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2 **Under Desert Dunes: the Kunin-van der Veer Legacy Revisited**

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31 **Analytic Solutions for Fresh Groundwater Lenses Floating on Saline Water**
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34 **Abstract.** A fully-saturated lens of steady fresh groundwater floating in a homogeneous and
35 isotropic desert sandy aquifer is analytically studied based on a hydrological model by Kunin
36 and interface solution by Van Der Veer. A static saline groundwater is beneath the lens. A
37 phreatic surface of moving fresh water inside the lens is partially recharged (either naturally or
38 by managed aquifer recharge) from the vadose zone and partially exfiltrates to it. A spatially
39 focused recharge and intensive evapotranspiration preserve a steady downward-upward
40 topology of fresh water motion. In terms of the 1-D Dupuit-Forchheimer approximation in a
41 horizontal in-lens saturated flow a boundary value problem (BVP) for an ODE for the Strack
42 potential is solved. The shape of the water table and, based on the Ghijben-Herzberg assumption
43 , the interface are found. The total volume of the positive-pore pressure water flowing within the
44 lens is evaluated. Constant infiltration and evaporation rates as well as evaporation linearly
45 decreasing with depth of the water table (counted from the ground surface) are considered. The
46 case of 2-D flow is tackled by the Toth model. A triangular analytic element approximates a half
47 of the flow domain and consists of an isobaric side and two no-flow sides. Conformal mapping
48 of this triangle onto a reference plane and solution of the Dirichlet BVP in a half-plane deliver
49 the distribution of infiltration-exfiltration intensity along the water table, total flow rate and locus
50 of the hinge point. A mathematically more cumbersome approximation of the flow domain
51 assumes the water table to be a tilted straight line but the interface to be found as a free
52 boundary. Solution of the corresponding BVP uses a curvilinear triangle in the hodograph
53 plane.

54 **Key words:** fresh-saline groundwater, interface, phreatic surface, conformal mappings,

55 2-D Tothian and 1-D Dupuit-Forchheimer flow models, hydrology of arid desert dunes.

56

57 *“When you're in the muck you can only see muck. If you somehow*
58 *manage to float above it, you still see the muck but you see it from a*
59 *different perspective. And you see other things too. That's the consolation*
60 *of philosophy.”*

61

David Cronenberg

62 **1. Introduction**

63 Fresh groundwater lenses are formed in coastal aquifers, oceanic islands and wetlands due to
64 rainfall-induced recharge to the lens phreatic surface (see e.g. Chesnaux and Allen, 2008, Eeman
65 et al., 2011, Houben et al. , 2014a, Houben et al., 2014b, Houben et al., 2018, Schot et al., 2004,
66 Vacher, 1988, van der Veer, 1977, Werner and Laattoe, 2016). Fresh-saline interface is a part of
67 the boundary of such lenses, found and investigated in Bermuda, Germany, Holland, Paraguay,
68 and other countries. The interface crops out into a well-defined surface water body (sea, river,
69 lake or drainage ditch) into which a light fresh groundwater discharges.

70 Terrestrial freshwater lenses (TFLs) in arid or semi-arid climates are not in a direct and
71 permanent contact with surface water (see Fig.2a, d in Laattoe et al., 2017). Fresh floating
72 groundwater lenses (hereafter abbreviated as FFGLs) are a special type of TFLs. FFGLs are
73 subtended by an interface and capped by a phreatic surface recharged from the vadose zone.
74 Post-rain infiltration or ephemeral ponding of topographic depressions filled after occasional
75 runoff is balanced by evaporation from the phreatic surface, if it is shallow enough. For example,
76 Hampton (1963) for lenses in Oregon estimated evaporation from the phreatic surface of lenses
77 in coastal dunes as 7 inches/year and pointed out: *“In parts of the area where the water table is*
78 *shallow and the vegetation is dense, the evapotranspiration doubtless is much greater;*

79 *conversely, in the barren dune areas, and at places where the water table is relatively deep, the*
80 *evapotranspiration probably is considerably less than 7 inches per year.” Emikh (1963)*
81 analytically studied channel-fed TFLs in Central Asia (see the corresponding sketch in Fig.2b
82 of Laattoe et al. 2017) for which evaporation was assumed the only factor discharging
83 groundwater from the lens.

84 FFGLs have been detected and studied in Australia, Kuwait, Namibia, Oman, UAE (see e.g.
85 Al-Weshah and Yihdego, 2016, Cendón et al., 2010, Kwarteng et al., 2000, Laattoe et al., 2017,
86 Macumber, 2003, Milewski et al., 2014, Rizk and Al-Sharhan, 2003, Young et al., 2004).
87 Unlike humid lenses in Europe (Voortman et al., 2015), infiltration-evaporation on the water
88 table of these FFGLs and ensued groundwater motion inside is much less investigated (Laattoe et
89 al., 2017).

90 In this paper, we use the studies of Kunin (1959)¹ in the Karakum desert of Turkmenistan. A
91 large-scale water supply from FFGLs in Central Asia started in the 1960s, with the arrival of the
92 Soviet oil-gas industry to the deserts of this region that required much more fresh water than the
93 local traditional animal husbandry. Bairamova (2013) reports that pumping from one of the
94 largest TFLs in Turkmenistan commenced in 1963, with 56 well galleries drilled in 1963-1969;
95 49 of them are being exploited around 40 years. A common interface delineating TFLs is a 1g/l
96 isoconcentric surface. Bairamova (2013) emphasizes that the genesis of TFLs (including FFGLs)
97 in Turkmenistan is still not well understood.

98 Pumping from TFLs of these deserts was compounded with managed aquifer recharge
99 (MAR) through engineered infiltration ponds placed just above the lenses (see the photos of the
100 Sansyz MAR project, retrieved from Kunin, 1959, supplementary file 1). These excavations
101 harnessed the natural occasional runoff and converted it into an intensive infiltration spot

¹ Dr. Vladimir Kunin (1906-1976), the corresponding member of the Academy of Sciences of USSR, one of the founders of the Institute of Deserts (city of Ashkhabad), Turkmen Academy of Sciences of the Turkmen Soviet Socialist Republic, published extensively on hydrology of deserts (see e.g. Kunin, 1968).

102 (analogous to ones shown in Fig.2 of Stuyfzand and van der Schans, 2018 for similar lenses in
 103 Holland). Without such simply engineered rainwater collectors the desert runoff is either lost to
 104 draining channels or gets diffused as a low-intensity infiltration to those parts of the desert,
 105 where the sand moisture redistributes to evaporation i.e. stable FFGLs can not be formed.
 106 Kunin (1959) and Schevchenko (1965), depending on topography and desert soil cover,
 107 distinguished three types of desert TFLs, viz. small sub-dune FFGLs (Fig.1a), small FFGLs
 108 under topographic depressions, called takyr (Fig.1b) and megalenses. Natural hydraulic
 109 gradients of several unconfined aquifers (salinity of chloride-type saline groundwater is 20-40
 110 g/l), common in Turkmenistan (Kunin, 1959, Rogovskaya et al., 1986), are adverse to retaining
 111 FFGLs even if infiltration is sufficient and spatially focused. Unlike under hydrological
 112 conditions of Holland, FFGLs in Turkmenistan can not be replenished by diversion of surface
 113 fresh water from perennial streams because they do not exist in the desert.

114 If a bedrock in Fig.1 is not deep enough (see also Danin, 2012, Fig.78) and the pristine
 115 unconfined saline groundwater is not thick enough, then a lens is not FFGL. In Fig.1b the
 116 corresponding lens is demarcated by the contour $E_1E_2E_3E_4E_1$. A horizontal segment of the
 117 bedrock, E_4E_1 , subtends this lens. It can be stable even without evaporation from the phreatic
 118 surface E_2E_3 , provided fresh groundwater seeps through a slightly permeable line E_4E_1 , as in
 119 Kacimov and Obnosov (2015). A deep borehole drilled across FFGL would show the following
 120 zonation: vadose zone –fresh water table – fresh groundwater zone - interface between fresh and
 121 saline water - saline groundwater -deep bedrock.

122 The topographic depressions (natural ones or engineered excavations) of an appropriate
 123 locale, sizes and placed above right phreatic saline phreatic surfaces convert a runoff of
 124 sufficiently strong rain into FFGLs which can be diligently abstracted (Kunin and
 125 Leszhinsky, 1960). As a prerequisite of existence of FFGL, Rogovskaya et al. (1979) pointed
 126 out that too small thicknesses of the vadose zone above the lens “lead to dissipation of the
 127 ground water by plant uptake, with some substantial salt accumulation” and “an aeration zone of

128 *thickness less than 3 m in conjunction with lack of plant cover results in highly saline water due*
129 *to physical evaporation.”*

130 Intelligent MAR operations in replenishing the exploited FFGLs of Turkmenistan, in
131 addition to natural recharge, incorporated skimming of fresh water by vertical wells, collated
132 with a simultaneous pumping of a subjacent saline water of FFGLs. This pioneering dual-
133 pumping controlled the fresh-saline water interface. Such mitigation of upconing of saline
134 water in FFGLs and ensuring their resilience commenced in 1963, prior to many similar
135 projects in the USA or Holland (see e.g. Singh and Stammers, 1989, Zuurbier et al., 2014).

