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METHODS OF PROJECTING A POINT IN NORMED SPACES AND THEIR APPLICATIONS

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ABSTRACT
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Scientific Secretary of the Dissertation Council
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GENERAL CHARACTERIZATION OF THE WORK

Timeliness of the topic. The problems of projecting a point onto a set offers a specific class of optimization problems, occurring in a lot of applied research.

In general, a mathematical model of this problem may be written in the following way:

Let us assume that $C \subset E$ is a set, where E is a normed space, and $a \in E$, $a \notin C$ is a given point. It is needed

$$\rho(x, a) \to \min,$$

with constraints $x \in C$, where $\rho(x, a)$ is a distance between points a and x.

One of the first developed iteration projecting algorithm that may be mentioned is the one described in the work of V.F. Demyanov and V.N. Malozemov, algorithm for finding a projection of the origin of a space onto convex polyhedron. Later in the works of L.G. Gurin, B.T. Polyak, E.V. Rayk, Ph. Wolfe, A.A. Dax, H.H. Bauschke and J.M. Borwein, E.A. Nurminski, C.H.J. Pang various point projecting algorithms onto polyhedrons, convex sets, described by intersection of convex sets and etc., were offered. L.M. Bregman considers the problem of projecting in some sense of a point onto an intersection of closed convex sets of linear topological space. It is also worth mentioning a procedure consisting of a finite number of steps for finding projection of a point onto a convex set developed by Z.R. Gabidullina.

A.N. Kulikov and V.R. Fazylov have considered some generalization of the problem of point projecting onto closed convex set of finite-dimensional Euclidean space and build a finite step by step procedure of its solution in their work.

Since not every set can be expressed in terms of convex sets, the approach suggested by H. Tuy, F. Al-Khayyal and P.T. Thach may be useful. Accuracy of set C approximation is increasing with every step in this approach, however, the structure of a new approximating set is becoming complicated, and auxiliary problem may also be quite laborious.

Of particular interest is the case where the stated set is a level surface of some function.

There also may be mentioned the works of A.S. Strekalovsky, who considers more general methods for solving a mathematical programming problem, in which objective functions and functions defining set C are so-called d.c.-functions.

In the case of sufficient smoothness of a function defining set C, Newton-type methods can be applied. In the case when C is defined as a set of zeros of a function satisfying on some convex compact Lipschitz-Hölder condition, V.I. Zabotin and A.M. Dulliev have proposed iterative projecting algorithm.

Projecting problems often arise in solving optimization problems, in particular, they are auxiliary in implementing some gradient projection methods. Mechanism of projection is essential in solving mathematical programming problems with preconvex constraint set by methods developed in several works of Yu.A. Chernyaev and V.I. Zabotin.

Algorithms offered in this work are ideologically close to cutting methods, proposed in works of V.P. Bulatov and O.V. Khamisov, because on each iteration

step a set is being built, which does not contain points of the set, onto which the projection is being carried out.

The main **work objectives** are:

1) formulating and analyzing problem of projecting onto the set of zeros (or some approximation of it) of a function, defined on some convex compact set and satisfying on it the condition of a more general type than Lipschitz condition, in particular, which is ε-Lipschitz;
2) the development and proof of convergence of numerical methods for solving assigned projecting problem;
3) defining various models for realization of developed methods;
4) the creation of a set of programs that implement the proposed algorithms.

The thesis research involves solution of the following **tasks:**

- development and proof of the convergence of the iteration algorithm for solving problem of projecting onto the set of zeros of a function, satisfying on some convex compact the condition generalizing Lipschitz condition;
- development and substantiation of approximate and exact algorithms for projecting of the set of zeros of a function satisfying on some convex compact ε-Lipschitz (continuity) condition;
- development and substantiation of algorithms and the techniques for solving auxiliary problems, arising from implementation of the proposed projecting methods, in particular:
  - algorithm of global minimization of a continuous function on an interval, based on an idea of the method, developed by Yu.G. Evtushenko;
  - approaches to the reduction of the dimension of the auxiliary multi-dimensional problem of minimization on the sphere arising from implementation of the developed projecting algorithms;
- substantiation of applicability and a proposal of implementation of the developed projecting methods for solving the problem of searching the first zero on the left of a continuous function on the interval;
- proposition and substantiation of the penalty method variety with a penalty function in a form of a distance from iteration point to constraint set;
- development of an economic-mathematical model of the consumer choice problem with goods prices depending on the volume of purchases, and its numerical analysis using the proposed methods;
- implementation of the proposed algorithms and the development on their basis of a software for solving the problems of assigned types;
- carrying out numerical tests on the developed software so that to justify the working capacity of the proposed algorithms.

Methods of research. Mathematical and functional analysis, extremum problem theories, linear algebra and also methodologies of structural and object-oriented programming were used to solve the research tasks.

Scientific novelty:

- Unlike currently existing projection algorithms, methods developed in this thesis allow find projection of a point onto set of a more complex geometry (e.g., non-convex sets with nonsmooth boundary).
- The developed methods may be applied for solving nonlinear equations with functions satisfying quite weak continuity condition on the compact.
- Simple in realization, based on the idea of Yu.G. Evtushenko and also representing a separate interest method for finding global minimum of a continuous function on the interval was proposed and substantiated.
- There has been built an economic and mathematical model of the consumer choice problem, the possibility of its solution with proposed algorithms has been shown, numerical experiments have been performed.

