Finite element approximation and iterative method solution of elliptic control problem with constraints to gradient of state

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Abstract

An optimal control problem with distributed control in the right-hand side of Poisson equation is considered. Pointwise constraints on the gradient of state and control are imposed in this problem. The convergence of finite element approximation for this problem is proved. Discrete saddle point problem is constructed and preconditioned Uzawa-type iterative algorithm for its solution is investigated.

Key words: optimal control, finite element method, iterative method, constrained saddle point problem

Introduction

Many physical processes modeled by partial differential equations require constraints on their solutions which a play role of state in the corresponding optimization problems. Pointwise constraints on the gradient of the state are important, in particular, in cooling and heating processes in order to avoid damage of the products caused by large material stresses (cf., e.g. [1] — [3] and bibliography therein for cooling in continuous casting process).

The state constraints in general deliver low regularity of adjoint variables and this fact brings difficulties in studying the discrete approximations of optimal control problems with state constraints. A series of articles is devoted to investigation of the approximation and iterative solution methods for the optimal control problem with pointwise constraints to the state ([4] — [9]). Compared to pointwise constraints on the state the gradient constraints involve the gradient operator, which has a non-trivial kernel, and this further complicates the problem. There is a few articles dealing with such kind of problems ([10]—[16]). Thus, in [10] a theoretical analysis of an optimal control of semilinear elliptic equation with pointwise constraints on the gradient of the state is made. The investigation of the convergence and rate of convergence of finite element approximations to optimal control problems with the constraints on the gradient of the state is the topic of articles [11] — [13]. In [11] variational discretization of the controls is considered combined with the lowest order Raviart-Thomas finite element approximations of a mixed formulation of the state equation. Controls are not discretized explicitely, but implicitly through the optimality conditions associated with the discrete approximation to the optimal control problem. This in particular leads to piecewise constant approximations to the state and the adjoint state. In [12] the L_r-norm of the control is included in cost functional with

$r > d$ ($d = 2, 3$ is the dimension of the problem) to guarantee the required regularity of the state. Variational discretization of the control problem then is investigated, as well as piecewise constant approximations of the control. In both cases standard piecewise linear and continuous finite elements for the discretization of the state is used. Error bounds for control and state are obtained depending on the value of r. Similar estimates are obtained in [13], where L_r-norm in cost functional is included as well. In [14] semi-smooth Newton methods and regularized active set methods are discussed for the solution of an elliptic equation with gradient constraints. An analysis for a barrier method for optimization with constraints on the gradient of the state can be found in [15]. Adaptive finite element methods for optimization problems for second order linear elliptic partial differential equations subject to pointwise constraints of the gradient of the state are considered in [16]. In a weak duality setting, i.e. without assuming a constraint qualification such as the existence of a Slater point, residual based a posteriori error estimators are derived.

In this paper we consider an elliptic optimal control problem with distributed control, observation in a subdomain and pointwise constraints on the gradient of state. We approximate this problem by finite element scheme with piecewise constant elements for control function and piecewise linear and continuous finite elements for the discretization of the state function. Pointwise bounds on the gradient of the discrete state are enforced element-wise. We prove the strong convergence of finite element approximation for this problem by using well-known approach to convergence theory for variational inequalities and minimization problems (cf., e.g. [17]).

Further we construct discrete saddle point problem and its iterative solution method. For these purposes we use the theory of preconditioned Uzawa-type iterative methods for saddle point problems developed in [18], [22], and applied for variational inequalities and optimal control problems in [19] — [21].

Let us emphasize that the main advantage of the proposed iterative method is its easy implementation: every iterative step includes only pointwise projections and solutions of the linear algebraic equations with the same matrices for all iterations.

1 Optimal control problem and its approximation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\Omega_1 \subseteq \Omega$ — its polygonal subdomain. Define arbitrary functions $y_d, u_d \in L_2(\Omega)$, and

$$\begin{gathered}\text{functions } y^*(x), u_1^*(x), u_2^*(x) \text{ from } C(\overline{\Omega}), \text{ such that} \\ y^*(x) > 0,\ u_1^*(x) < 0 < u_2^*(x) \text{ at } x \in \overline{\Omega}.\end{gathered} \tag{1}$$