136 The very existence of a stable FFGL is conditioned by the following factors:

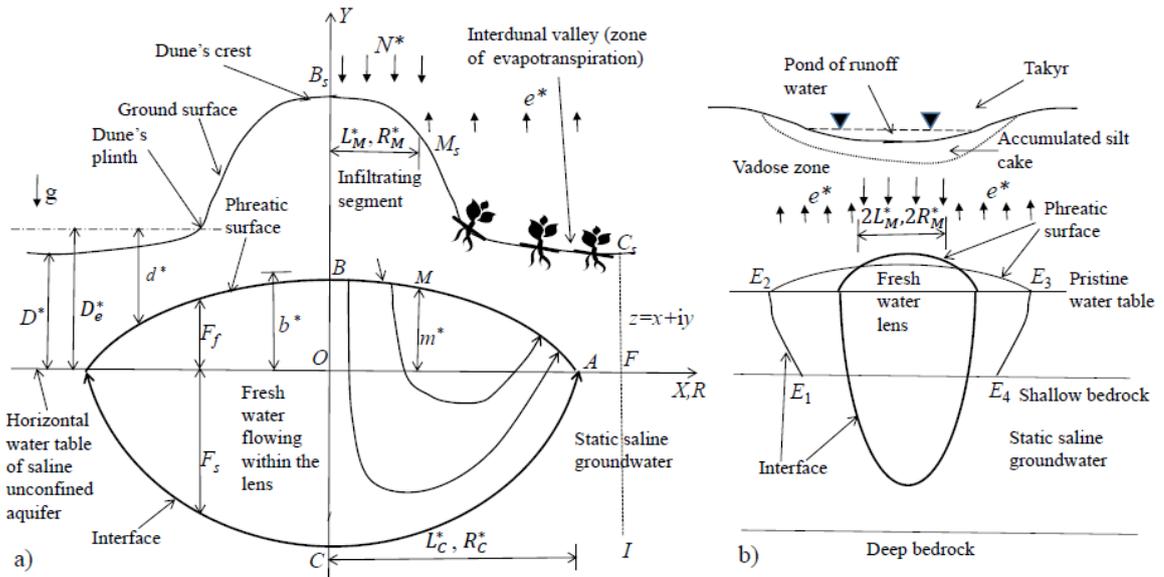
- 137 • There should be saline/brine background water table on which the lenses emerge as
138 hydrogeological buoys that is illustrated in Fig.1 a-f of Laattoe et al., 2017;
- 139 • The natural gradient of this subjacent background saline groundwater should be
140 small, i.e. away from the lens the water table should be almost flat because otherwise
141 FFGL slips downgradient (Kunin, 1959) and/or gets dispersed by the subjacent
142 saline groundwater;
- 143 • The thickness of the vadose zone above the phreatic surface of the lens should be
144 appropriate. Otherwise, infiltration and evaporation do not make a favourable source-
145 sink hydrodynamic couple similar to one in Kacimov et al. (2018). Juxtaposition of
146 recharge to the water table and losses from it maintains a continuous motion of fresh
147 water within the lens.
- 148 • Transpiration intensity by natural hydromorphic plants (if any) on the ground surface
149 above the lens should be bounded both from above and below. No transpiration
150 may result in high water table and its salinization. Too intensive interception of
151 groundwater by the roots may eventually deplete the lens and, consequently,
152 exterminates the plants.

153 Natural recharge and MAR to and pumping from FFGLs for drinking (both by humans and
154 livestock) are going on in modern Turkmenistan (Babaev, 1999). A post-Kunin review of
155 hydrology, hydrogeology, climate, topography, geomorphology, soil cover and water
156 consumption in deserts of Turkmenistan is reported by Babaev (1999), Bairamova (2013),
157 Fleskens et al. (2007), Hydrogeology...(1972), Kolodin (1981).

158 Typical concentrations of total dissolved solids (TDS) of fresh and saline groundwater in
159 FFGLs are about 1 g/l and 20-60 g/l (respectively). Kunin (1959) modeled a thin transition
160 zone by a Ghijben-Herzberg interface. The field data on the vertical profile of TDS in the
161 Sansyz FFGL is shown in supplementary file 2 (retrieved from Kunin, 1959). This cross-section
162 illustrates that close to point C in our Fig.1a the thickness of the transition zone between
163 isoconcentric contours 1 g/l and 10 g/l is about 1-2 meters (the maximal thickness of the lens is
164 about 12 m). Towards point A in our Fig.1a the Kunin interface is blurred i.e. the distance
165 between isolines of 1 g/l and 10 g/l becomes large.

166 The thickness of small FFGLs can be several tens of centimeters only. These mini-lenses
167 are tapped by dug wells and fresh water is occasionally and carefully consumed by local
168 nomads. The bed of takyr gets gradually clogged due to mechanical colmatation (see the
169 photos of a dry takyr retrieved from Kunin, 1959, supplementary file 3).

170 In a vertical cross-section of FFGL (Fig.1a) the dune is symmetric with respect to a
171 vertical axis OY of Cartesian (XOY) or cylindrical (ROY) coordinate systems. A horizontal axis
172 is OX or OR , respectively. This axis coincides with the background level of saline groundwater.
173 If no infiltration and no fresh water present, the saline water table is a plane. The curve $B_sM_sC_s$
174 in Fig.1a is the ground surface contour. Due to symmetry, below only the right half of the dune
175 is considered. FFGL is bounded from above by a phreatic surface BMC and from below by an
176 interface AC . The separatrix BOC in Fig.1a is a vertical segment.



177

178 Fig.1 Vertical cross-section of FFGL recharged from a linear dune (a) and from a takyr (b).

179 The following questions intrigued arid zone hydrologists in Turkmenistan: why does a stable
 180 FFGL in Fig.1a exist and why is it digon-shaped? Why the curve BMA is a subdued replica of
 181 $B_s C_s$ (the topography of the dune) and AC is an amplified, Ghijben -Herzbergian replica of BMA ?
 182 In other words, can one extend the criterion of Haitjema and Mitchell-Bruker (2005, their
 183 eqn.(8)) of topography-recharge controlled phreatic surface flows to BMA in Fig.1a, which has
 184 a focused infiltration strip (disk) under a dune (or ponded takyr bed)?

185 Unlike the situations of common deep or shallow unconfined aquifers in Haitjema and
 186 Mitchell-Bruker (2005) and their assumption of a constant accretion on the water table, FFGLs
 187 in our Fig.1 have a large variation of its fresh water thickness, $F_s + F_f$. Flow in our Fig.1 is a
 188 conjugation of flows in Figs.1 and 8 of Van Der Veer (1977), i.e. a source-sink boundary
 189 condition along a phreatic surface (recharge under the crest of the dune and evaporation on the
 190 flanks) makes the situation different from one in Haitjema and Mitchell-Bruker (2005). The
 191 groundwater systems modeled by Haitjema and Mitchell-Bruker (2005) and Toth (2009) are
 192 common for humid climates where evaporation from the phreatic surface is not a major factor. In
 193 our case, the evaporation rate, e^* , from segment AM of the phreatic surface in Fig.1a is high and
 194 may depend on d^* , the depth of the phreatic surface under an average terrain surface.

195 In this paper, we propose a plausible explanation of the existence of FFGLs: a stable lens
 196 is a result of a spatial alteration (with the X or R coordinate in Fig.1a) of the boundary
 197 condition along the water table. Namely, infiltration occurs from the crest zone, B_sM_s , of a
 198 fixed width and evapotranspiration takes place from the plinth-valley zone, M_sC_s the width of
 199 which is a part of solution. We emphasize that – similarly to Emikh (1963) - the lens in Fig.1a
 200 is much wider than the hill and zone of infiltration from above, B_sM_s . This is different from
 201 oceanic lenses the shape of which is determined by the topographic hill shape. In desert TFLs,
 202 the width of the lens is controlled by evapotranspiration which dominates in a relatively flat zone
 203 M_sC_s of Fig.1a.

