results can be found in last papers. Hovewer sometimes base order can not be retrieved. In paper [30] was given an example of ring R, which is isomorphic to rings $M_2(R)$, $T_2(R)$, $R \oplus R$. Isomorphism problem was studied and for groupoids [31], semirings [32]. Explicit form of isomorphisms of incidence algebras was found [33] [34] [35].

This dissertation continues study of isomorphism problem for formal matrix rings over ring. During research it became clear that it is natural to study formal matrix rings with every bimodule being either 0 or ${}_RR_R$. Since it is similar to incidence algebras, it was called generalized incidence algebras.

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Goals and problems of dissertation. Goals of dissertations:

1. study of isomorphism problem for formal matrix rings;
2. explicit form of automorphism of formal matrix rings.

Main problems of dissertation:

1. finding isomorphism criteria of formal matrix rings in terms om multiplicative coefficients (similar to Krylov's theorem);
2. explicit form of automorphism of upper-triangular formal matrix rings and formal matrix rings with zero trace ideals;
3. describing formal matrix rings up to isomorphisms;
4. studying automorphisms of formal matrix rings, conditions for all automorphisms being inner, finding outer automorphisms group.

Results applied for defense. Main results applied for defense are:

1. Solution of isomorphism problem for formal matrix rings of the form $M_{\beta,0,\ldots,0}(R)$ and $M_{\beta,\beta,\ldots,\beta}(R)$ under some restrictions on ring R.
2. Gaining explicit form of isomorphisms of upper triangular matrix rings and formal matrix rings with zero trace ideals. Sufficient isomorphism conditions in term of multiplicative elements. As an application, description of automorphisms for some algebras of upper triangular matrix rings we provided.
3. Natural extension of incidence algebras on formal matrix rings were introduced and studied.
4. Solution of isomorphism problem for formal matrix incidence rings over commutative local ring. Generalization of Krylov theorem on this case. For formal matrices of order 3 and 4 counter examples are given.
5. Classification of generalized incidence algebras of order less or equal 4 up to isomorphism.

Scientifical contribution of thesis results Main results of thesis are new and proved by author himself. In collaboration papers with scientific supervisor, A.N. Abyzov author not included in thesis results, problems definition and research methods. Propositions 1.1.15 and 1.2.10 are prooved in collaboration with scientific supervisor. All other results are proved by author himself. In **conclusion** open problems that arise in this research area were formulated.

Research methods. Thesis utilize classical methods of theory of rings and modules. Accuracy of results provided are guaranteed by correct proofs.

Creditability and approbation. Main results of thesis published in 9 (nine) papers [1-9], 4 (four) of them [1-4] published in ‘’Перечень ВАК при Минобрнауки России рецензируемых научных изданий, в которых должны быть опубликованы основные научные результаты диссертаций на соискание ученой степени кандидата наук, на соискание ученой степени доктора наук”.

Theoretical and practical results of thesis. Thesis results are pure theoretic. They can be used in future research in theory of rings and modules. Moreover, thesis results an be used for writing manuscripts and teaching additional courses at universities.

Approbation Main results of thesis were presented at following conferences and seminars:

1. XII International conference ‘’Algebra, number theory and discrete geometry: modern problems and applications”, Tula, May 25–30 2015
2. XIV Russian junior conference ‘’Lobachevsky meetings–2015”, Kazan, October 22–27, 2015
3. International conference on algebra, analysis and geometry, Kazan, June 26 – July 2, 2016
4. Scientific seminar at chair of algebra of faculty of mechanics and mathematics of TSU, Tomsk, November 17, 2016
5. International conference ‘’Maltcev meetings”, Novosibirsk, November 21–25, 2016
6. XIV Russian junior conference ‘’Lobachevsky meetings–2017”, Kazan, November 24–29, 2017
7. Scientific seminar at chair of algebra of faculty of mechanics and mathematics of MSU, Moscow, November 27, 2017
8. Scientific seminars and overall conferences at chair of algebra and mathematical logic of Institute of mathematics and mechanics Kazan Federal University, Kazan, 2015—2018

Thesis structure and size. Thesis consists of introduction, three chapters, which are organized in paragraphs, conclusion and bibliography with 53 (fifty three) titles, included works by author on the same research. Thesis size – 156 (one hundred fifty six) pages.