Let state problem is the Dirichlet problem for the Poisson equation:

$$y \in H_0^1(\Omega) : \int\limits_\Omega \nabla y \cdot \nabla z \, dx = \int\limits_\Omega uz dx \quad \forall\, z \in H_0^1(\Omega), \tag{2}$$

where $u(x)$ is the control function and solution $y(x)$ of equation (2) is state of the system. Define the convex and closed sets of the constraints for control and state functions:

$$U_{ad} = \{u \in L_2(\Omega) : u_1^*(x) \leqslant u(x) \leqslant u_2^*(x) \text{ a.e. in } \Omega\},$$
$$Y_{ad} = \{y \in H_0^1(\Omega) : |\nabla y(x)| \leqslant y^*(x) \text{ a.e. in } \Omega\}.$$

Let $\alpha > 0$. Consider the following optimal control problem:

$$\min_{(y,u)\in K} \Big\{ J(y,u) = \frac{1}{2} \int\limits_{\Omega_1} (y - y_d)^2\, dx + \frac{\alpha}{2} \int\limits_{\Omega} (u - u_d)^2\, dx \Big\},$$
$$K = \{(y,u) :\ y \text{ is a solution of (2) and } y \in Y_{ad},\ \ u \in U_{ad}\}. \tag{3}$$

Lemma 1. *Problem* (3) *has a unique solution.*

Proof. Set K is a non-empty convex compact set in $H_0^1(\Omega) \times L_2(\Omega)$, containing zero function, and the functional J is continuous. Therefore the existence of the minimum point of J on the set K follows from Weierstrass theorem.

To prove the uniqueness of the solution, we prove that the functional J is strictly convex on the set K. In fact, let $(y_1, u_1) \neq (y_2, u_2)$ be two different elements of K. Then $u_1 \neq u_2$, because otherwise $y_1 = y_2$ according to the equation (2). Now, from the convexity of the functional in y and strict convexity in u follows inequality $J((y_1 + y_2)/2, (u_1 + u_2)/2) < 1/2 J(y_1, u_1) + 1/2 J(y_2, u_2)$, i.e. strict convexity of J on K.

Below we use the notation $\|\cdot\|_{0,p}$ for norms of Lebesgue spaces $L_p(\Omega)$ and $\|\cdot\|_{l,p}$ for norms of Sobolev spaces $W_p^l(\Omega)$ for $1 \leqslant p \leqslant \infty$ and integers $l > 0$.

Let $\mathcal{T}_h = \bigcup e_i$ be a conforming and regular triangulation of the domain Ω, h be the maximum diameter of elements $e \in \mathcal{T}_h$ ([23]). We assume that the triangulation is compatible with Ω_1 in the sense that $\overline{\Omega}_1$ consists of a number of triangles $e \in \mathcal{T}_{1h} \subseteq \mathcal{T}_h$. We define the finite element spaces

$$H_h = \{y_h \in H_0^1(\Omega) : y_h(x) \in P_1 \text{ on } e \in \mathcal{T}_h\},$$
$$U_h = \{u_h \in L_2(\Omega) : u_h(x) \in P_0 \text{ on } e \in \mathcal{T}_h\},$$

where P_k is the set of polynomials of degree at most k in all variables. We denote by π_h the operator of integral averaging of functions from $L_1(\Omega)$, with values in U_h:

$$\pi_h u(x) = |e_i|^{-1} \int\limits_{e_i} u(t)dt \ \text{ for } \ x \in e_i, \quad |e_i| = \text{meas } e_i.$$

Let $y_{dh} = \pi_h y_d, u_{dh} = \pi_h u_d, y_h^* = \pi_h y^*, u_{1h}^* = \pi_h u_1^*, u_{2h}^* = \pi_h u_2^*$. Then $y_h^*(x) > 0$, $u_{1h}^*(x) < 0 < u_{2h}^*(x)$. By the continuity in average of functions y_d and u_d the following limit relations hold:

$$\|y_{dh} - y_d\|_{0,2} \to 0, \quad \|u_{dh} - u_d\|_{0,2} \to 0,$$

and from uniform continuity of functions y^*, u_1^* and u_2^*, it follows that

$$\|y_h^* - y^*\|_{0,\infty} \to 0, \quad \|u_{1h}^* - u_1^*\|_{0,\infty} \to 0, \quad \|u_{2h}^* - u_2^*\|_{0,\infty} \to 0. \tag{4}$$

We define a convex and closed sets of the constraints on the mesh control and state functions:

$$Y_{ad}^h = \{y_h \in U_h : |\nabla y_h| \leqslant y_h^* \text{ on } \Omega\}, \quad U_{ad}^h = \{u_h \in U_h : u_{1h}^* \leqslant u_h \leqslant u_{2h}^* \text{ on } \Omega\}.$$