204 The near-surface sand at the crest of tall desert dunes is coarser and more permeable than that
 205 at the slopes and interdunal depressions. The crests are commonly not covered by biocrust,
 206 vegetation or fine sediment, while the lower slopes and interdunal valleys are (e.g. Chamizo et
 207 al., 2016, Kidron and Yair, 2008, Fig. 17.2, Pye and Tsoar, 2008, Fig.7.28, 9.6; Yair et al, 2008,
 208 Figs.2.2,24). The infiltration rate on the crest is higher than along the dune slopes and in the
 209 valley (see e.g. Lopez, et al., 2015, Tao et al., 2001), provided the sand is not hydrophobic.
 210 Annual evaluations of N^* in loose sands give values which are several times higher than in
 211 vegetated, silt-clay clogged and biocrusted sands (see e.g. Gael and Smirnova, 1999). In humid
 212 climates, however, the dune crests can be covered by moss (see e.g. Voortman, 2018) and the water-
 213 repellent sand makes the infiltration distribution to subjacent fresh groundwater lenses more complex:
 214 less recharge comes through a zone on the dune crest as compared with the slopes and valley (see e.g.
 215 Houben, et al., 2014a).

216 In satellite images or aerial views, the green cover depicted in Fig.1a, makes a distinct
 217 spatial pattern of ecotones monitored also by botanists on the ground (see e.g. Dedkov, 2011).
 218 For example, the black *saksaol* trees on the slopes and interdunal areas of the Repetek
 219 National Park in Turkmenistan strongly depend on the subsurface hydrology. As these plants
 220 grow, transpiration increases and climatic conditions (i.e. recharge through the curve BM and

221 distributed losses to the vadose zone along MC in Fig.1a) remain stable. The lens starts to
 222 shrink because of a more intensive subsurface water uptake by *saksol* roots that results in
 223 degradation of the trees. That biome-lens dynamics is in comport with what Toth (2009) reported
 224 in his Section 4.4.2.3. Devegetation also impacts the hydraulic and capillary properties of the
 225 vadose zone (see e.g. Wang et al., 2015).

226 It is noteworthy that dunes in our Fig.1a are sufficiently tall. For less-bulging desert
 227 landforms, *viz.* small-size sand piles, the hillocks (the so-called nebkhas), plants grow on the
 228 crest of the hillocks, see e.g. Danin (2012, Figs. 16, 21, 51, 62), Khalaf et al. (1995).

229 On an annual and decadal time-scale, the segment BM of the phreatic surface in Fig.1a is
 230 grossly a zone of recharge of FFGL of lateral extension L_M^* or R_M^* (2-D and axisymmetric cases,
 231 respectively). The segment MC of horizontal sizes L_C^* or R_C^* is a zone of a distributed
 232 exfiltration (evapotranspiration) from the lens. Correspondingly, the recharge-discharge
 233 distribution along the phreatic surface in Fig.1a is exactly the same as in Strack (1978), who
 234 considered an unconfined aquifer (without a subjacent interface and bedrock deep at infinity).
 235 Van Der Veer (1977) studied a semi-infinite wedge of a fresh water commingled with a static sea
 236 water intrusion tongue. We combine the analytic components of Strack's and Van Der Veer's
 237 flow models: the curve BMA caps FFGL in Fig.1a and AC reflects its confining and
 238 floatational features. We note that transient variable-density flows in the takyr MAR-fed FFGLs
 239 of Turkmenistan have been numerically modeled by FEFLOW (Kuznetsov et al., 2006,
 240 Yakirevich et al., 2005).

241 Dynamics of fresh groundwater in Fig.1a is similar to what has been studied in the Toth
 242 (2009) regional flow model, in which a rigid bedrock commonly serves as a confining boundary
 243 of fresh groundwater flows (see e.g. his Figs. 2.1, 2.2, 3.1,3.2, 3.6-3.14, 3.22, 4.1, 4.16, 5.26,
 244 5.34-5.38, 5.51, 5.67). The Toth-type topology of streamlines shown in Fig.1a is corroborated by
 245 Yair (2008): "*The long-term effect of subsurface lateral water flow is supported by data on*

246 *changes in electrical conductivity and amount of fine-grained material at the base of the dune*
 247 *slopes. Wedges of increased salinity and content of fine-grained particles, parallel to the sloping*
 248 *surfaces, were detected at the slope bases (Fig. 18.11).” Yair experimentally proved that near*
 249 *point B_s (recharge zone) in our Fig.1a soil water inside the dune is an order of magnitude less*
 250 *saline than near the dune pedestal (evaporation zone). Obviously, for artificially recharged*
 251 *FFGL depicted in Fig.1b (see also Kunin and Lezshinsky, 1960) the schematization of Fig.1a*
 252 *is adopted to a takyr topography, viz. a zone of engineeringly controlled L_M^* or R_M^* specified by*
 253 *the size of a infiltration basin (natural or constructed). The basin can be linear, circular or of a*
 254 *more complex shaped (Yakirevich et al., 2005, Kuznetsov et al., 2006). In Fig.1b, instead of an*
 255 *elevated highly permeable dune crest above FFGL of Fig1a, a low permeable and depressed*
 256 *colmatation layer feeds the lens.*

257 One of the motives in studying FFGL in Fig.1 stems from the necessity to prepare for a
 258 possible water shortage and corresponding planning for the extraction of water resources in the
 259 deserts. As Porter (2015) pointed out, the whole world is not flat i.e. in our application to
 260 topography in Fig.1a, FFGLs stored under undulating sandy ground surface may become a
 261 vital reservoir for the desert settlements when regular aquifers are depleted or contaminated .

262 **2. Conceptual Model**

263 We assume a steady-state, Darcian, capillarity-free (one-phase, fully saturated) 2-D flow in
 264 FFGL formed in a sand of hydraulic conductivity k . A light pore water of density ρ_f floats on a
 265 static dense saline water of density ρ_s with an interface AC . Dispersion is ignored.

266 FFGL in Fig.1a is recharged by infiltration of intensity $N^*(X) > 0$ which arrives at a segment
 267 BM of a given length L_M^* of the phreatic surface from a zone B_sM_s of the dune crest, similarly to
 268 the case of infinite strip islands (Chesnaux and Allen, 2008, Vacher, 1988). The evaporation
 269 of intensity $e^*(X) > 0$ – unlike the island lenses - takes place from MA . The ordinate of point A is

270 *a priori* unknown. Fresh groundwater evaporates from FFGL and from a horizontal segment AF
 271 of a static saline water table. On the ground surface, the corresponding curve $M_s C_s$ extends to
 272 the point C_s , the middle of the interdunal valley. The ray $C_s F I$ in Fig.1a separates the modeled
 273 dune from its neighbor on the right. Measurements or even assessments of e^* are more
 274 problematic than of N^* (see e.g. Hampton, 1963).

275 We consider a solitary lens i.e. we assume that N^* is small enough and e^* is high enough such
 276 that L_C^* is narrow enough i.e. the lenses under two neighbouring dunes do not interfere. If
 277 $N^* = e^* = 0$, then the water table of the ambient static saline groundwater coincides with the axis
 278 Ox .

279 The streamlines in Fig.1a, bounded by a separatrix BOC , illustrate the Toth-type flow of
 280 fresh groundwater within the lens, with a hinge point M (point on the phreatic surface where
 281 infiltration changes to exfiltration). The position of the phreatic surface of the lens $F_f(X)$, in
 282 particular, the elevations b^* and m^* of the points B and M , the horizontal size of the lens L_C^* (the
 283 ordinate of the point C where the phreatic surface and interface intersect) and the locus of the
 284 interface $F_s(X)$ are to be found.

285

286 3. Dupuit-Forchheimer Approximation

287 3.1 Planar flow

288 In the Dupuit-Forchheimer (DF) model (see e.g. Bakker, 2000, Eeman, 2017, Polubarinova-
 289 Kochina, 1977, Strack, 1989, Van Der Veer, 1977, Van Duijn and Schotting, 2017), steady 1-D
 290 flow in FFGL obeys the ODE:

$$291 \quad \frac{d}{dX} \left(k (F_f + F_s) \frac{dF_f}{dX} \right) = \begin{cases} -N^*(X), & \text{at } 0 \leq X \leq L_M^*, \\ e^*(X), & \text{at } L_M^* \leq X \leq L_C^*. \end{cases} \quad (1)$$

292 where F_s is the interface ordinate, which is related to the phreatic surface as

$$293 \quad F_s = F_f / \gamma, \gamma = (\rho_s - \rho_f) / \rho_s. \quad (2)$$

294 For water supply schemes and MAR, the volume, Ar^* , of fresh water stored in FFGL is
 295 of primary interest.

296 We introduce dimensionless quantities:

$$297 (g_s, g_f, x, y, b, m, L_C) = (F_s, F_f, X, Y, b^*, m^*, L_C^*) / L_M,$$

$$298 Ar = Ar^* / L_M^2, \quad (N, e) = (N^*, e^*) / (k(1 + 1/\gamma)).$$

299 which represent the geometric and hydrological characteristics of FFGL.