Content of dissertation.

In **introduction** relevancy of work is justified, goals and problems are formulated, a brief overview of content is provided.

In **chapter 1** basic notions of theory of rings and modules are introduced. Formal matrix rings over ring are studied.

In paragraph 1.1 introduced notions of formal matrix ring and formal matrix ring over ring.

In definition 1.1.1 notion of Morita context is introduced. Morita context rings over R can be described by single element $s \in C(R)$ [36]. Such tings are marked as $K_s(R)$. In proposition 1.1.4 [37] stated theorem of P.A. Krylov on isomorphism of $K_s(R)$, $K_t(R)$ in terms of s, t.

Similarly, one can introduce notions of formal matrix ring of order n [38] and formal matrix ring of order n over R. In later case we have a set of coefficients.

Let $K_n(R : \{\varphi_{ijk}\})$ be a formal matrix ring over R of order n. Define $\eta_{ijk} = \varphi(1 \otimes 1)$ for all $1 \leq i, j, k \leq n$. Formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ is uniquely define bet set of central elements $\{\eta_{ijk} \mid 1 \leq i, j, k \leq n\}$. In this case formal matrix ring $K_n(R : \{\varphi_{ikj}\})$ will be marked as $K_n(R : \{\eta_{ikj}\})$.

Definition 1.1.11. Let $K = K_n(R : \{\eta_{ikj}\})$ be a formal matrix ring. Set $\eta = \{\eta_{ikj}\}$ will be called a set of *multiplicative coefficients*(or *multiplicative set*) and we will write $K = K_n(R; \eta)$.

In paragraph 1.2 special cases of formal matrix ring are considered. Notion of $M_n(R; s)$ [39] introduced, for which holds true the result similar to Krylov's theorem.

[36] Krylov P.A., Tuganbaev A.A. Formal matrices. – Springer, 2017. – 165 pp.

[37] Krylov P.A. Isomorphisms of generalized matrix rings // Algebra Logic. – 2008. – V.47 – № 4. – P. 258—262.

[38] Krylov P.A., Tuganbaev A.A. Formal matrices. – Springer, 2017. – 165 pp.

[39] Tang G., Zhou Y. A class of formal matrix rings // Linear Algebra and its Appl. – 2013. – V. 438. – № 12. – P. 4672—4688.

Definition 1.2.7 Let R be a ring, $n \geq 2$ и $\beta_1, ..., \beta_n \in C(R)$. Let

$$\eta_{ijk} = \beta_i^{\delta_{ik}-\delta_{ij}} \beta_j^{1-\delta_{jk}} = \begin{cases} 1, & \text{если } i = j \text{ или } j = k, \\ \beta_j, & \text{если } i, j, k \text{ различны}, \\ \beta_i \beta_j, & \text{если } k = i \neq j. \end{cases}$$

It is straightforward that $K_n(R; \{\eta_{ikj}\})$ is a formal matrix ring, which we will write as $\mathbb{M}_{\beta_1,...,\beta_n}(R)$.

Rings $M_n(R; s)$ are a special case of this construction: $M_n(R; s) = \mathbb{M}_{s,...,s}(R)$. For this class of formal matrix ring isomorphism problem was partly solved. In every case solveed, results similar to Krylov's theorem hold true.

Theorem 1.2.12. *Let R be a commutative ring, $n \geq 3$, $\beta, \gamma_1, ..., \gamma_n \in R$ and $\mathrm{ann}_R(\beta) \subseteq J(R)$. Then $\mathbb{M}_{\underbrace{\beta, 0, ..., 0}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,...,\gamma_n}(R)$ iff $\gamma_i = \alpha(\beta) v_i a_i$ for all $1 \leq i \leq n$, where $\alpha \in \mathrm{Aut}(R)$, $v_i \in U(R)$, и $1 = a_1 + a_2 + ... + a_n$ is a decomposition to a sum of orthogonal idempotents a_i.*

Theorem 1.2.15. *Let R be a commutative ring, such that $Z(R) \subseteq J(R)$. Let $n \geq 3$ and $\beta, \gamma_1, ..., \gamma_n \in R$. Then $\mathbb{M}_{\underbrace{\beta, \beta, ..., \beta}_{n}}(R) \cong \mathbb{M}_{\gamma_1,\gamma_2,...,\gamma_n}(R)$ iff $\gamma_i = \alpha(\beta) v_i$ for all $1 \leq i \leq n$, where $\alpha \in \mathrm{Aut}(R)$ and $v_i \in U(R)$.*

In paragraph 1.3 automorphisms of algebra $K_s(\mathbb{Z})$ are considered. In case of $s = \pm 1$ this algebra is isomorphic to $M_2(\mathbb{Z})$. It is straightforward that for $s \neq \pm 1$ there exists non-inner automorphism.