Practical utility:

- There have been calculated numerical algorithms for projecting onto sets from classes, which have not been studied before;
- The proposed model of the consumer choice problem describes the strategy of the supplier of goods on pricing more accurately than a classical model;
- There have been developed software applications for solving the tasks formulated in the research work:
  - projecting problems of up to 4 dimensionality;
  - problem of finding a global minimum of the continuous function on the interval;
  - the problem of searching the first zero on the left of a continuous function on the interval;
  - the consumer choice problem with goods prices depending on the amount of purchases by the proposed laws.

The reliability of the received results is provided by the mathematical formulation of the formulated problems, strictness and completeness of argumentations and proofs, and confirmed by obtaining adequate and satisfying the problem statement results of numerical calculations.

Approbation of the work. The main results of the work were presented in reports at the following conferences: The XXVth International Scientific-Technical Conference "Mathematical methods and informational technologies in economics, sociology, and education" (Penza, 2010), The XVIIIth International Youth Scientific Conference "Tupolev readings” (Kazan, 2010), The XXXIst and XXXIIIrd International Scientific-Technical Conference "Mathematical methods and information technologies in economics, sociology and education" (Penza, 2013 and 2014), as well as a scientific seminar of RAS Academician Yu.G. Evtushenko in CC RAS (2015).

Publications. 9 publications have been published on the theme of work, 4 of which are scientific articles in peer-reviewed journals recommended by HAC (Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki, Vestnik KGTU im. A.N. Tupoleva and The Review of Economy, the Law and Sociology), the

English-language version of one of them – the journal, part of the international system of citation SCOPUS (Computational Mathematics and Mathematical Physics).

Structure and volume of work. Thesis work consists of introduction, four chapters, conclusion, list of abbreviations and symbolic notation, a reference list including 54 names, list of illustrative material and appendices. The work is presented on 97 pages of typewritten text, contains 7 tables and 10 figures.

Personal input of the author.

Chapter 1. 1 Statement of ***task 1*** and proof of proposition 1.1 have been completed in cooperation with V.I. Zabotin. The rest of the results of the chapter have been obtained by the author of the dissertation.

Chapter 2. Statement of ***task 2.2*** and proof of lemmas 2.2, 2.3 have been completed in cooperation with V.I. Zabotin, formulation of algorithm 2.3 and proof of proposition 2.10 have been completed in cooperation with A.M. Dulliev. The rest of the results of the chapter belong to the author.

Chapter 3. All results belong to the author.

Chapter 4. Proposition 4.1 has been proved in cooperation with V.I. Zabotin. Assertion of the consumer choice problem with a step function of prices (type 1) has been offered by the author, V.I. Zabotin and N.P. Zabotina. Construction of mathematical models of type 1 and type 2 and their analysis as well as development of algorithms have been completed by the author. The rest of the results of the chapter belong to the author.

Development of all operational algorithms, software and calculations of test activities are made by the author.

SCOPE OF WORK

Throughout the paper, unless otherwise stated, the following symbols are used: E is a normed space; $A \subset E$ is convex compact (in the sense of compactness in itself); int B, cl B, ∂B – interior, closure and boundary of set $B \subset E$, respectively; $X^* \subset A$ – set of zeros of function f, defined on A. All functions are assumed to be real-valued.

The Introduction provides a brief review of the scientific literature on the topic of the thesis, the degree of its status and settles the timeliness of the work. Also, a brief description of the content of the chapters of the dissertation and a list of published scientific works on the topic are given in the introduction.

Algorithm for projecting of a point onto the set of zeros of the function satisfying the conditions of a more general form than the Lipschitz condition is proposed in **the first chapter**.

Let us assume that convex function $q(x)$ is defined on E so that $q(x) = 0$ then and only then, when $x = 0$. Let f satisfy the condition:

$$\forall x, y \in A: \left| f(x) - f(y) \right| \le q(x - y). \tag{1}$$

If for $a \in A$, $f(a) > 0$, and some $x^* \in X^*$ takes place $\nu(a, X^*) = q(x^* - a)$, where

$$\nu(a, X^*) = \inf_{x \in X^*} q(x - a), \tag{2}$$

point x^* is named *a projection in the sense of* (2) of point a onto set X^*.

Task 1 is set: *To develop and substantiate an algorithm for finding projection in the sense of* (2) *of a given point a onto the set* X^*.

For solving this task ***algorithm 1*** is proposed:

Step 0. $k := 0$, $x_0 := a$ are given.

Step 1. The following set is constructed $K_k = \left\{ x \in E : q(x-a) \le \sum_{i=0}^{k} f(x_i) \right\}$.

Step 2. The following iteration point is found:

$$x_{k+1} = \arg\min \{ f(x) : x \in \partial K_k \cap A \}.$$

Step 3. If $f(x_{k+1}) = 0$,

then x_{k+1} is regarded as desired projection,

else – set $k := k+1$ and pass to step 1.

Proposition 1.1. *Any of limit points (in the sense of a norm of space E) of the sequence* x_k *generated by algorithm 1, belongs to the set* X^*.