Discrete state problem is the approximation by the finite element method of the boundary value problem (2):

$$y_h \in H_h : \int\limits_{\Omega} \nabla y_h \cdot \nabla z_h \, dx = \int\limits_{\Omega} u_h z_h \, dx \quad \forall\, z_h \in H_h, \quad u_h \in U_h. \tag{5}$$

Objective function $J_h : H_h \times U_h \to \mathbb{R}$ is defined by the equality

$$J_h(y_h, u_h) = \frac{1}{2} \int\limits_{\Omega_1} (y_h - y_{dh})^2 \, dx + \frac{\alpha}{2} \int\limits_{\Omega} (u_h - u_{dh})^2 \, dx.$$

It is easy to verify that the discrete optimal control problem

$$\begin{gathered} \min_{(y_h,u_h)\in K_h} J_h(y_h, u_h), \\ K_h = \{(y_h, u_h) : \ y_h \text{ is a solution of (5) and } y_h \in Y_{ad}^h, \ u_h \in U_{ad}^h\} \end{gathered} \tag{6}$$

has a unique solution (y_h, u_h). The reasoning is the same as that for problem (3), namely, set K_h is a nonempty convex compact, and the function J_h is continuous and strictly convex on K_h.

2 Convergence of the discrete scheme

Let (y_h, u_h) be the solution of problem (6) for a fixed h while (y, u) be the solution of problem (3). We prove the strong convergence $(y_h, u_h) \to (y, u)$ as $h \to 0$ by using the traditional approach to the study of the convergence of discrete approximations for variational inequalities and minimization problems (see eg., [17], Chapter 1, §4.3, 4.4). This approach is based on the proving the approximation of K by the family of sets $\{K_h\}_h$ and functional J by the family of functions $\{J_h\}_h$.

The fact that sets K_h, defined in (6), approximate the set K, defined in (3), delivered in the following two lemmas.

Lemma 2. *If $\{(y_h, u_h)\} \in K_h$ and $(y_h, u_h) \to (y, u)$ weakly in $H_0^1(\Omega) \times L_2(\Omega)$, then $(y, u) \in K$.*

Proof. Let $\{(y_h, u_h)\} \in K_h$ and $(y_h, u_h) \to (y, u)$ weakly in $H_0^1(\Omega) \times L_2(\Omega)$. We first prove that (y, u) satisfies equation (2), and then that $(y, u) \in Y_{ad} \times U_{ad}$.

For any $z \in H_0^1(\Omega)$, take the sequence $\{z_h\} \in H_h$, which is strongly converges to z in $H_0^1(\Omega)$. Passing to the limit as $h \to 0$ in the equation (5), we get that (y, u) satisfies (2).

Take an arbitrary $\varepsilon > 0$ and consider the sets $Y_{ad}^{\varepsilon} = \{y \in L_2(\Omega) : |\nabla y(x)| \leqslant y^*(x) + \varepsilon$ a.e. in $\Omega\}$ and $U_{ad}^{\varepsilon} = \{u \in L_2(\Omega) : u_1^*(x) - \varepsilon \leqslant u(x) \leqslant u_2^*(x) + \varepsilon$ a.e. in $\Omega\}$. Due to the limit relations (4) it is obvious that $Y_{ad}^h \subset Y_{ad}^{\varepsilon}$ and $U_{ad}^h \subset U_{ad}^{\varepsilon}$ for sufficiently small $h \leqslant h(\varepsilon)$. Since the convex and closed sets Y_{ad}^{ε} and U_{ad}^{ε} are weakly closed, so $y \in Y_{ad}^{\varepsilon}$ and $u \in U_{ad}^{\varepsilon}$. It remains to note that $Y_{ad} = \bigcap\limits_{\varepsilon>0} Y_{ad}^{\varepsilon}$ and $U_{ad} = \bigcap\limits_{\varepsilon>0} U_{ad}^{\varepsilon}$ and $y \in Y_{ad}^{\varepsilon}$, $u \in U_{ad}^{\varepsilon}$ for all $\varepsilon > 0$, so $y \in Y_{ad}$ and $u \in U_{ad}$.