Theorem 1.3.8. *Let $\mathbb{Z} \ni s \neq 0, \pm 1$, $A = K_s(\mathbb{Z})$. Then for $s = p_1^{\alpha_1} ... p_n^{\alpha_n}$ with p_i being pairwise different primes, we have $\mathrm{Out}_{\mathbb{Z}}(A) \cong \underbrace{\mathbb{Z}_2 \times ... \times \mathbb{Z}_2}_{n}$.*

Chapter 2 concerns automorphisms of formal matrix ring with zero trace ideals. Paragraph 2.1 provide brief overview of history of this problem as well as latest known results. In theorems 2.1.9 [40] and 2.1.12 [41] stated results that gives explicit form of isomorphisms between formal matrix rings of order 2: in upper triangular case and in case of zero trace ideals respectively.

In theorem 2.1.12 [42] stated similar result for upper triangular formal matrix rings of order n. However, last theorem solves isomorphism problem in iterative form: it is equivalent to existing a series of special isomorphisms.

We will denote ring of upper triangular formal matrices of the form

$$\begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ 0 & R_2 & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}$$

as $T_n(\{R_i\}, \{M_{ij}\})$.

Definition 2.2.1. Ring of formal matrices of the form

$$\begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ M_{21} & R_2 & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & R_n \end{pmatrix}$$

is called *formal matrix ring with zero trace ideal*, if for every $i < j$ we have $M_{ij}M_{ji} = 0 = M_{ji}M_{ij}$.

Theorems 2.2.9 and 2.2.19 provide explicit form of isomorphisms of upper triangular formal matrix rings and formal matrix rings with zero trace ideals respectively.

Theorem 2.2.9 *Let $n \in \mathbb{N}$, $A_1 = T_n(\{R_i\}; \{M_{ij}\})$, $A_2 = T_n(\{R'_i\}; \{M'_{ij}\})$ and rings $R_1, ..., R_n$ be semicentral reduced. Let $\Phi : A_1 \to A_2$ be an isomorphisms. Then*

[40] Anh P.N., van Wyk L. Automorphism group of generalized triangular matrix rings // Linear Algebra and its Appl. – 2011. – V. 434. – P. 1018—1026.

[41] Boboc C., Dascalescu S., van Wyk L. Isomorphisms between Morita context rings // Linear and Multilinear Algebra. – 2012. – V. 60. – P. 545—563.

[42] Anh P.N., van Wyk L. Isomorphisms between strongly triangular matrix rings// Linear Algebra and its Appl. – 2013. – V. 438. – P. 4374—4381.

there exist permutation $\tau \in S_n$ and invertible matrix $U \in U(A_2)$, such that

$$\Phi([a_{ij}]) = U\left[\chi_{ij}(a_{\tau(i)\tau(j)})\right]U^{-1},$$

with

(1) $\chi_{ii} : R_{\tau(i)} \to R'_i$ *is a ring isomorphism,* $1 \le i \le n$*;*

(2) $\chi_{ij} : M_{\tau(i)\tau(j)} \to M'_{ij}$ – $R_{\tau(i)}$-$R_{\tau(j)}$ *is a bimodule isomorphisms with respect to* χ_{ii} *and* χ_{jj}, $1 \le i < j \le n$*;*

(3) *for every* $1 \le i, k, j \le n$, $a \in M_{\tau(i)\tau(k)}$, $b \in M_{\tau(k)\tau(j)}$,

$$\chi_{ij}(a \circ b) = \chi_{ik}(a) \circ \chi_{kj}(b).$$

Moreover, if conditions 1–3 hold, then the mapping $\Phi([a_{ij}]) = U\left[\chi_{ij}(a_{\tau(i)\tau(j)})\right]U^{-1}$ *is a ring isomorphism.*