Proposition 1.2. *If* $x^* \in X^*$ *is a limit point of the sequence* x_k *generated by algorithm 1, then it is a projection in the sense of* (2) *of point a onto* X^*.

Proposition 1.3. *If* $x^* \in X^*$ *is the single nearest in the sense of* (2) *point to a from* X^*, *then* $x_k \xrightarrow[k]{} x^*$.

Proposition 1.4. *If set of zeros* X^* *of function f on compact A is empty, then algorithm 1 recognizes that in finite numbers of steps.*

Let E be n-dimensional Euclidian space and function $f(x_1, x_2, \ldots, x_n)$ satisfies a Lipschitz-Hölder condition coordinate-wise:

$$\forall i = \overline{1,n}\ \exists L_i, \alpha_i > 0\ \forall x_i', x_i'' \in A: \quad \left| f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i'', x_{i+1}, \ldots, x_n) \right| \le L_i \left| x_i' - x_i'' \right|^{\alpha_i}. \tag{3}$$

Proposition 1.5. *Condition* (3) *is equivalent to the fact that there exist* $L_i > 0$ *and* $\alpha_i > 0$, $i = 1, 2, \ldots, n$, *where for all* $x, y \in A$ *the following inequation takes place*

$$\left| f(x) - f(y) \right| \le \sum_{i=1}^{n} L_i \left| x_i - y_i \right|^{\alpha_i}.$$

It may be noticed, that when $\alpha_i \ge 1$, $i = 1, 2, \ldots, n$, function

$$q(x) = \sum_{i=1}^{n} L_i \left| x_i \right|^{\alpha_i} \tag{4}$$

satisfies all the conditions, set in the formulation of task 1.

However, when $\alpha_i > 1$, $i = 1, 2, \ldots, n$, it is of no interest as in this case the function is constant. Thus, condition (1) with function $q(x)$ of type (4) when $\alpha_i = 1$ is equivalent to condition (3).

In case when $E = R^2$ and $E = R^3$, and function f satisfies condition (3), algorithm 1 for solving task 1 is software-programmed and it showed its working capacity and expected results in test calculations.

The task of projecting of a point onto set, defined by function *f*, which satisfies the following condition:

$$\forall \varepsilon > 0\ \exists L(\varepsilon) > 0\ \forall x, y \in A : |f(x) - f(y)| \le L(\varepsilon)\|x - y\| + \varepsilon,$$

is set and considered in the **second chapter**. This condition was named ε-Lipschitz condition and proposed by Robert J. Vanderbei.

In the same work Robert J. Vanderbei shows, that *every real-valued continuous function defined on a convex compact of finite-dimensional Euclidean space is ε-Lipschitz function*, and proposes approaches to define estimate $L(\varepsilon)$.

Task 2.1: *To develop and substantiate a numerical algorithm for solving the problem of projecting a point* $a \in A$ $(f(a) > 0)$ *onto the set* $X^{\varepsilon} = \{x \in A: 0 \le f(x) \le \varepsilon\}$, *in the sense of norm of space*;

Task 2.2: *To develop and substantiate a numerical algorithm for solving the problem of projecting a point* $a \in A$ $(f(a) > 0)$ *onto the set* $X^{*} = \{x \in A: f(x) = 0\} \ne \varnothing$ *in the sense of norm of space.*

For solving task 2.1 the following algorithm that is named approximate in the thesis is proposed.

Algorithm 2.1.

Step 0. $k = 0$, $x_0 = a$ are given.

Step 1. If $f(x_k) \le \varepsilon$, *then* x_k is taken as a desired point,
else – transition to step 2 is performed.

Step 2.

$$r_k = \frac{f(x_k) - \varepsilon}{L(\varepsilon)}, \quad K_k = \left\{ x \in E : \|x - a\| \le \sum_{i=0}^{k} r_i \right\}, \quad x_{k+1} = \arg\min\{f(x) : x \in \partial K_k \cap A\}.$$

are defined.

Step 3. $k := k + 1$ is given and transition to step 1 is performed.

A range of propositions have been received for proof of convergence of an algorithm.

Proposition 2.1. *If* $f(x_k) > \varepsilon$, *then* $f(x) > 0$ *on* $\operatorname{int} K_k \cap A$.

Proposition 2.2.

1°. *If there exists k, for which* $f(x_k) > \varepsilon$ *and* $f(x_{k+1}) \le \varepsilon$, *then* $x_{k+1} \in X^{\varepsilon}$.

2°. *If there holds* $f(x_k) > \varepsilon$ *for any k, then any limit point of generated sequence* x_k *is a point from* X^{ε}.

Proposition 2.3. *If* $X^{\varepsilon} = \varnothing$, *then algorithm 2.1 recognizes that in finite number of steps.*

For every fixed $\varepsilon \in (0, f(a))$ let us denote $l(\varepsilon) = \inf\{L(\varepsilon)\}$.

Constant $L(\varepsilon)$ for function *f* is named ε-*attainable*, if

$$\exists x_0, y_0 \in A, x_0 \ne y_0 : |f(x_0) - f(y_0)| = L(\varepsilon)\|x_0 - y_0\| + \varepsilon.$$

Lemma 2.1. *For any* $\varepsilon \in (0, f(a))$ *there holds* $l(\varepsilon) > 0$, $l(\varepsilon) \in \{L(\varepsilon)\}$, *or, in other words, the lower bound of the set* $\{L(\varepsilon)\}$ *is attainable.*

Lemma 2.2. *Estimate* $l(\varepsilon)$ *is* ε-*attainable for any fixed* $\varepsilon \in (0, f(a))$. *If* $L(\varepsilon)$ ε-*attainable, then* $L(\varepsilon) = l(\varepsilon)$.