300 Then, using eqn.(2) the governing eqn.(1) is written:

$$301 \frac{d^2 \Phi_f}{dx^2} = \begin{cases} -N(x), & \text{at } 0 \leq x \leq 1, \\ e(x), & \text{at } 1 \leq x \leq L_C, \end{cases} \quad (3)$$

302 where $\Phi_f = g_f^2 / 2$ is Strack's (1989) potential, and $Q(x) = -d\Phi_f / dx$ is the vertically
 303 integrated dimensionless discharge. At $x=0$ and $x=L_C$ the discharge $Q=0$. Three other boundary
 304 conditions for solving eqn.(3) are: $Q(x)$ and $\Phi_f(x)$ are continuous functions at the hinge point M
 305 ($x=1$). We note that $N(1) \neq e(1)$ i.e. the sink-source intensity in the RHS of eqns.(1),(3) is
 306 discontinuous. Also in Fig.1a the lens tapers to point A i.e. $\Phi_f(L_C) = 0$.

307 From annual hydrological balances of precipitation-evapotranspiration we assume that
 308 $N=N_0=\text{const} > 0$, $e=e_0=\text{const} > 0$. Then, similarly to Kacimov et al. (2009), after simple algebra
 309 solution to the BVP (3) with the above stated boundary conditions is:

$$310 g_f = \begin{cases} \sqrt{N_0(1 + N_0/e_0) - N_0x^2}, & \text{at } 0 \leq x \leq 1, \\ \sqrt{e_0(1 + N_0/e_0 - x)}, & \text{at } 1 \leq x \leq L_C = 1 + N_0/e_0, \end{cases} \quad (4)$$

311 Obviously, from eqn. (4) the infiltrating branch BM of the phreatic surface is an arc of an
 312 ellipse with its summit at point B where $b = \sqrt{N_0(1 + N_0/e_0)}$ and the evaporating branch MC is
 313 a straight tilted line (as was the evaporating phreatic surface of Van Der Veer, 1977, his Fig.8).
 314 Clearly, according to the Ghijben-Herzberg approximation $g_s = g_f/\gamma$.

315 A dimensionless cross sectional area, Ar , of the curvilinear triangle $ABCA$ is:

$$316 \quad Ar = (1 + 1/\gamma) \int_0^{1+N_0/e_0} g_f(x) dx = (1 + 1/\gamma) \left(\frac{N_0}{2\sqrt{e_0}} \left(1 + \frac{N_0}{e_0} \right) + \frac{\sqrt{N_0}}{2} \left(\frac{N_0}{e_0} + \left(1 + \frac{N_0}{e_0} \right) \arctan \left[\frac{1}{\sqrt{e_0}} \right] \right) \right), \quad (5)$$

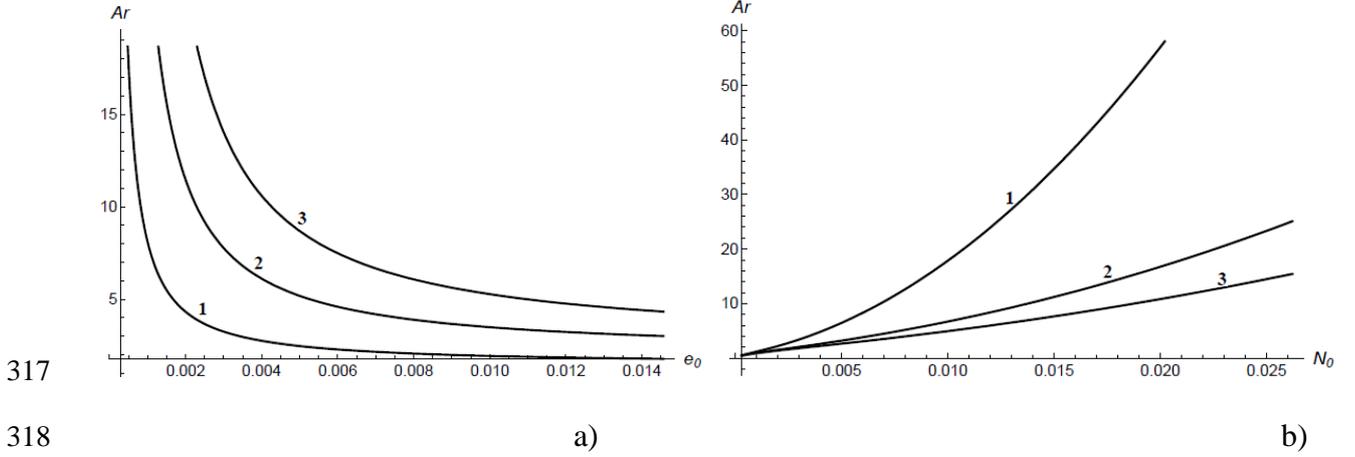


Fig.2. Cross-sectional area Ar of the right half of FFGL in Fig.1a (curvilinear triangle $ABCA$) as a function of evaporation rate e_0 from the discharge zone at the density contrast $\gamma = 0.03$ and infiltration rates $N_0 = 0.1, 0.2, 0.3$ (curves 1-3, correspondingly) (a); $Ar(N_0)$ at $\gamma = 0.03$ and $e_0 = 0.1, 0.3, 0.5$ (curves 1-3, correspondingly) (b).

Fig.2a and Fig.2b show (respectively) the graphs $Ar(e_0)$ at $\gamma = 0.03$ and $N_0 = 0.1, 0.2, 0.3$ (curves 1-3, correspondingly) and $Ar(N_0)$ at $\gamma = 0.03$ and $e_0 = 0.1, 0.3, 0.5$ (curves 1-3, correspondingly).

PK-77 (pp.485-486) pointed out that evaporation from the phreatic surface may not be constant (as assumed above) but rather depends on the depth of this surface counted from the local ground surface. For simplicity, we ignore the relief of the topography above the evapotranspiring arc of FFGL and assume that $M_s C_s$ in Fig.1a is a horizontal line (dashed-dotted) located at an effective depth D_e^* above the undisturbed saline water table. Then the evapotranspiration intensity depends on $d[F_f(x)]$. PK-77 provided field data for $e^*[d^*]$ as a linear function i.e. $e^* = e_0^* - \beta D_e^* + \beta F_f$ where β is a given constant. Then in dimensionless quantities eqn. (3) transforms to

$$335 \quad \frac{d^2 \Phi_f}{dx^2} = \begin{cases} -N(x), & \text{at } 0 \leq x \leq 1, \\ e_1 + e_2 \sqrt{\Phi_f(x)}, & \text{at } 1 \leq x \leq L_C, \end{cases} \quad (6)$$

336 where the constants e_1 and e_2 are directly expressed via e_0 , β and D_e . The boundary

337 conditions in BVP (6) are the same as in BVP (3).

338 The first (linear) equation in (6) integrates into the same elliptical arc as the first eqn.(4)

339 (branch MC). The second ODE in (6) is nonlinear with respect to Strack's potential but it

340 integrates in elliptic functions. For the sake of brevity we skip over computations of g_f and g_s

341 obtained as solutions of BVP (6).

342

343 3.2 Axisymmetric lenses

344 If the dune surface $B_s C_s$ (Fig.1a) resembles a cupola (barchan dunes, see e.g. Bishop, 2013)

345 and infiltration originates from a disk-shaped spot of rainfall at the crest of this cupola (co-axial

346 with the cupola), then flow in FFGL is axisymmetric. The saturated water storage is equal to

347 the volume V_f^* of the corresponding body of revolution. Kunin reported axisymmetric shapes of

348 MAR-fed FFGLs (see the field data on the Sansyz lens from Kunin, 1959, supplementary file 4).

349 The DF model for axisymmetric flows reduces to the following nonlinear ODE:

$$350 \quad \frac{1}{R} \frac{d}{dR} \left(Rk (F_f + F_s) \frac{dF_f}{dR} \right) = \begin{cases} -N^*(R), & \text{at } 0 \leq R \leq R_M^*, \\ e^*(R), & \text{at } R_M^* \leq R \leq R_C^*. \end{cases} \quad (7)$$

351 We introduce dimensionless quantities:

$$352 \quad (g_s, g_f, r, y, b, m, r_C) = (F_s, F_f, R, Y, b^*, m^*, R_C^*) / R_M^*, \quad V_f = V_f^* / R_M^{*2}.$$

353 Eqn.(7) reads:

$$354 \quad \frac{d}{d} \left[r \frac{d g_f}{d r} \right] = \begin{cases} -N(r)r, & \text{at } 0 \leq r \leq 1, \\ e(r)r, & \text{at } 1 \leq r \leq r_C. \end{cases} \quad (8)$$

355 If the infiltration and evaporation rates are constant (N_0 and e_0), then - similarly to the case of a

356 linear dune - eqn.(8), with the same boundary conditions as in the previous Section, integrates

357 this BVP to:

$$g_f = \begin{cases} \sqrt{(e_0 + N_0) \text{Log}(1 + N_0 / e_0) - N_0 r^2} / \sqrt{2}, & \text{at } 0 \leq r \leq 1, \\ \sqrt{(N_0 + e_0) \text{Log}[r\sqrt{1 + N_0 / e_0}] + e_0 r^2 - (N_0 + e_0)} / \sqrt{2}, & \text{at } 1 \leq r \leq r_c = \sqrt{1 + N_0 / e_0}. \end{cases} \quad (9)$$

A dimensionless volume of FFGL is

$$V_f = 2\pi(1 + 1/\gamma) \int_0^{\sqrt{1 + N_0 / e_0}} r g_f(r) dr. \quad (10)$$

We put eqn.(9) as an integrand in eqn.(10) and computed V_f with the routines of Wolfram's (1991) *Mathematica*.