Theorem 2.2.19 *Let* $t \in \mathbb{N}$, $A_1 = D_t(\{R_i\};\{M_{ij}\}) = (X_{ij})$, $A_2 = D_t(\{R'_i\};\{M'_{ij}\}) = (Y_{ij})$ *and rings* $R_1, ..., R_t$ *does not contain nontrivial idempotents. Let* $\varphi : A_1 \to A_2$ *be an isomorphism. Then there exust permutation* $\tau \in S_t$ *and invertible matrix* $U \in U(A_2)$*, such that*

$$\Phi([a_{ij}]) = U\left[\chi_{ij}(a_{\tau(i)\tau(j)})\right]U^{-1},$$

with

(1) $\chi_{ii} : R_{\tau(i)} \to R'_i$ *is a ring isomorphism,* $1 \le i \le t$*;*

(2) $\chi_{ij} : X_{\tau(i)\tau(j)} \to Y_{ij}$ – $R_{\tau(i)}$-$R_{\tau(j)}$ *is a bimodule isomorphism with respect to* χ_{ii} *u* χ_{jj}, $1 \le i, j \le t$*;*

(3) *for every* $1 \le i, k, j \le n$, $a \in X_{\tau(i)\tau(k)}$, $b \in X_{\tau(k)\tau(j)}$,

$$\chi_{ij}(a \circ b) = \chi_{ik}(a) \circ \chi_{kj}(b).$$

Moreover, if conditions 1–3 hold, then the mapping $\Phi([a_{ij}]) = U\left[\chi_{ij}(a_{\tau(i)\tau(j)})\right]U^{-1}$ *is a ring isomorphism.*

Paragraph 2.3 provides applications of results above to describing automorphisms of formal matrix rings.

As $Q(R)$ we will denote total quotient ring of R.

Theorem 2.3.3 *Let R be a commutative ring without nontrivial idempotents, $n \in \mathbb{N}$ и $\{I_{ij}\}_{1 \le i < j \le n}$ be a set of ideals of R, each of them containing at least of non-zero divisor, with the property $I_{ij} I_{jk} \subseteq I_{ik}$ for every $i < j < k$. Let $A = T_n((R); \{I_{ij}\}) \subseteq T_n(Q(R))$ be a formal matrix ring with usual operations of matrix addition and multiplication. Then every R-automorphism of ring A can be decomposed as $C_U \circ C_V$, with $U \in U(A)$, $V = \operatorname{diag}(h_1, ..., h_n) \in U(T_n(Q(R)))$, $C_V(A) = A$.*

As it shown in theorem 2.3.3, R-automorphisms of $T_n((R); \{I_{ij}\})$ are "almost" inner. However, in general case this result can not be specified to being just inner.

Example 2.3.4 Let $S = \mathbb{Z}[x, x^{-1}]$ be a ring of integer Laurent polynomials, and $I = 2\mathbb{Z}[x, x^{-1}]$ be an ideal of S. Then $R = \mathbb{Z} + 2\mathbb{Z}[x, x^{-1}]$ is a subring of S, consisting of all Laurent polynomials, with even coefficients at nonzero powers of x. R is a commutative ring without nontrivial idempotents. Mapping

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & xb \\ 0 & c \end{pmatrix}$$

is a non-inner R-automorphism of a ring $A = \begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$.

However, for R being Dedekind of factorial ring theorem 2.3.3 can be specified to all automorphisms of $T_n((R); \{I_{ij}\})$ being inner.

In **chapter 3** isomorphism problem is considered for formal matrix rings over ring. Paragraph 3.1 provide brief overview of ring-theoretic properties of incidence algebras.

Paragraph 3.2 consider formal matrix rings, with every bimodule M_{ij} being either zero or ${}_R R_R$ for a fixed ring R. Basic properties are given.

Definition 3.2.1. Let R be a ring, X be a set with binary relation $\le$ and $\eta = \{\eta_{abc} \mid a, b, c \in X\}$ be a set of central elements of R, such that following hold:

(1) $a \le a$ for every $a \in X$ (reflexivity);

(2) $a \leq b$, $b \leq c$, $\eta_{abc} \neq 0$ implies $a \leq c$ for every $a, b, c \in X$ (η-transitivity);

(3) a set $\{c \in X \,|\, a \leq c \leq b,\ \eta_{acb} \neq 0\}$ is finite for every $a, b \in X$ (generalize local finiteness);

(4) $\eta_{aab} = \eta_{abb} = 1$ for every $a \leq b \in X$;

(5) $\eta_{abc}\,\eta_{acd} = \eta_{abd}\,\eta_{bcd}$ for every $a \leq b \leq c \leq d \in X$.

