

Adelic φ_d^4 -theory

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Formal massless euclidean φ_d^4 -theory on the adèle group is defined. It is shown that φ_d^4 -theory is renormalizable up to the third order in perturbation theory.

Recently there have been a number of papers which discuss a possible use of p -adic numbers and adeles in string theory, statistical physics, quantum mechanics [1–10]. There is a conjecture, coming from number theory, predicting an eventual synthesis of real and p -adic mathematical physics based on the adèle language [3,4,9,10]. For example, the descriptions of the real and p -adic fixed points of the Wilson renormalization group are very similar [7–10]. The aim of this letter is to define φ_d^4 -theory on the adèle space within the frame work of perturbation theory. We shall see that this theory has some non-trivial features.

Let A^d be a d -dimensional adèle group, i.e., A^d is a set of all sequences $(x_\infty, x_2, \dots, x_p, \dots) = x$ with $x_\infty \in \mathbb{R}^d$, $x_p \in \mathbb{Q}_p^d$, and $x_p \in \mathbb{Z}_p^d$ for almost all prime p . Here \mathbb{R}^d is a real d -dimensional space, \mathbb{Q}_p^d is a p -adic d -dimensional space, $\mathbb{Z}_p^d = \{x \in \mathbb{Q}_p^d: |x_p|_p \leq 1\}$, where $|x_p|_p = \max_j |x_p^j|_p$, $x_p = (x_p^1, \dots, x_p^d)$, $|\cdot|_p$ is a p -adic norm on \mathbb{Q}_p . We also denote by A^d the set of sequences (x_2, \dots, x_p, \dots) , where $x_p \in \mathbb{Z}_p^d$ for almost all p (adeles without real components). Let us define the module of the adèle x by

$$|x| = |x_\infty|_\infty |x_2|_2 \dots |x_p|_p \dots, \quad (1)$$

where $|\cdot|_\infty$ is the usual real norm on \mathbb{R}^d . One can show that for $d > 1$ the Haar measure of adeles with zero module is equal to zero, but for $d = 1$ the measure of the corresponding set is equal to one. The function $|x|^\alpha$, $\alpha \in \mathbb{R}$ is well defined as a distribution on the Schwartz–Bruhat space $S(A^d)$ of all functions on A^d that are representable as finite linear combinations

of elementary functions. We recall that the elementary function $f(x)$ is defined as

$$f(x) = f_\infty(x_\infty) f_2(x_2) \dots f_p(x_p) \dots, \quad (2)$$

where $f_\infty(x_\infty) \in S(\mathbb{R}^d)$, $f_p(x_p) \in S(\mathbb{Q}_p^d)$, $p = 2, 3, 5, \dots$, and for all p , except a finite number $f_p(x_p) = \chi_p(x_p)$ is a characteristic function of \mathbb{Z}_p^d . We denote by $S(\mathbb{R}^d)$ the Schwartz space on \mathbb{R}^d and by $S(\mathbb{Q}_p^d)$ the space of test functions $f_p(x_p)$ satisfying the following two requirements: (1) the function $f_p(x_p)$ is finite, i.e., $\text{supp } f_p$ is a compact set; (2) there exists n (depending on f) such that $f_p(x_p) = f_p(y_p)$ if $|x_p - y_p|_p \leq p^n$. If $f(x)$ is an elementary function then

$$\begin{aligned} \langle |x|^\alpha, f \rangle &= \int |x_\infty|_\infty^\alpha f_\infty(x_\infty) dx_\infty \\ &\times \prod_p \int |x_p|_p^\alpha f_p(x_p) dx_p, \end{aligned} \quad (3)$$

where dx_∞ is the Lebesgue measure on \mathbb{R}^d , dx_p is the Haar measure on \mathbb{Q}_p^d , normalized such that $\text{mes}(\mathbb{Z}_p^d) = 1$. It is easy to see that

$$\begin{aligned} \langle |x_p|_p^\alpha, \chi_p \rangle &\equiv \int |x_p|_p^\alpha \chi_p(x_p) dx_p \\ &= \Psi_p(\alpha + d) [\Psi_p(d)]^{-1}. \end{aligned}$$

So, if $d > 1$ and $\alpha > 0$, then the infinite product $\prod_p \langle |x_p|_p^\alpha, \chi_p \rangle$ converges and is equal to $\zeta(\alpha + d) \zeta^{-1}(d)$, where $\zeta(\alpha)$ is the Riemann zeta-function. One can prove that the infinite product (3) also converges and extends to a meromorphic function of α . Moreover, $\Psi(d) \Psi^{-1}(\alpha + d) \langle |x|^\alpha, f \rangle$ is an entire function of α , where $\Psi(\alpha) = \Psi_\infty(\alpha) \zeta(\alpha)$,