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MATHEMATICS

Uniformization of Simply Connected Ramified Coverings of the Sphere by Rational Functions[1]

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Abstract—We deduce a system of ODEs describing the behavior of critical points and poles of a smooth one-parametric family of rational functions uniformizing a given family of ramified coverings of the Riemann sphere.

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Consider a simply connected Riemann surface S that is a ramified covering of the Riemann sphere $\overline{\mathbb{C}}$. It is well known that there exists a rational function R uniformizing S. An important problem is to develop approximate methods for finding R. Here, we suggest a method based on an investigation of the dependence of its critical points and poles on the critical values.

Let S have p sheets and $N+1$ points of multiplicities $n_1, n_2, \ldots, n_{N+1}$ lie over the point at infinity. Let us have M branch points of multiplicities $m_1, m_2, \ldots, m_M$ over the finite plane; we denote by $A_1, A_2, \ldots, A_M$ their projections on the plane.

Note that, as a rule, the surface S is not uniquely determined by given $A_1, A_2, \ldots, A_M$ and multiplicities $m_1, m_2, \ldots, m_M, n_1, n_2, \ldots, n_{N+1}$. Hurwitz was the first to pay attention to this fact. He posed the problem of determining the number of nonequivalent coverings with a given branch type and suggested some ways to investigate the problem [1, 2], see also [3]. Among the works devoted to the problem, we note the papers by Mednykh [4–6], who obtained effective formulas for the number of all nonequivalent coverings with fixed branch data. Some results concerning the study of the problem can be found in [7]. There are also some geometric approaches based on the theory of vector bundles (see, e.g., [8, 9]). In spite of the nonuniqueness mentioned above, for fixed branch multiplicities $m_1, m_2, \ldots, m_M$ and $(n_1, \ldots, n_{N+1})$, we can consider the numbers $A_1, A_2, \ldots, A_M$ as local coordinates in the space of branched coverings.

The essence of our approximate method for determining a rational function R uniformizing a given surface S is as follows. Consider the set $\mathfrak{S}$ of all Riemann surfaces over the sphere with a branch type equivalent to that of S. (This means that the numbers m_k $(1 \le k \le M)$ and n_j $(1 \le j \le N+1)$ characterizing the multiplicities over finite points and infinity for S coincide with those for an arbitrary element $S' \in \mathfrak{S}$.) Denote by $\pi : \mathfrak{S} \to \mathbb{C}^M$ the map relating a surface $S' \in \mathfrak{S}$ to the projections A_j' of its branch points lying in the finite part of the plane.

Assume that $S_0 \in \mathfrak{S}$ is a surface with a known uniformizing function R_0. We connect the surface S_0 to $S = S_1$ in $\mathfrak{S}$ by a smooth curve $S = S(t)$, $0 \le t \le 1$, such that $S(0) = S_0$ and $S(1) = S_1$; the smoothness means that $\pi(S(t))$ is a smooth curve in $\mathbb{C}^M$.

We show that there exists a smooth curve

$$\gamma(t) = (a_1(t), \ldots, a_M(t), b_1(t), \ldots, b_N(t)), \quad 0 \le t \le 1,$$

in $\mathbb{C}^{M+N}$ such that a rational function $R(z,t)$ with critical points $a_1(t), \ldots, a_M(t)$ and poles $b_1(t), \ldots, b_N(t)$, $b_{N+1} = \infty$ uniformizes $S(t)$; under an appropriate normalization, $R(z,t)$ is uniquely defined. The curve γ can be determined as a solution of some definite autonomous system of ODEs (see Theorem 2) with initial data corresponding to the critical points and poles of R_0. Solving this Cauchy problem, for $t = 1$, we obtain the parameters of the desired rational function uniformizing S_1.

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