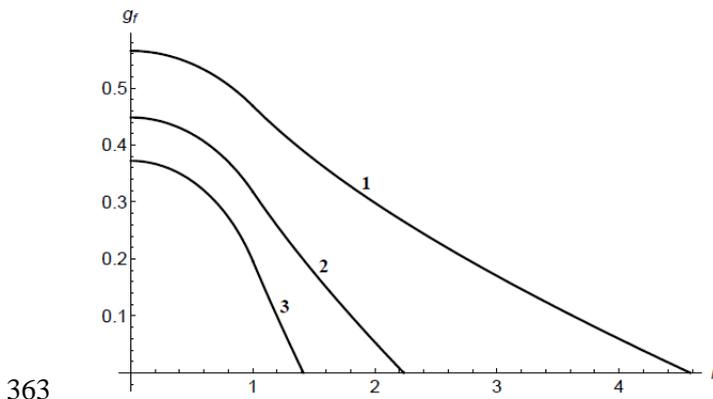


Fig.3a

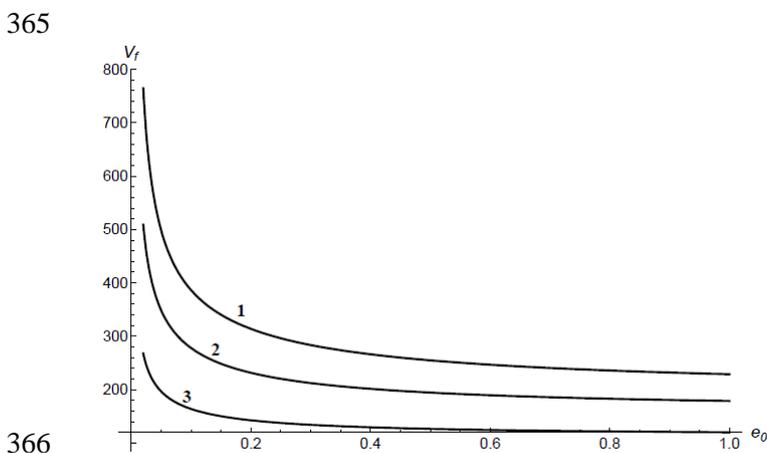


Fig.3b

Fig. 3. The water table $g_f(r)$ axisymmetric FFGL for $N_0=0.2$, $\gamma=0.03$ and $e_0=0.01, 0.05, 0.2$ (curves 1-3, correspondingly) (a). Saturated water storage in FFGL $V_f(e_0)$ for $\gamma=0.03$ and $N_0=0.1, 0.2, 0.3$ (curves 1-3) (b).

372 Fig.3a shows the water table of axisymmetric FFGL for $N_0=0.2$, $\gamma=0.03$ and $e_0=0.01, 0.05,$
 373 0.2 (curves 1-3, correspondingly). Apparently, the evaporating branch of the phreatic surface is
 374 almost straight, especially, close to the point $r=r_C$ where the fresh water zone tapers to the saline
 375 phreatic horizon. Fig 3b shows $V_f(e_0)$ for $\gamma=0.03$ and $N_0=0.1, 0.2, 0.3$ (curves 1-3). Here –
 376 similarly to Fig.2b – we see that the saturated storage of FFGL is most sensitive to the intensity
 377 of evaporation at small e_0 .

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4. Two-Dimensional Flow in a Linear Dune

380 Now we consider 2-D flow in a vertical plane of FFGL under a linear dune. Similarly
 381 to groundwater systems modeled by PK-77, Strack, 1978 and Van Der Veer, 1977, a rigorously
 382 stated mathematical problem involves three free boundaries, *viz.* a branch of the phreatic surface
 383 with accretion, a branch with evaporation and interface. Then, the corresponding domain in the
 384 hodograph plane is a curvilinear quadrangle. For such polygons, an analytical solution by the
 385 PK-77 method becomes prohibitively complicated². In this Section, we circumvent conformal
 386 mappings of a curvilinear polygon in the hodograph plane (see PK-77, her Fig. 224 and 227) by
 387 approximating the free boundaries by straight segments.

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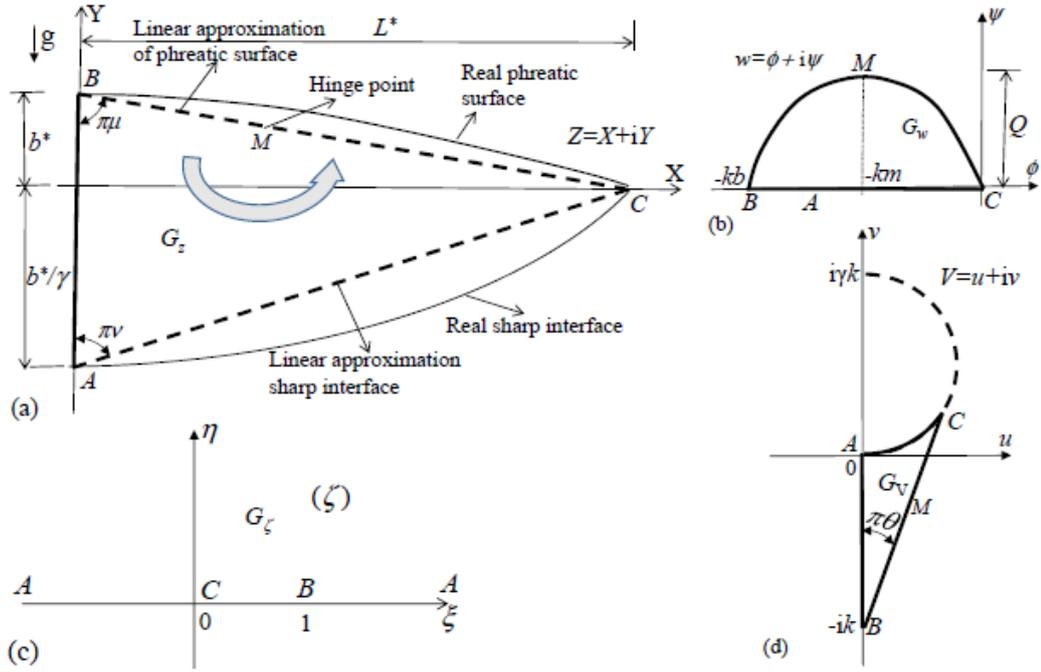
The interface CA can be approximated reasonably by a straight line. In the DF model
 the interface under infiltrated phreatic surface usually has a high degree of curvature close to
 point A (see the elliptical phreatic surfaces and interfaces found in PK77 and van der Veer
 (1977) in terms of 2-D potential and DF models for an island lens). Similarly, the DF model
 predicts an elliptic phreatic surface in Fig.3 of Haitjema and Mitchell-Bruker (2005) in case of
 uniform infiltration and equal levels of water in the reservoirs. However, in case of a uniform
 evaporation the phreatic surface in Fig.3 of Haitjema and Mitchell-Bruker (2005) at small
 evaporation rates becomes a hyperbola. At high evaporation the hyperbola splits into two

² We recall that PK-77 effectively solved simpler 2-D problems with either constant recharge or constant evaporation intensity along the whole water table, see her Figs. 223 and 225, with a curvilinear triangle in the hodograph plane (see also Maas, 2007). Strack (1978) solved the problem with a variable infiltration-evaporation but in an unconfined aquifer without a subjacent bedrock.

396 straight lines bounding two hydraulically disconnected saturated wedges (see e.g. Kacimov et
 397 al., 2003, Youngs, 1974). Similarly, for the case of an interface rather than a horizontal bedrock
 398 of Haitjema and Mitchell-Bruker (2005) at point A of our Fig.1a a wedge depicted in Fig.8 of
 399 van der Veer (1978) emerges. Kunin's (1959) field observation (see the supplementary file 2)
 400 illustrate that both phreatic surface and interface flatten towards point A in our Fig.1a. Therefore,
 401 the slope of BMC is zero at point B , relatively high at point M and small again at point A . This is
 402 also illustrated by the behavior of curves 1-3 in Fig.3. Our straight-line BMA and Ghijben-
 403 Herzberg CA are, hence, a reasonable approximation of a real phreatic surface and interface,
 404 which have inflection points. From field observations-records of the water table of numerous
 405 FFGLs (Kunin, 1959), one can easily get the loci of points B and C in the physical plane i.e. b^*
 406 and L^* in Fig.1a. Next, we assume that the Ghijben-Herzberg relation holds at point A , i.e. $Y_A =$
 407 b^* / γ . We also approximate the curved phreatic surface and interface (solid lines in Fig.4a) by
 408 straight segments BC and BA (dashed lines), i.e. we get a triangle BAC (G_z which is a Tothian
 409 unit basin of this problem) as the right half of FFGL. We solve a 2-D problem in G_z .