Lemma 3. *For every $(y, u) \in K$ there exists a sequence $\{(y_h, u_h)\} \in K_h$ such that $(y_h, u_h) \to (y, u)$ strongly in $H_0^1(\Omega) \times L_2(\Omega)$.*

Proof. The proof is divided into two parts. First we prove that any $(y, u) \in K$ is the limit of the functions $(y_n, u_n) \in K$, having additional smoothness, and such that $(y_n, u_n) \in \operatorname{int} Y_{ad} \times \operatorname{int} U_{ad}$. Then, for such functions we construct the sequence $\{(y_h, u_h)\} \in K_h$ converging to (y, u) strongly in $H_0^1(\Omega) \times L_2(\Omega)$.

In the proof of the first assertion we will use the following two facts ([24], [25]):

1. If $g \in L_\infty(\Omega)$, then the solution y of the boundary value problem
$$-\triangle y = g(x), \quad x \in \Omega, \quad y(x) = 0, \quad x \in \partial\Omega, \tag{7}$$
belongs to $W_p^2(\Omega)$ with some $p = 2 + \varepsilon$, $\varepsilon > 0$, i.e. $y \in W_\infty^1(\Omega)$, and the following estimate holds
$$\|y\|_{1,\infty} \leqslant c\,\|u\|_{2,p} \leqslant c\,\|g\|_{0,\infty}. \tag{8}$$

2. If $y \in W_p^2(\Omega)$ is the solution of (7) with $g \in L_\infty(\Omega)$, and y_h is the FEM solution
$$y_h \in H_h : \int\limits_{\Omega} \nabla y_h \cdot \nabla v_h \; dx = \int\limits_{\Omega} g v_h \; dx \quad \forall\; v_h \in H_h,$$
then
$$\|y - y_h\|_{1,\infty} \leqslant c\, h^{1-2/p}\, \|g\|_{0,\infty} \to 0 \quad \text{при } h \to 0. \tag{9}$$

By conditions (1) there exists a sufficiently small $\delta > 0$ such that
$$u_1^*(x) + 3\delta \leqslant 0, \quad u_2^*(x) - 3\delta \geqslant 0 \quad \text{and} \quad y^*(x) - 3\delta \geqslant 0 \ \text{ a.e. in } \ \Omega.$$

Take an arbitrary pair of $(y, u) \in K$ and let $(y_\rho, u_\rho) = (\rho\, y, \rho\, u)$ with $0 < \rho < 1$. Then $(y_\rho, u_\rho) \in K$ and $(y_\rho, u_\rho) \to (y, u)$ in $H_0^1(\Omega) \times L_2(\Omega)$ when $\rho \to 1 - 0$. We fix a maximum value ρ such that
$$u_\rho(x) \in [u_1^*(x) + 3\delta, u_2^*(x) - 3\delta] \quad \text{and} \quad |\nabla y_\rho(x)| \leqslant y^* - 3\delta \ \text{ a.e. in } \ \Omega.$$

We now take a sequence $u_n \in C^\infty(\overline{\Omega})$, which converges to u_ρ in $L_\infty(\Omega)$ and let y_n solution of (7) with right hand side u_n. These functions belong to $W_p^2(\Omega)$, $p = 2+\varepsilon$, $\varepsilon > 0$, and by virtue of (8) satisfy the relations:

$$\|y_n - y_\rho\|_{1,\infty} \leqslant c\,\|u_n - u\|_{0,\infty} \to 0 \quad \text{when } n \to \infty.$$

Therefore, there exists a number $n = n(\delta)$ such that holds the following inequalities $|\nabla y_n(x)| \leqslant y^*(x) - 2\delta$ and $u_n(x) \in [u_1^*(x) + 2\delta, u_2^*(x) - 2\delta]$ a.e. in Ω.

So, to prove the lemma it suffices to construct the sequence $\{(y_h, u_h)\} \in K_h$, which is converges to a pair $(y, u) \in K$ strongly in $H_0^1(\Omega) \times L_2(\Omega)$ such that

$$u(x) \in [u_1^*(x) + 2\delta, u_2^*(x) - 2\delta], \quad |\nabla y(x)| \leqslant y^*(x) - 2\delta \quad \text{a.e. in } \Omega,$$
$$u \in C(\overline{\Omega}), \quad y \in W_p^2(\Omega), \quad p = 2+\varepsilon,\ \varepsilon > 0.$$