Lemma 2.3.

1°. $l(\varepsilon)$ *strictly increases as* ε *tends monotonically to zero.*

2°. *If* $l(\varepsilon)$ *is bounded above on* $(0, f(a))$, *then* f *is Lipschitz continuous.*

Algorithm2.2.

Because of the property 2° we will assume unboundedness of $l(\varepsilon)$ above.

Step 0. Initial point $x_0 = a \in A$, algorithm parameter value $\lambda \in (0; 1)$, number of a step $k := 0$ are given.

Step 1. The following values are computed: parameter $\varepsilon_k = \lambda f(x_k)$, corresponding estimate $l(\varepsilon_k)$, radius $r_k = \dfrac{f(x_k)}{l(\varepsilon_k)}(1-\lambda)$.

Step 2. x_{k+1} is defined as any point of the set

$$\arg\min\{f(x): x \in \partial K_k \cap A\}, \text{where } K_k = \left\{x : \|x-a\| \le \sum_{i=0}^{k} r_i\right\}.$$

Step 3. If $f(x_{k+1}) = 0$,

then x_{k+1} is taken as a solution of the given task,

else – $k := k + 1$ and transition to step 1 is performed.

Proposition 2.4. *If* $x \in \operatorname{int} K_k \cap A$, *then* $f(x) > 0$.

Proposition 2.5. *If* x_k *is iterative sequence generated by algorithm 2.2, then* $f(x_k) \xrightarrow[k]{} 0$.

Proposition 2.6. *If* $y^* \in X^*$ *and* y^* *is a limit point of the sequence* x_k, *then* $\rho(a, X^*) = \|a - y^*\|$, *i.e.* y^* *is the nearest point to* a *from* X^*.

Proposition 2.7. *If* X^* *consists of a single point or the point from* X^* *nearest to* a *is unique, then* x_k *converges to this point.*

Proposition 2.8. *If* $X^* = \varnothing$, *then algorithm 2.2 recognizes that in finite number of steps.*

Algorithm2.3.

Step 0. Initial parameter value $\varepsilon_0 > 0$, initial point $x_0 = a \in A$, algorithm parameters $\gamma, \lambda \in (0; 1)$ are given. It is set, that $y_0 := x_0$, $k := 0$, $m := 0$.

Step 1. If $f(x_k) < \varepsilon_k(1 + \gamma)$,

then consequently $x_{k+1} := x_k$, $y_{m+1} := x_k$, $m := m + 1$ are set, and transition to step 2 is performed;

else – transition to step 3.

Step 2. If $f(x_k) \le \varepsilon_k$, *then* $\varepsilon_{k+1} := \lambda f(x_k)$,

else $\varepsilon_{k+1} := \lambda\varepsilon_k$;

and transition to step 4 is performed.

Step 3. x_{k+1} is defined according to the following scheme:

$$r_k = \frac{f(x_k) - \varepsilon_k}{L(\varepsilon_k)},\ K_k = \left\{x \in E : \|x-a\| \le \sum_{i=0}^{k} r_i\right\},\ x_{k+1} = \arg\min\{f(x): x \in \partial K_k \cap A\};$$

$\varepsilon_{k+1} := \varepsilon_k$ and transition to step 4 is performed.

Step 4. If $f(x_{k+1}) = 0$ or $f(y_m) = 0$,
then x_{k+1} or y_m, are taken as the solution of the task, respectively;
else $k := k + 1$ is set and transition to step 1 is performed.

Proposition 2.9. *Let the function $f(x)$ be continuous on the convex compact A and $X^* \neq \varnothing$ as before, then for any limit point y^* of the sequence y_m generated by algorithm 2.3, there holds $y^* \in X^*$.*

In case of finite number of values of *the sequence* x_k, the last found value x_k^* or y_m^* is used as a solution, according to step 4 of algorithm 2.3 belonging to the set X^*.

Proposition 2.10. *Points y^*, x_k^* or y_m^* depending on the finiteness of the number of values of the sequence x_k are the projections of the initial point $a \in A$ onto the set X^*.*

Proposition 2.11. *If $X^* = \varnothing$, then algorithm 2.3 recognizes that in finite number of steps.*

On the base of principle algorithms 2.1 – 2.3 there have been created operational algorithms for cases, when space E has dimensionality up to $n = 4$ and when metric given in it is Euclidean or coordinate-wise, there has been developed a software and calculated a number of test tasks.

As an example there have been considered the following functions with a corresponding estimate of ε- Lipschitz constant:

$$f_1(x,y) = \sqrt{|x|} + \sqrt{|y|}, \quad L_1(\varepsilon) = \frac{1}{\sqrt{2}\varepsilon}; \qquad f_2(x,y) = \sqrt{|x| + |y|}, \quad L_2(\varepsilon) = \frac{1}{2\sqrt{2}\varepsilon}.$$

And these values $L(\varepsilon)$ as a result of satisfaction the conditions described in lemmas 2.1 – 2.3 can be used as $l(\varepsilon)$.