410 4.1. Toth's model with a fixed straight phreatic surface and interface as free surface

411 As usually, we introduce a complex physical coordinate $z = x + i y$, velocity potential
 412 $\phi^* = -kh_f$ of the moving fresh water where $h_f(x, y)$ is the hydraulic head in the fresh water
 413 counted from point C . The complex potential $w^* = \phi^* + i\psi^*$, where ψ^* is a stream function.
 414 Point C in Fig.4a is selected as fiducial and the complex potential domain G_{w^*} corresponding to
 415 G_z is shown in Fig.4b. The curve BMC of the digone G_{w^*} is not known. The specific discharge
 416 vector is $\vec{V} = \nabla \phi^*$. The pressure head and pressure in fresh water are $p_f(x, y) = -(\phi^* / k + y)$ and
 417 $P_f = \rho_f g p_f$, correspondingly. The pressure head and pressure in saline water are $p_s(y) = -y$, and
 418 $P_s = \rho_s g p_s$, correspondingly.



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Fig.4. Saturated triangle G_z (half of FFGL) in vertical cross-section of the Toth-type flow (a), digon G_w in the complex potential domain corresponding to G_z (b), reference plane G_ζ (c), curvilinear triangle G_v in the hodograph plane for BVP in the Appendix (d).

423

Thus, the BVP is:

$$\begin{cases} \Phi^* + k_s y = 0, & \text{along } BC, \\ \Psi^* = 0, & \text{along } AC, \\ \Phi^* = 0, & \text{along } AB. \end{cases} \quad (11)$$

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We recall that in the 1960-s Toth (2009) solved a similar BVP in a rectangle (by the method of separation of variables in the Laplace equation); Kacimov (1996) and Al-shukeili et al. (2019) solved a mixed BVP in a trapezium and pentagon, correspondingly. Unlike the DF approximation of Section 2, the widths of the infiltration and exfiltration segments in Fig.4a are not specified but are found as a part of solution i.e. the locus of point M is determined. Moreover, by assuming isobaric condition along a given line (BC in our case) the Toth model automatically requires determination of the intensities of inflow through BM and exfiltration through MC , which were also input parameters of the DF model.

433 We use the Schwarz-Christoffel formula and map conformally the triangle G_z onto the
 434 half plane G_ζ in Fig.4c with the correspondence of points $(C, B, A) \rightarrow (0, 1, \infty)$

$$435 \quad Z(\zeta) = ie^{i\pi\mu} c \int_0^\zeta t^{-\mu-\nu} (1-t)^{\mu-1} dt + L_C^* = ie^{i\pi\mu} c B_\zeta(1-\mu-\nu, \mu) + L_C^*, \quad (12)$$

436 where B_ζ is the incomplete Beta-function and a real constant c is determined from the condition

$$437 \quad \text{Im } Z(1) = b^* \text{ as}$$

$$438 \quad c = b^* / (\cos \pi\mu B(1-\mu-\nu, \mu)), \quad (13)$$

439 The angles of G_z are $\tan[\pi\mu] = L^* / b^*$, $\tan[\pi\nu] = \gamma L^* / b^*$, and $B(\mu, \nu)$ is the complete Beta-
 440 function.

441 From eqn.(10) along BC

$$442 \quad Y(\xi) = b^* \frac{B_\xi(1-\mu-\nu, \mu)}{B(1-\mu-\nu, \mu)}, \quad 0 < \xi < 1, \quad (14)$$

443 where $B_\xi(1-\mu-\nu, \mu)$ is the incomplete Beta-function.

444 From eqn.(11) the analytic functions $w^*(\zeta) / \sqrt{\zeta(1-\zeta)}$ obeys a nonhomogeneous

445 Dirichlet's boundary condition

$$446 \quad \text{Re} \left[w^*(\xi) / \sqrt{\xi(1-\xi)} \right] = -kY(\xi) / \sqrt{\xi(1-\xi)} \text{ at } \xi \in (0, 1),$$

$$447 \quad \text{Re} \left[w^*(\xi) / \sqrt{\xi(1-\xi)} \right] = 0 \text{ at } \xi \in (-\infty, 0) \cup (1, \infty).$$

448 The function $w^*(\zeta) / \sqrt{\zeta(1-\zeta)}$ vanishes at infinity and has integrable singularities at the

449 points $\zeta = 0$ and $\zeta = 1$. Therefore (see Henrici, 1993) the last Dirichlet problem is uniquely

450 solvable and its solution gives the Schwarz operator for a half-plane. Thus, we obtain

$$451 \quad w^*(\zeta) = \frac{kb^* \sqrt{\zeta(1-\zeta)}}{\pi i B(1-\mu-\nu, \mu)} \int_0^1 \frac{B_t(1-\mu-\nu, \mu) dt}{\sqrt{t(1-t)(t-\zeta)}}. \quad (15)$$

452 From eqns. (14), (15), along the isobar BC we apply the Plemelj-Sokhotski formula (see
453 Henrici, 1993):

$$454 \quad \psi^*(\xi) = \frac{kb^* \sqrt{\xi(1-\xi)}}{\pi B(1-\mu-\nu, \mu)} \int_0^1 \frac{B_t(1-\mu-\nu, \mu) dt}{\sqrt{t(1-t)(t-\xi)}}, \quad 0 < \xi < 1. \quad (16)$$

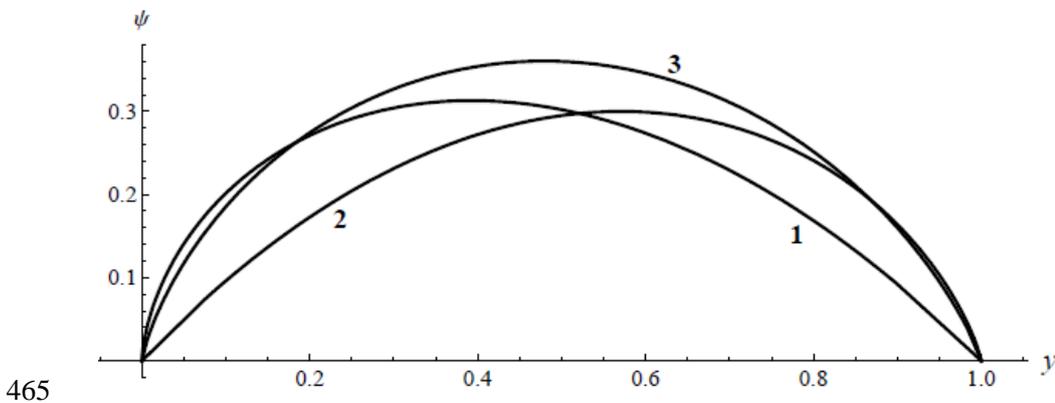
455 The integral in eqn. (16) is singular and is understood in the sense of principal value (Henrici,
456 1993). The stream function attains its unique and global maximum at a hinge point M (see
457 Fig.4b) that quantifies the total flow rate, Q , of fresh groundwater flow through FFGL:

$$458 \quad Q^* = \psi^*(\xi_M).$$

459 We introduce dimensionless quantities:

$$460 \quad (x, y, m, L_C, L_M) = (X, Y, m^*, L_C^*, L_M^*) / b^*, \quad (\phi, \psi, Q) = (\phi^*, \psi^*, Q^*) / (kb^*)$$

461 For computations of the integrals in eqn.(16) we used the **NIntegrate** function of
462 Wolfram's (1991) *Mathematica*, with the option **PrincipalValue->True** for singular integrals.
463 The flow rate Q was computed by applying the *Mathematica* built-in function **FindMaximum** to
464 eqn. (16).



466 Fig.5. Distribution of the streamfunction in 2-D flow of Fig.4a as a function of the vertical
 467 coordinate for three half-width of FFGL, $L_C=1, 4.3, 30$ (curves 1-3) at $\gamma=0.03$.

468 Fig.5 plots the distribution $\psi(y)$ along BC for $L_C=1, 4.3, 30$ (curves 1-3) at $\gamma=0.03$. These
 469 curves show that with steepening of the phreatic surface the maximum of ψ (attained at the
 470 hinge point M) shifts towards point B . We also varied continuously L_C and found the maximum
 471 of the function $Q(L_C)$ at $\gamma=0.03$. This maxi-max is attained at $L_C \approx 4.3$ and is $Q_{\max} \approx 0.36$. The
 472 elevation of point M is $y_M \approx 0.48$. This simple optimization can be used for *a priori* estimations
 473 of Q , viz. for a measured peak b^* of the water table we can immediately write the bound
 474 $Q < Q_{\max}$, whatever is the FFGL width $2L_C$. This width, unlike the height b of the groundwater
 475 mound (commonly located somewhere in the middle of the MAR pond) is not easy to precisely
 476 measure (see the vertical profile of the Sansyz FFGL, Kunin, 1959, supplementary file 2).