Let $u_h = \pi_h u$. Then $u_h(x) \in [u_{1h}^*(x) + 2\delta, u_{2h}^*(x) - 2\delta]$ in Ω and $\|u_h - u\|_{0,\infty} \to 0,\ h \to 0$. Denote by $\tilde{y}_h$ solution of (7) with the right hand side u_h. Then the estimate (8) implies that $\|y - \tilde{y}_h\|_{1,\infty} \leqslant c\,\|u - u_h\|_{0,\infty} \to 0,\ h \to 0$. Thus, with $h \leqslant h_1(\delta)$ following inequalities holds:

$$|\nabla(y - \tilde{y}_h)(x)| \leqslant \delta \quad \Rightarrow \quad |\nabla \tilde{y}_h(x)| \leqslant y^*(x) - \delta \ \text{ in } \Omega.$$

Now let y_h is the FEM solution

$$y_h \in H_h : \int\limits_\Omega \nabla y_h \cdot \nabla v_h \; dx = \int\limits_\Omega u_h v_h \; dx \quad \forall\ v_h \in H_h.$$

In accordance with (9)

$$\|\tilde{y} - y_h\|_{1,\infty} \leqslant c\,h^{1-2/p}\,\|u_h\|_{0,\infty} \leqslant c\,h^{1-2/p} \to 0 \text{ when } h \to 0.$$

This means that the inequality

$$|\nabla(y_h - \tilde{y})(x)| \leqslant \delta \quad \Rightarrow \quad |\nabla y_h(x)| \leqslant y^*(x) \ \text{ in } \Omega$$

is true for $h \leqslant h_2(\delta)$. Thus, the pair (y_h, u_h) belongs to K_h for sufficiently small h and strongly converges to (y, u) in $H_0^1(\Omega) \times L_2(\Omega)$ when $h \to 0$.

Theorem 1. *Solutions $\{(y_h, u_h)\}$ of the problem* (6) *strongly converge to the solution (y, u) of* (3) *in $H_0^1(\Omega) \times L_2(\Omega)$ when $h \to 0$.*

Proof. a) *Weak convergence.* Let $(y_h, u_h) \in K_h$ be the solution of (6). Then (y_h, u_h) is bounded in $H_0^1(\Omega) \times L_2(\Omega)$ uniformly with respect to h. This allows to select from the sequence $\{(y_h, u_h)\}$ weakly converging in $H_0^1(\Omega) \times L_2(\Omega)$ subsequence. Keep for it the notation $\{(y_h, u_h)\}$. By Lemma 2 its limit (y, u) belongs to K. Further, since $y_h \to y$,

$y_{dh} \to y_d$, $u_{dh} \to u_d$ strongly and $u_h \to u$ weakly in $L_2(\Omega)$, and the quadratic functional $j(w) = \int_\Omega w^2\,dx$ is weakly lower semicontinuous, then

$$\liminf_{h\to 0} J_h(y_h, u_h) = \liminf_{h\to 0} \Big\{\frac{1}{2}\int\limits_{\Omega_1} (y_h - y_{dh})^2\,dx + \frac{\alpha}{2}\int\limits_{\Omega} (u_h - u_{dh})^2\,dx\Big\} \geqslant J(y,u).$$

Take an arbitrary $(z,w) \in K$. By Lemma 3 there exists $\{(z_h, w_h)\} \in K_h : (z_h, w_h) \to (z, w)$ strongly in $H_0^1(\Omega) \times L_2(\Omega)$, so $\lim\limits_{h\to 0} J_h(z_h, w_h) = J(z, w)$. As a result,

$$J(y,u) \leqslant \liminf_{h\to 0} J_h(y_h, u_h) \leqslant \lim_{h\to 0} J_h(z_h, w_h) = J(z,w) \quad \forall\,(z,w) \in K,$$

and (y, u) is the solution of problem (3). Since the solution (y, u) is unique, then the whole sequence $\{(y_h, u_h)\}$ of solutions to (6) weakly in $H_0^1(\Omega)\times L_2(\Omega)$ converges to (y, u). Indeed, suppose that $\{(y_h, u_h)\}$ does not converge weakly to (y, u). This means that there exists a subsequence $\{(y_{h_n}, u_{h_n})\}$, the number $\varepsilon_0 > 0$ and an element $(z, w) \in H_0^1(\Omega) \times L_2(\Omega)$ such that

$$\Big|\int\limits_{\Omega} \nabla(y_{h_n} - y)\cdot\nabla z\,dx + \int\limits_{\Omega} (u_{h_n} - u)\,w\,dx\Big| > \varepsilon_0 \quad \forall\, h. \tag{10}$$

