Extending Wadge Theory to k-Partitions

Victor L. Selivanov[1,2(✉)]

[1] A.P. Ershov Institute of Informatics Systems SB RAS, Novosibirsk, Russia
vseliv@iis.nsk.su

[2] Kazan (Volga Region) Federal University, Kazan, Russia

Abstract. We extend some results about Wadge degrees of Borel subsets of Baire space to finite partitions of Baire space. A typical new result is the characterization up to isomorphism of the Wadge degrees of k-partitions with $\mathbf{\Delta}^0_3$-components.

Keywords: Baire space · Wadge reducibility · Lipschitz reducibility · Backtrack reducibility · k-partition · h-preorder · Well preorder · Infinite game

1 Introduction

For subsets A, B of the Baire space $\mathcal{N} = \omega^\omega$, A is *Wadge reducible* to B ($A \leq_W B$), if $A = f^{-1}(B)$ for some continuous function f on $\mathcal{N}$. The quotient-poset of the preorder $(P(\mathcal{N}); \leq_W)$ under the induced equivalence relation $\equiv_W$ on the power-set of $\mathcal{N}$ is called *the structure of Wadge degrees* in $\mathcal{N}$. W. Wadge [15,16] characterized the Wadge degrees of Borel sets up to isomorphism, in particular this poset is well-founded and has no 3 pairwise incomparable elements.

Let $2 \leq k < \omega$. By a *k-partition of* $\mathcal{N}$ we mean a function $A : \mathcal{N} \to k = \{0, \ldots, k-1\}$ often identified with the sequence $(A_0, \ldots, A_{k-1})$ where $A_i = A^{-1}(i)$ are the components of A. Obviously, 2-partitions of $\mathcal{N}$ can be identified with the subsets of $\mathcal{N}$ using the characteristic functions. The set of all k-partitions of $\mathcal{N}$ is denoted $k^{\mathcal{N}}$, thus $2^{\mathcal{N}} = P(\mathcal{N})$. The Wadge reducibility on subsets of $\mathcal{N}$ is naturally extended to k-partitions: for $A, B \in k^{\mathcal{N}}$, $A \leq_W B$ means that $A = B \circ f$ for some continuous function f on $\mathcal{N}$. In this way, we obtain the preorder $(k^{\mathcal{N}}; \leq_W)$. For any pointclass $\mathbf{\Gamma} \subseteq P(\mathcal{N})$, let $\mathbf{\Gamma}(k^{\mathcal{N}})$ be the set of k-partitions of $\mathcal{N}$ with components in $\mathbf{\Gamma}$.

In contrast with the Wadge degrees of sets, the structure $(\mathbf{\Delta}^1_1(k^{\mathcal{N}}); \leq_W)$ for $k > 2$ has antichains of any finite size. Nevertheless, a basic property of the Wadge degrees of sets may be lifted to k-partitions, as the following very particular case of Theorem 3.2 in [4] shows:

Proposition 1. *For any $2 \leq k < \omega$, the structure $(\mathbf{\Delta}^1_1(k^{\mathcal{N}}); \leq_W)$ is a well preorder, i.e. it has neither infinite descending chains nor infinite antichains.*

Although this result gives an important information about the Wadge degrees of Borel k-partitions, it is far from a characterization. Our aim is to obtain

J. Kari et al. (Eds.): CiE 2017, LNCS 10307, pp. 387–399, 2017.
DOI: 10.1007/978-3-319-58741-7_36