The analysis of results showed that algorithm 2.1 has the most high-speed, algorithm 2.2 has the least.

The results of experiments also showed that the number of iterative steps is rather accurately estimated as $O(1/\varepsilon^*)$.

Methods for solving auxiliary problems from proposed projecting algorithms are developed in **chapter 3.**

Task 3: *To find a value* $f^* = \min\limits_{x \in [a;b]} f(x)$ *for function* $f = f(x)$ *satisfying the ε-Lipschitz condition on interval* $[a; b]$.

The essence of the proposed ***modification of Evtushenko's method for continuous functions on the interval*** is generating such a sequence of points $x_1, x_2, \ldots, x_n$ ($a \le x_1 < x_2 < \ldots < x_n \le b$), that the value of the function in one of it may be taken as some approximation to f^* with some accuracy ε^*, i.e.

$$f(x_k) = \min_{1 \le i \le n} f(x_i) \le f^* + \varepsilon^*. \tag{5}$$

For the work of algorithm additional parameter $\varepsilon \in (0; \varepsilon^*)$ is introduced, for which estimate $L(\varepsilon)$ is known.

Let us denote: $h = 2(\varepsilon^* - \varepsilon)/L(\varepsilon)$; $F_i = \min\limits_{1 \le j \le i} f(x_j)$, $\forall i = \overline{1,n}$.

Algorithm 3.

1) Set an initial point: $x_1 = a + h/2$.
2) For each $i = \overline{1, n-2}$ define $x_{i+1} = x_i + h + (f(x_i) - F_i)/L(\varepsilon)$.
3) Set $x_n = \min\{x_{n-1} + h + (f(x_{n-1}) - F_{n-1})/L(\varepsilon), b\}$.
4) Define a number n satisfying the condition:

$$x_{n-1} < b - h/2 \le x_{n-1} + h + (f(x_{n-1}) - F_{n-1})/L(\varepsilon).$$

5) Take a value F_n as a desired minimal value f^*.

Proposition 3.1. *Described method of sequential search solves task* (5).

Algorithm 3 is implemented as a separate software application and used as an auxiliary tool for solving the problem of global minimization of ε-Lipschitz function on the sphere arising in one of the steps of algorithms 2.1 – 2.3.

It should be noted that the proposed modification is of interest in itself, because it gives the possibility of numerical minimization of a continuous function on the interval.

The proposed modification has advantages over generalization of Piyavskii's method proposed by Robert J. Vanderbei, similar to the advantages of Evtushenko's method over the original Piyavskii's method.

Evtushenko's method and its modification are important in creating operational algorithms of the above mentioned projecting methods and are needed to solve auxiliary problems of minimization of a function on a sphere, if the minimum is finding by successive fixation of coordinates. Therefore, it is essential to decrease the dimension of the problem at least by one.

For coordinate-wise norm it is natural to find the minimum on each face of a hypercube.

In the case of Euclidean norm extended spherical substitution with corresponding re-calculation of ε-Lipschitz constant estimate allows to reduce the dimension of the problem. Let:

$$\begin{cases} x_1 = a_1 + \rho \cdot \cos\varphi_{n-1} \cdot \cos\varphi_{n-2} \cdot \ldots \cdot \cos\varphi_2 \cdot \cos\varphi_1, \\ x_2 = a_2 + \rho \cdot \cos\varphi_{n-1} \cdot \cos\varphi_{n-2} \cdot \ldots \cdot \cos\varphi_2 \cdot \sin\varphi_1, \\ x_3 = a_3 + \rho \cdot \cos\varphi_{n-1} \cdot \cos\varphi_{n-2} \cdot \ldots \cdot \sin\varphi_2, \\ \vdots \\ x_{n-1} = a_{n-1} + \rho \cdot \cos\varphi_{n-1} \cdot \sin\varphi_{n-2}, \\ x_n = a_n + \rho \cdot \sin\varphi_{n-1}, \end{cases} \quad \begin{array}{l} 0 \le \varphi_1 \le 2\pi, \\ -\dfrac{\pi}{2} \le \varphi_i \le \dfrac{\pi}{2}, \quad i = \overline{2, n-1}, \end{array}$$

where a_i, x_i, $i = \overline{1,n}$, – are Cartesian coordinates of points a and x, respectively, and $(\rho, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ – are spatial polar coordinates of point x, while $\rho = const$ for all points of sphere ∂K, which decreases dimension of the problem.

Let $F(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ denote the values of function f on sphere ∂K, after mentioned substitution.

Lemma 3.1. *If function f is ε-Lipschitz with respect to* $x_1, \ldots, x_n$ *with constant* L_f *, then function F is ε-Lipschitz with respect to each* φ_i, $i=\overline{1,n-1}$, *while others are fixed with constant* $L_{\varphi_i} = \sqrt{2}\rho L_f$.

Let us define the functions:

$$\Phi_i(\varphi_1,\varphi_2,\ldots,\varphi_{n-i-1}) = \min_{\varphi_{n-i}} \min_{\varphi_{n-i+1}} \ldots \min_{\varphi_{n-1}} F(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}), \quad i=\overline{1,n-2}.$$

Lemma 3.2. *Each function* $\Phi_i(\varphi_1,\varphi_2,\ldots,\varphi_{n-i-1})$, $i=\overline{1,n-2}$, *is ε-Lipschitz with respect to* φ_{n-1-i} *for any fixed* $\varphi_1 \in [0;2\pi]$, $\varphi_j \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, $j=\overline{2,n-2-i}$, *with a constant* $L_\Phi = \sqrt{2}\rho L_f$.