Such set η is called *multiplicative set*, and its elements are called *multiplicative coefficients.* Elements η_{aab} and η_{abb} are called *trivial* for $a \leq b$.

Relation $\leq$ is called *η-qoset.* η-qoset on X is called *total*, if $a \leq b$ for every $a, b \in X$.

Definition 3.2.2. Let R be a ring, and on set X we have η-qoset. Consider set

$$K(X, R; \eta) = \{f : X \times X \to R \,|\, f(x, y) = 0, \text{ если } x \not\leq y\}.$$

Define on $K(X, R; \eta)$ element-wise operations of addiction and multiplication on a scalar. Define multiplication as:

$$(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y)\, \eta_{xzy}.$$

It is straightforward that under those operations set $K(X, R; \eta)$ becomes a ring, that we will call *formal matrix incidence ring*

Definition 3.2.3. *η-poset* on X is an η-qoset on X, such that $a \leq b$, $b \leq a$, $\eta_{aba} \neq 0$ implies $a = b$.

Definition 3.2.4. Let R be a commutative ring. Then $K(X, R; \eta)$ becomes R-algebra, that we will call *generalized incidence algebra* and will denote by $I(X, R; \eta)$.

Let R be a ring, X be an η-qoset. Define equivalence relation on X as: $x \sim y \Leftrightarrow x \leq y, y \leq x, \eta_{xyx} \in U(R)$.

Corollary 3.2.25. *Let R be a commutative local ring, X be a finite η-qoset. Then*

$$J(I(X, R; \eta)) = \{f \in I(X, R; \eta) \,|\, f(x, y) \in J(R), \text{ if } x \sim y\}.$$

On the set $\widehat{X} = \{[r_1], ..., [r_k]\}$ of equivalence classes we can define relation $\leq$ as: $[r_i] \leq [r_j] \Leftrightarrow r_i \leq_X r_j$. Then for $x' \sim x$, $y' \sim y$, $z' \sim z$ we have $\eta_{xyz} = v\, \eta_{x'y'z'}$, $v \in U(R)$. In particular, for $x \sim y \sim z$ we have $\eta_{xyz} \in U(R)$.

In paragraph 3.3 isomorphism problem is considered for generalized incidence algebras over commutative local ring.

Theorem 3.3.4. *Let R be a commutative local ring, X be a finite η-qoset, Y be a finite μ-qoset. Then algebras $A = I(X, R; \eta)$ and $B = I(Y, R; \mu)$ are isomorphic as R-algebras iff following hold:*

(1) there exists a bijection $\varphi : X \to Y$, that keep orders $'\leq'$ and $'\sim'$;

(2) there exists a function $g : Y \times Y \to U(R)$ with

$$\eta_{xyz}\, g(\varphi(x)\varphi(z)) = \mu_{\varphi(x)\varphi(y)\varphi(z)}\, g(\varphi(x)\varphi(y))\, g(\varphi(y)\varphi(z)),$$

for every $x \leq y \leq z \in X$.

It is relatively hard to check for existence of function g. But if we drop sufficiency condition last theorem can be greatly simplified.

Theorem 3.3.11. (isomorphism theorem in Krylov's form) *Let R be a commutative local ring, X be a finite η-qoset, and Y be a finite μ-qoset. If $I(X, R; \eta) \cong I(Y, R; \mu)$ as algebras, then there exists a bijection $\varphi : X \to Y$ that keep order $\leq$, such that for every $x \leq y \leq z \in X$ we have $\eta_{xyz} = v_{xyz}\, \mu_{\varphi(x)\varphi(y)\varphi(z)}$, $v_{xyz} \in U(R)$.*

It is natural to ask whether theorem 3.3.11 provides a sufficient condition of isomorphism. For $|X| = 2$ it is holds due to theorem by Krylov.