477 5. Suggestions for future work

478 Our analytical solutions allow to carry out the analysis of vulnerability of recharge-
 479 evaporation controlled FFGLs to variation of the basic input parameters: intensities of the
 480 distributed sink-source along phreatic surfaces, similarly to what has been done for other TFLs
 481 (Laattoe et al. , 2017) and for lenses in oceanic islands and coastal dunes (Morgan and Werner,
 482 2014, Stuyfzand and van der Schans, 2018). Exploitation of desert FFGLs, especially, ones
 483 only naturally recharged, requires smart skimming of the fresh water because overabstraction
 484 causes a rapid brackization (Kunin,1959). A potential can be used for modeling of horizontal or
 485 vertical wells tapping FFGLs (Hocking and Forbes, 2004). As in Morgan and Werner (2014),
 486 Stuyfzand and van der Schans (2018), our analytical models assume steady flows, although a
 487 real focused post-rain infiltration to BM in Fig.1a comes in short pulses. The real dunes are not
 488 homogeneous and isotropic media as we assumed but are structured (Lancaster, 1995) that
 489 complicates the infiltration-evaporation paths of pore water and determination of the interface-

490 phreatic surface inside the dune bodies (Al-Shukaili et al., 2019, Bagnold, 2012, Berndtsson
491 and Chert, 1994, Berndtsson et al., 1996, Dose et al., 2014, McCord et al., 1991, McCord and
492 Stephens, 1987, Pye and Tsoar, 2008, Chapter 7). Also, dispersion blurs the interface of FFGL.
493 Further laboratory (e.g. sandbox) and numerical modeling, as well as field piezometry,
494 measurements of TDS and matrix potential in desert FFGLs should be done, similarly to
495 analogous SWI problems (see e.g. Houben et al., 2018).

496 **6. Concluding Remarks**

497 Kunin's FFGLs are unique hydrological entities which emerge in desert climates with a
498 spatially focused natural recharge to a part of the water table from a dune crest or with a MAR
499 infiltration from a surface pond. An intensive evaporation from the tail (not recharged part) of
500 the water table supports an FFGL-scale water motion in sand. In order to float these lenses
501 require a substrate of saline groundwater (an unconfined aquifer) of a sufficient depth and small
502 natural hydraulic gradient. In these conditions, FFGL is subtended by a trough-shaped interface
503 between a moving fresh water inside the lens and static saline water. This interface has the same
504 confining role as the bedrock in the Toth (2009) regional flow but unlike the fixed lower
505 boundary of the Toth unit basin (determined by geological stratification) the interface supporting
506 FFGL is a free boundary, i.e. not fixed but depending on hydraulic conductivity of the sand,
507 recharge and evaporation intensities, the width of the infiltration spot and densities of fresh and
508 saline groundwater. FFGL resembles a fresh-water MAR "bubble" completely submerged into
509 saline groundwater (Kacimov et al. 2018). Analytical solutions to BVPs for steady state regimes
510 described by ODEs (DF model) for Strack's potential give the shapes of the water table and
511 interface (including the locus of the crest and total width of FFGL). Remarkably, an evaporating
512 segment of the FFGL and therefore the DF interface are straight lines, provided the recharge-
513 evaporation intensities are constant, as was the case in the Van Der Veer (1977) solution to an
514 SWI problem. For 2-D flows in FFGL, holomorphic characteristic functions are used with the
515 help of Toth's approximation of the upper boundary of the flow domain as a fixed isobar in

516 gravity-driven groundwater flows. Half of the flow domain is considered as a Tothian unit basin.
 517 This triangle or curvilinear triangle is made of an isobar and streamline.

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522 **Appendix**

523 **Toth's model with a fixed straight phreatic surface and interface as a free surface**

524 In this Appendix, we extend the Toth model of Subsection 4.1 by assuming that the
 525 interface, AC , is not a straight segment but a free boundary. The upper boundary of the lens,
 526 BMC , as in Subsection 4.1, is still not a free boundary but a given straight tilted isobaric
 527 segment of a slope s and elevation b above the background saline water table. Even with this
 528 simplification, to get an analytical solution we engage the machinery of integral representations
 529 of the Riemann-Hilbert BVP (Henrici, 1993, PK-77). This BVP is formulated as following:

$$530 \quad \begin{cases} \psi = 0, & \phi - k\gamma y = 0 \text{ along } AC, \\ y = b - s x, & \phi + ky = 0 \text{ along } BMC, \\ \psi = 0, & x = 0 \text{ along } BA, \end{cases} \quad (A1)$$

531 where b and L_c are given constants,

$$532 \quad \gamma = \rho_s / \rho_f - 1, \quad s = b / L_c = \tan \theta. \quad (A2)$$

533

534 From (A1) the hodograph, G_v , is a circular triangle shown in Fig.4d (see PK-77 for
 535 details of hodographs in interface flows, in particular her Fig.227). First, we map conformally
 536 the circular triangle symmetric to G_v with respect to the real axis in Fig.4d onto the same

537 reference half-plane G_ζ in Fig.4c. The correspondence of points is $(C, B, A) \rightarrow (0, 1, \infty)$. The
 538 mapping function is (Koppenfels and Stallman, 1959) is:

$$539 \quad \omega(\zeta) = ik - ikc \left(\frac{\zeta - 1}{\zeta} \right)^\theta \frac{F(1/4 + \theta/2 - \lambda/2, 1/4 + \theta/2 + \lambda/2; 1 + \theta; 1 - 1/\zeta)}{F(1/4 - \theta/2 - \lambda/2, 1/4 - \theta/2 + \lambda/2; 1 - \theta; 1 - 1/\zeta)}, \quad (A3)$$

540 where F stands for a hypergeometric function ${}_2F_1$, $0 < \arg(1 - 1/\zeta) < \pi$,

$$541 \quad c = \frac{\Gamma(3/4 + \theta/2 + \lambda/2)\Gamma(3/4 + \theta/2 - \lambda/2)\Gamma(1 - \theta)}{\Gamma(3/4 - \theta/2 + \lambda/2)\Gamma(3/4 - \theta/2 - \lambda/2)\Gamma(1 + \theta)}, \quad \pi\lambda = \arccos \left[\frac{2 + \gamma}{\gamma} \sin \pi\theta \right]. \quad (A4)$$

542 Here $\pi\lambda$ is the angle at the vertex C of the triangle G_v . From (A4) and (A2) follows that γ

543 must satisfy the inequality $\gamma \leq 2s / (\sqrt{s^2 + 1} + s)$.

544

545 **Remark.** The hypergeometric functions in (A3) are defined for $\zeta \in (1/2, \infty)$. If $\zeta \in (-\infty, 1/2)$

546 then we have to use the analytic continuation (Abramowitz and Stegun, 1969, formula 15.3.7),

547 which gives

$$548 \quad \omega(\zeta) - ik = -ike^{i\pi\theta} \frac{f(\zeta; \theta, \lambda) + f(\zeta; \theta, -\lambda)}{f(\zeta; -\theta, \lambda) + f(\zeta; -\theta, -\lambda)}, \quad (A5)$$

549 where

$$550 \quad f(\zeta; \theta, \lambda) = e^{-i\pi\lambda/2} \left(\frac{\zeta - 1}{\zeta} \right)^{\lambda/2} \frac{\Gamma(\lambda)\Gamma\left(\frac{3}{4} + \frac{\theta - \lambda}{2}\right)F\left(\frac{1}{4} + \frac{\theta - \lambda}{2}, \frac{1}{4} - \frac{\theta + \lambda}{2}; 1 - \lambda; -\frac{\zeta}{\zeta - 1}\right)}{\Gamma(1/4 + (\theta + \lambda)/2)}. \quad (A6)$$

551 Next, we introduce two functions $F(\zeta) = dw/d\zeta$ and $Z(\zeta) = dz/d\zeta$, and reformulate the

552 boundary conditions (A1) in G_ζ as the two-dimensional Riemann BVP (Gakhov, 1966)

$$553 \quad \begin{aligned} \operatorname{Im}[iF(\xi) - k\gamma Z(\xi)] &= 0, & \operatorname{Im}F(\xi) &= 0, & -\infty < \xi < 0; \\ \operatorname{Im}[iF(\xi) + kZ(\xi)] &= 0, & \operatorname{Im}[e^{i\pi\theta} Z(\xi)] &= 0, & 0 < \xi < 1; \\ \operatorname{Im}[iZ(\xi)] &= 0, & \operatorname{Im}F(\xi) &= 0, & 1 < \xi < \infty. \end{aligned} \quad (A7)$$