The fourth chapter describes some applications of the proposed projecting algorithms, including the ones for the analysis of an economic and mathematical model, set out in this thesis.

Projecting algorithms described in Chapters 1 – 2 can be used as a tool for solving various equations containing in their formulations continuous functions.

The special case of task 2.2 of projecting a point is the problem of searching the first zero from the left of continuous function on the interval.

In this case algorithm 2.2 is much simplified. ***Algorithm 4.1*** has been developed for this case. The essential element of which is the absence of the need to solve the auxiliary minimization problem – the next iterative point is uniquely determined according to the following formula: $x_{k+1} = x_k + f(x_k)(1-\lambda)/l(\varepsilon_k)$.

Algorithm 4.1 is also implemented as a software application and experimental numerical calculations have been made.

Proposition 4.1. *If A is a compact set in* R^n, *then function of distance* $\rho(x, A)$ *is Lipschitz with respect to x with constant equal to* 1.

Now (as, for instance, it was proposed by F.P. Vasil'ev) as a penalty function there may be taken $\rho(x, A)$ and penalty method may be used according to the scheme proposed by V.G. Karmanov, constructing a penalty function according to the formula: $\Phi_k = f(x)/\beta_k + \rho(x, A)$, where $\beta_k \xrightarrow[k]{} \infty$. On the assumption that f is Lipschitz, function Φ_k becomes Lipschitz continuous with constant $L_k = L_f/\beta_k + 1$.

Now, to find the required solution in addition to function Φ_k there can be used a known method of global minimization of Lipschitz function, determining the distance using one of the proposed in the work projecting algorithms.

Described type of penalty method has been applied to the numerical analysis of the proposed economic-mathematical model of ***the consumer choice problem*** with prices of goods, depending on the amount of purchase, as described below.

The proposed formulation of the consumer choice problem differs from the classic one, that pricing strategies of a seller are not assumed to be constant, but depend on the volume of sold goods.

The dissertation examines the so-called direct problem (Marshallian type), the classical formulation of which has the following form:

$$\begin{aligned} & u(x) = u(x_1, x_2, \ldots, x_n) \to \max, \\ & x \in G, \\ & \sum_{j=1}^{n} p_j x_j - M = 0. \end{aligned}$$

Here:

- $x = (x_1, x_2, \ldots, x_n)^T$, $x_j \geq 0$, $j = \overline{1,n}$, – *a bundle of goods*, purchased by a consumer, where x_j, $j = \overline{1,n}$, are measured in «ever varying» units;
- $G = \left\{ x = (x_1, x_2, \ldots, x_n) \in [0;\infty)^n : \forall j = \overline{1,n} \left(x_j^{\min} \leq x_j \leq x_j^{\max} \right) \right\}$ – feasible sets of goods ($x_j^{\min}$ and $x_j^{\max}$ – fixed minimal and maximal quantity of j-th good, available for purchase);
- $p = (p_1, p_2, \ldots, p_n)$, p_j = const, $p_j \geq 0$, $j = \overline{1,n}$, – *goods price vector*, fixed on the purchased goods;
- $M > 0$ – *consumer's income (wealth)*, i.e. a limited amount of money that a consumer can use to purchase the goods;
- $u = u(x)$ – *utility function*, reflecting the level (or degree) that satisfies his wants by getting a bundle of goods x, asserted isotonic.

Linearity of restrictions of the problem is provided by the assumption of the constancy of price values p_j, $j = \overline{1,n}$.

The problem with the simplest dependences $p_j = p_j(x_j)$, $j = \overline{1,n}$, is discussed in the work of Ikuma Issombo Yan. However, this work has a number of significant inaccuracies.

It is considered in this dissertation that $p_j = p_j(x_j)$, $j = \overline{1,n}$, is nonascending with respect to variable x_j function on the interval $\left[x_j^{\min}; x_j^{\max}\right]$. The seller's strategy to drop the price $p_j(x_j)$ is considered known to a customer for each $j = \overline{1,n}$.

Now the task has the following form:

$$\begin{aligned} & u(x) = u(x_1, x_2, \ldots, x_n) \to \max, \\ & S(x) - M = 0, \\ & x \in G, \end{aligned} \tag{6}$$

where $S(x) = S(x, p(x))$ is a *cost function* of all purchased bundle of goods x.

We will calculate function $S(x)$ as the total area of hypographs of $p_j = p_j(x_j)$, on the interval $[0; x_j]$, $j = \overline{1,n}$, i.e.:

$$S(x) = \sum_{j=1}^{n} S_j(x_j), \quad \text{where} \quad S_j(x_j) = \int_0^{x_j} p_j(y)dy, \quad j = \overline{1,n}.$$

Two types of functions $p_j = p_j(x_j)$ and corresponding functional dependences $S_j = S_j(x_j)$, $j = \overline{1,n}$ are proposed in the work. Index j is omitted below for

convenience, and upper indexing of prices and quantities of goods is used.