Theorem 3.3.12. *Condition in theorem 3.3.11 is not sufficient for commutative local rings even for $|X| = 3$.*

Example in the proof 3.3.12 is not just any generalized incidence algebra, but a formal matrix ring over ring R. However, this example greatly utilize nontrivial idempotents in ring R. So we should consider case of $R = F$ be a field. In this case we can formulate the problem if inversing theorem 3.3.11 in explicit form.

Definition 3.3.14. Let R be a commutative ring, X be an η-qoset. Multiplicative set η is called $\{0,1\}$-*multiplicative set*, if every $\eta_{ijk} \in \eta$ is either 0 or 1. Generalized incidence algebra $I(X, R; \eta)$ with $\{0,1\}$-multiplicative set η is called $\{0,1\}$-*algebra*.

Given generalized incidence algebra $I(X, F; \eta)$ over field F one can construct $\{0,1\}$-algebra $I(X, F; \overline{\eta})$ as: $\eta_{ijk} \neq 0 \;\Leftrightarrow\; \overline{\eta}_{ijk} = 1$. It is straightforward that inversibility of theorem 3.3.11 is equivalent to algebras $I(X, F; \eta)$ and $I(X, F; \overline{\eta})$ being isomorphic.

Theorem 3.3.14. *For a field F and $|X| = 3$ theorem 3.3.11 provides sufficient condition of isomorphism.*

In paragraph 3.4 considered problem of inversing theorem 3.3.11 for generalized incidence algebras over field with $|X| = 4$. Two one-parameter classes of generalized incidence algebras of order 4 over a field F are introduced: $A_4(s; F)$ and $A_4^r(s; F)$ with $s \in F$. For every algebra in both of this classes every multiplicative coefficient, except one, in equal either to 0 or 1.

Theorem 3.4.3. *Let F be a field, $s, t \in F$. Algebras $A_4(s; F)$ and $A_4(t; F)$ isomorphic iff $s = t$.*

Corollary 3.4.4. *Let F be a field, containing at least 3 elements, and $|X| = 4$. Then theorem 3.3.11 does not provide sufficient condition of isomorphism for algebras $I(X, F; \eta)$.*

Due to this negative result it became interesting to find all isomorphism classes. In particular, if $A = I(X, F; \eta)$, $B = I(X, F; \overline{\eta})$ its $\{0,1\}$-algebra and $A \ncong B$, could algebra A be isomorphic to some other $\{0,1\}$-algebra of order $|X|$?

Proposition 3.4.6. *Let R be a commutative local ring, X be a finite η-qoset, $A = I(X, F; \eta)$, $B = I(X, F; \overline{\eta})$ its $\{0,1\}$-algebra. If algebra A is isomorphic to some $\{0,1\}$-algebra $I(Y, R; \mu)$, then $|Y| = |X|$ and $A \cong B$.*

Next theorem describes all generalized incidence algebras of order 4 over a field up to isomorphism.

Theorem 3.4.8. *Let F be a field and $|X| \leq 4$. Let $A = I(X, F; \eta)$ be a generalized incidence algebra and $A' = I(X, F; \bar{\eta})$ its $\{0,1\}$-algebra. Then one and only one of the following conditions hold:*

(1) algebras A and A' are isomorphic;

(2) algebra A is isomorphic to $A_4(s; F)$ for some unique s not equal to 0 and 1;

(3) algebra A is isomorphic to $A_4^r(t; F)$ for some unique t not equal to 0 u 1.

Paragraph 3.5 provided a technique for extending all results from chapter 3 from generalized incidence algebras to formal matrix incidence rings. Those results gives a partial answer to problems by P.A. Krylov and A.A. Tuganbaev[43].

In **conclusion** open problems that arise in this research area were formulated.

Conclusion

In thesis isomorphism problem for formal matrix rings were considered. As a brief overview of thesis results, one can distinguish those:

1. Solving isomorphism problem for formal matrix rings $\mathbb{M}_{\beta,0,\ldots,0}(R)$ and $\mathbb{M}_{\beta,\ldots,\beta}(R)$.
2. Describing outer automorphism group of Krylov ring over integers.
3. Explicit form of isomorphisms of formal matrix rings with zero trace ideals.
4. Introduction and study of natural extension of incidence algebras to formal matrix rings.
5. Solving isomorphism problem for formal matrix rings over commutative local ring.
6. Describing generalized incidence algebras of order 4 over field up to isomorphism.

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