554

555 Now, using the relation $F(\zeta) = \omega(\zeta)Z(\zeta)$, from eqn. (A7) we can obtain two equivalent scalar

556 Riemann BVPs with respect to $F(\zeta)$ or $Z(\zeta)$. The latter BVP takes the form

$$\begin{aligned}
& \text{Im}[\omega(\zeta)Z(\xi)] = 0, \quad -\infty < \xi < 0; \\
557 \quad & \text{Im}[e^{i\pi\theta} Z(\xi)] = 0, \quad 0 < \xi < 1; \\
& \text{Im}[iZ(\xi)] = 0, \quad 1 < \xi < \infty.
\end{aligned} \tag{A8}$$

558

559 We introduce a new function $Z_1(\zeta)$ by

$$560 \quad Z(\zeta) = e^{-i\pi\theta} (1-\zeta)^{1/2-\theta} Z_1(\zeta), \tag{A9}$$

561 where the branch of the function $(1-\zeta)^{1/2-\theta}$ is fixed in the upper half-plane to be positive at
562 $0 < \xi < 1$. The function (A9) transforms BVP (A8) to the following homogeneous Riemann
563 BVP:

564

$$\begin{aligned}
565 \quad & \text{Im}[e^{-i\pi\theta} \omega(\zeta)Z_1(\xi)] = 0, \quad -\infty < \xi < 0; \\
& \text{Im}[Z_1(\xi)] = 0, \quad 0 < \xi < \infty.
\end{aligned} \tag{A10}$$

566 Solution to problem (A10) is sought in the class of functions with an integrable singularity at
567 the point $\zeta = 0$ and a zero at infinity, the order of this zero is more than one. From the second
568 condition (A10) and the representation (A9) follows that $\zeta = 1$ is a simple pole of $Z_1(\zeta)$.
569 Next, we introduce the function

$$570 \quad \chi(\zeta) = \exp\left[\frac{1-\zeta}{2\pi i} \int_{-\infty}^0 \log\left(e^{i2\pi\theta} \frac{\overline{\omega(t)}}{\omega(t)}\right) \frac{dt}{(t-1)(t-\zeta)}\right] = \zeta^{-\theta} \exp\left[\frac{\zeta-1}{\pi} \int_{-\infty}^0 \frac{\arg \omega(t) dt}{(t-1)(t-\zeta)}\right]. \tag{A11}$$

571 This function vanishes at infinity. If ζ approaches the real axis at $\xi \in (-\infty, 0)$ from the upper
572 half-plane, then the limit value of $\chi(\zeta)$, obtained via the Plemely-Sokhotski formula (Henrici,
573 1993), follows from eqn.(A11) as:

$$574 \quad \chi(\xi) = e^{-i\pi\theta} (-\xi)^{-\theta} \sqrt{\frac{\omega(\xi)}{\overline{\omega(\xi)}}} \exp\left[\frac{\xi-1}{\pi} \int_{-\infty}^0 \frac{\arg \omega(t) dt}{(t-1)(t-\xi)}\right]. \tag{A12}$$

575 At the vicinity of zero we have

$$576 \quad \exp\left[\frac{\zeta-1}{\pi} \int_{-\infty}^0 \frac{\arg \omega(t) dt}{(t-1)(t-\zeta)}\right] = \zeta^{-1/4+(\theta+\lambda)/2} \chi_0(\zeta), \tag{A13}$$

577 where $-1/4 + (\theta + \lambda)/2 = \arg \omega(0)/\pi$ and the function $\chi_0(\zeta)$ is continuous and not vanishing
578 at the point $\zeta = 0$.

579 Taking into account eqns.(A11), (A12), (A13) and the boundary condition $\text{Im } \chi(\xi) = 0$ for
 580 $\xi \in (0, \infty)$, we infer that the function

$$581 \quad Z_2(\zeta) = Z_1(\zeta)\chi(\zeta) \quad (\text{A14})$$

582 satisfies a homogeneous boundary condition $\text{Im } Z_2(\xi) = 0$ at $\xi \in (-\infty, \infty)$.

583 Clearly, $Z_2(\zeta)$ has simple poles at the points $\zeta = 0$, $\zeta = 1$, and zero of the second order at
 584 infinity. Hence

$$585 \quad Z_2(\zeta) = \frac{c_1}{\zeta(1-\zeta)}, \quad (\text{A15})$$

586 where c_1 is an arbitrary real constant to be determined later. Taking into account eqns. (A9),

587 (A11), (A14), and (A15) we obtain the following integral solution to the Dirichlet BVP:

$$588 \quad Z(\zeta) = c_1 e^{-i\pi\theta} (1-\zeta)^{-1/2-\theta} \zeta^{\theta-1} \exp\left[\frac{1-\zeta}{\pi} \int_{-\infty}^0 \frac{\arg \omega(t) dt}{(t-1)(t-\zeta)}\right], \quad F(\zeta) = \omega(\zeta)Z(\zeta). \quad (\text{A16})$$

589 We integrate $Z(\zeta)$ in eqn. (A16) and get

$$590 \quad z(\zeta) = \int_0^\zeta Z(t) dt + L_C, \quad w(\zeta) = \int_0^\zeta F(t) dt. \quad (\text{A17})$$

591 The constant c_1 is determined from eqns. (A16), (A17) using the condition $z(1) = ib$, or

$$592 \quad \text{Im} \int_0^1 Z(t) dt = -c_1 \sin \pi\theta \int_0^1 (1-t)^{-1/2-\theta} t^{\theta-1} \exp\left[\frac{1-t}{\pi} \int_{-\infty}^0 \frac{\arg \omega(\tau) d\tau}{(\tau-1)(\tau-t)}\right] dt = b.$$

593 Using the last relation, eqns. (A16), (A17) and taking into account that

594 $\omega(\xi) = ik - ie^{i\pi\theta} |\omega(\xi) - ik|$ for $\xi \in (0, 1)$, we obtain

$$595 \quad \psi(\xi) = b \frac{\int_0^\xi (|\omega(t) - ik| - k \cos \pi\theta) (1-t)^{-1/2-\theta} t^{\theta-1} \exp\left[\frac{1-t}{\pi} \int_{-\infty}^0 \frac{\arg \omega(\tau) d\tau}{(\tau-1)(\tau-t)}\right] dt}{\sin \pi\theta \int_0^1 (1-t)^{-1/2-\theta} t^{\theta-1} \exp\left[\frac{1-t}{\pi} \int_{-\infty}^0 \frac{\arg \omega(\tau) d\tau}{(\tau-1)(\tau-t)}\right] dt}. \quad (\text{A18})$$

596 It is clear that the function (A18) achieves its maximum at the point $\xi_0 \in (0,1)$ such that

$$597 \quad |\omega(\xi_0) - ik| = k \cos \pi\theta. \quad (\text{A19})$$

598 This completes solution of BVP (A1).

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789

790 **Figure Captions**

791 Fig.1 Vertical cross-section of FFGL recharged from a linear dune (a) and from a takyr (b).

792 Fig.2. Cross-sectional area A_r of the right half of FFGL in Fig.1a (curvilinear triangle $ABCA$)
 793 as a function of evaporation rate e_0 from the discharge zone at the density contrast $\gamma = 0.03$ and
 794 infiltration rates $N_0 = 0.1, 0.2, 0.3$ (curves 1-3, correspondingly) (a); $A_r(N_0)$ at $\gamma = 0.03$ and $e_0 = 0.1,$
 795 $0.3, 0.5$ (curves 1-3, correspondingly) (b).

796

797 Fig. 3. The water table $g_f(r)$ axisymmetric FFGL for $N_0=0.2$, $\gamma=0.03$ and $e_0=0.01, 0.05, 0.2$
 798 (curves 1-3, correspondingly) (a). Saturated water storage in FFGL $V_f(e_0)$ for $\gamma=0.03$ and
 799 $N_0=0.1, 0.2, 0.3$ (curves 1-3) (b).

800

801 Fig.4. Saturated triangle G_z (half of FFGL) in vertical cross-section of the Toth-type flow
 802 (a), digon G_w in the complex potential domain corresponding to G_z (b), reference plane G_ζ (c),
 803 curvilinear triangle G_V in the hodograph plane for BVP in the Appendix (d).

804

805 Fig.5. Distribution of the streamfunction in 2-D flow of Fig.4a as a function of the vertical
 806 coordinate for three half-width of FFGL, $L_C=1, 4.3, 30$ (curves 1-3) at $\gamma=0.03$.

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811 List of Acronyms

812 BVP=boundary value problem

813 FFGL=fresh floating groundwater lens

814 MAR=managed aquifer recharge

815 PK-77= Polubarinova-Kochina, P.Ya., 1977. Theory of Ground Water Movement. Nauka,
 816 Moscow (in Russian)

817 TFL=terrestrial freshwater lenses