Type 1 («stepwise function»). Given values are $p^m > p^{m-1} > \ldots > p^1 > {}^0 > 0$ and $0 < a^1 < a^2 < \ldots < a^m$. We assume that $a^0 = 0$, and introduce value a^{m+1} of the most amount of goods available for purchase ($a^{m+1} > a^m$). Law of price variation depending on the amount of purchased goods has the following form:

$$p(x) = p^{m-i}, \quad \text{если } x \in \left(a^i; a^{i+1}\right], \quad i = \overline{0,m}.$$

The total cost S of all the amount of purchased goods x, $x \in (0; a^{m+1}]$ will be calculated by the following formula:

$$S(x) = \sum_{j=0}^{i-1} p^{m-j}\left(a^{j+1} - a^j\right) + p^{m-i}\left(x - a^i\right), \quad x \in \left(a^i; a^{i+1}\right], \quad i = \overline{0,m}.$$

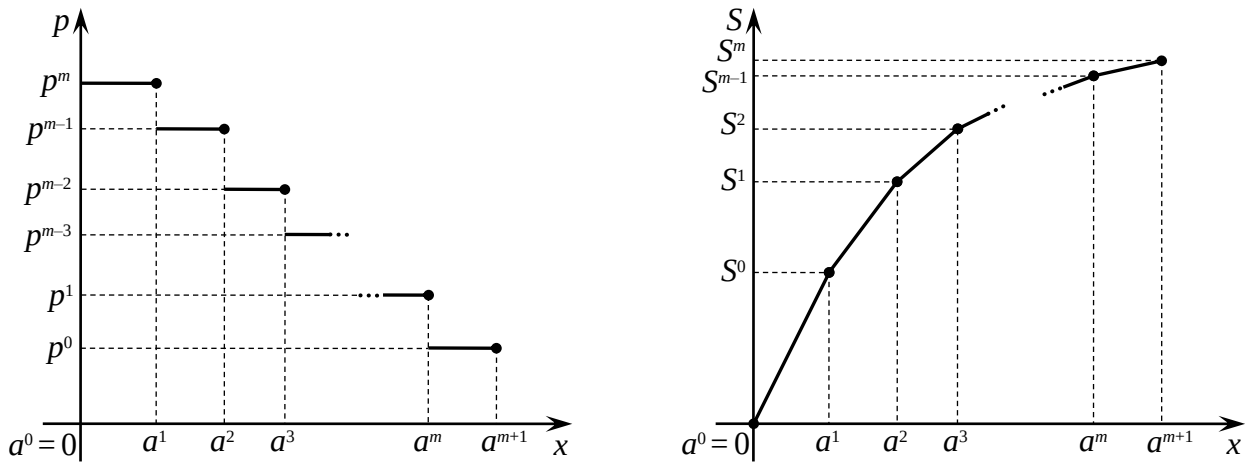
Graphical view of such connections is illustrated in Fig. 1.

It may be shown, that

$$S(x) = \min_{0 \le i \le m} \tilde{P}_i \tilde{A}_i(x),$$

where $\tilde{P}_i = \left(\tilde{p}_i^{\,j}\right), \quad \tilde{p}_i^{\,j} = \begin{cases} p^{m-i}, & j \le i, \\ 0, & j > i, \end{cases} \quad i, j = \overline{0,m};$

$$\tilde{A}_i(x) = \left(\tilde{a}_i^{\,j}\right)^T, \quad \tilde{a}_i^{\,j} = \begin{cases} a^{j+1} - a^j, & j < i, \\ x - a^j, & j = i, \\ 0, & j > i, \end{cases} \quad i, j = \overline{0,m}.$$



а) б)

Fig. 1. Type 1 («stepwise function») of price and cost of goods function

Type 2 («piecewise-linear»). It is given, that $0 = a^0 < a^1 < a^2 < \ldots < a^m < a^{m+1}$ and $p^{m+1} = p^m > p^{m-1} > \ldots > p^1 > p^0 > 0$.

Functions of price and cost in this case are defined in the following way:

$$p(x) = p^{m-i+1} + \frac{p^{m-i} - p^{m-i+1}}{a^{i+1} - a^i}\left(x - a_i\right), \quad x \in \left[a^i; a^{i+1}\right], \quad i = \overline{0,m},$$

$$S(x) = \frac{p^{m-i} - p^{m-i+1}}{2\left(a^{i+1} - a^i\right)}\left(x - a^i\right)^2 + p^{m-i+1}\left(x - a_i\right) + \sum_{j=0}^{i-1} \frac{p^{m-j} + p^{m-j+1}}{2}\left(a^{j+1} - a^j\right).$$

These functions are shown graphically in Fig. 2.

It is shown in the thesis, that functions of the cost of one good purchase are Lipschitz with the constant equal to p^m for two types of strategies. The function of the whole purchase cost consisting of several $n > 1$ goods will then satisfy Lipschitz condition with constant $L = \sum_{j=1}^{n} p_j^m$ and will satisfy coordinate-wise Lipschitz condition with respect to corresponding x_j with constants p_j^m. Thus, function of cost of the whole purchase $S(x)$ will satisfies condition (1) with function $q(x)$ of the form:

$$q(x) = \sum_{j=1}^{n} L_j |x_j|.$$

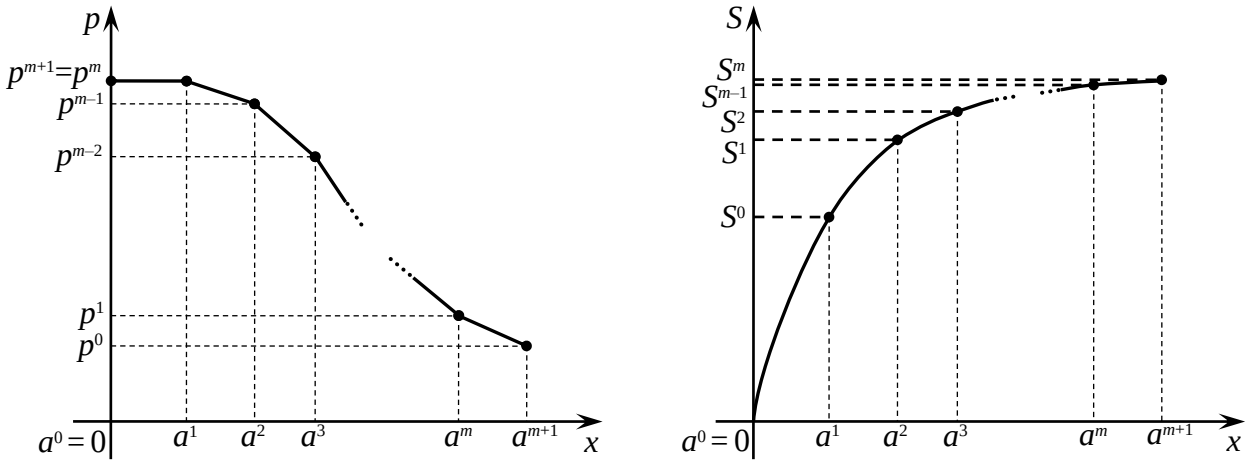


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Fig. 2. Type 2 («piecewise-linear») of price and cost of goods function

Now for solution of the problem (6) we can use the above described scheme of penalty method, where the estimate of proximity of iterative point x^k to constraint set is in the sense of (2) using algorithm 1.

Let us assume objective function $u(x)$ and constraint function $g(x) = S(x) - M$ to be Lipschitz with known Lipschitz constants L_u и L_g, respectively.

We will minimize the function $\Phi_k = -\alpha_k u(x) + |g(x)|$, where $\alpha_k > 0$, $\alpha_k \xrightarrow[k]{} 0$. Its Lipschitz constants calculated by the following formula: $L_k = \alpha_k L_u + L_g$.

Algorithm 4.2.

Step 0. Accuracy $\varepsilon > 0$; integer $r > 0$; number of a step $k := 0$ are given.

Step 1. α_k is calculated and point x^{k+1} is determined as any point of the set $\arg\min\{\Phi_k(x) : x \in G\}$ by one of the methods of minimization of Lipschitz functions.

Step 2. If k multiple of r,

then transition to step 3 is performed,

else – to step 4.

Step 3. If the termination criteria $\rho\left(x^{k+1}, A\right) \le \varepsilon$ or $\nu\left(x^{k+1}, A\right) \le \varepsilon$ is satisfied
then x^{k+1} is taken as a solution of the original problem,
else – transition to step 4 is performed.

Step 4. $k := k + 1$ is set and transition to step 1 is performed.

As an example, the consumer choice problem (6) is considered for $n = 2$ goods, prices for which are dropped according to one of the above strategies with utility function as Leontief function:

$$u(x_1, x_2) = \min\{\gamma_1 x_1, \gamma_2 x_2\}, \quad x_1, x_2, \gamma_1, \gamma_2 > 0.$$

Its Lipschitz constant can be calculated by the following formula $L_u = \max\{\gamma_1, \gamma_2\}$, and utility function $u(x_1, x_2)$ will not be smooth.

To solve this type of consumer choice problem on the basis of described principle algorithm 4.2 an operational algorithm was built, later software implemented. There were carried out calculations of test examples with different strategies of both types of drop of prices for bundle of two goods.

THE MAIN RESULTS OF THE WORK

The following problems were set and solved in the thesis:

1) development and proof of convergence of method for projecting of a point onto the level surface of a function satisfying conditions of a more general type than a Lipschitz-Hölder condition;
2) development and proof of convergence of approximate method for projecting of a point onto the level surface of ε-Lipschitz function;
3) development and proof of convergence of two exact methods for projecting onto the set of zeros of ε-Lipschitz function;
4) development and substantiation of method for global optimization of ε-Lipschitz functions on the interval;
5) proof of approaches of dimensionality reduction for problems of global optimization of ε-Lipschitz functions that occur in implementing the above mentioned algorithms for point projecting;
6) development of an operational algorithm for finding the first zero on the left of ε-Lipschitz function on the interval;
7) development of a penalty method version with a penalty function in the form of distance to constraint set, and substantiation of its use when implementing the above mentioned projecting methods;
8) development of economic-mathematical models of the consumer choice problem with prices depending on the volume of purchases, and justification of the use of the above discussed modification of penalty method for their numerical analysis.

For all the above mentioned methods there have been created operational algorithms, also software implemented. Demonstration examples have been calculated by developed software.

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