But the subsequence $\{(y_{h_n}, u_{h_n})\}$ is bounded, so, in accordance with proven above, it contains a subsequence which weakly in $H_0^1(\Omega) \times L_2(\Omega)$ converges to (y, u). This contradicts to (10).

b) *Strong convergence* $u_h \to u$. First we prove, that $\lim\limits_{h\to 0} J(y_h, u_h) = J(y, u)$. Let a sequence $(z_h, w_h) \in K_h$ be such, that $(z_h, w_h) \to (y, u)$ strongly in $H_0^1(\Omega)\times L_2(\Omega)$ (lemma 3). Then

$$\limsup_{h\to 0} J(y_h, u_h) \leqslant \lim_{h\to 0} J(z_h, w_h) = J(y,u).$$

Together with the inequality $J(y, u) \leqslant \liminf\limits_{h\to 0} J(y_h, u_h)$ this gives $\lim\limits_{h\to 0} J(y_h, u_h) = J(y, u)$. Since y_h strongly in $L_2(\Omega)$ converge to y, then $\int_\Omega (y_h-y_{dh})^2\,dx \to \int_\Omega (y-y_d)^2\,dx$. From this and limit relation $J(y_h, u_h) \to J(y, u)$ it follows that $\int\limits_\Omega (u_h - u_{dh})^2\,dx \to \int\limits_\Omega (u - u_d)^2\,dx$. Together with weak in $L_2(\Omega)$ convergence of u_h to u, this implies a strong convergence u_h to u in $L_2(\Omega)$.

c) *Strong convergence* $y_h \to y$ *in* $H_0^1(\Omega)$. Take a sequence $\{\tilde{y}_h\}$: $\tilde{y}_h \to y$ strongly in $H_0^1(\Omega)$ and use the state equations (5) and (2) and Friedrichs inequality

$$\int\limits_\Omega y_h^2\,dx \leqslant c_f^2 \int\limits_\Omega |\nabla y_h|^2\,dx \ \ \forall\, y_h \in H_h. \tag{11}$$

We obtain:

$$\int_\Omega |\nabla(y_h - \tilde y_h)|^2\,dx = \int_\Omega u_h(y_h - \tilde y_h)\,dx - \int_\Omega \nabla\tilde y_h\cdot\nabla(y_h - \tilde y_h)\,dx =$$
$$= \int_\Omega \nabla(y - \tilde y_h)\cdot\nabla(y_h - \tilde y_h)\,dx + \int_\Omega (u_h - u)(y_h - \tilde y_h)\,dx \leqslant$$
$$\leqslant \Big(\int_\Omega |\nabla(y_h - \tilde y_h)|^2\,dx\Big)^{1/2}\Big(\Big(\int_\Omega |\nabla(y - \tilde y_h)|^2\,dx\Big)^{1/2} + c_f\|u - u_h\|_{L_2(\Omega)}\Big).$$

Hence it follows $y_h - \tilde y_h \to 0$ in $H_0^1(\Omega)$ and, so, $y_h \to y$ in $H_0^1(\Omega)$.

3 Discrete saddle point problem

We introduce an auxiliary function $\bar p_h = \nabla y_h \in U_h \times U_h$ and define a set of constraints

$$P_{ad}^h = \{\bar p_h \in U_h \times U_h : |\bar p_h(x)| \leqslant y_h^*(x) \text{ a.e. in } \Omega\}.$$

Now the problem (6) can be written as

$$\min_{(y_h,u_h,\bar p_h)\in W_h}\Big\{J_h(y_h,u_h) = \frac12\int_{\Omega_1}(y_h - y_{dh})^2\,dx + \frac{\alpha}{2}\int_\Omega (u_h - u_{dh})^2\,dx\Big\}, \tag{12}$$
$$W_h = \{(y_h,u_h,\bar p_h):\ \bar p_h \in P_{ad}^h,\ u_h \in U_{ad}^h,\ \bar p_h = \nabla y_h,\ y_h \text{ is a solution of } (5)\}.$$

Define the corresponding Lagrangian function by the equality

$$\mathcal{L}_h(y_h,u_h,\bar p_h,\lambda_h,\bar\mu_h) = J_h(y_h,u_h) + \int_\Omega \nabla y_h\cdot\nabla\lambda_h\,dx -$$
$$- \int_\Omega u_h\lambda_h\,dx + \int_\Omega \bar\mu_h(\nabla y_h - \bar p_h)\,dx, \tag{13}$$

where the Lagrange multipliers $\lambda_h \in H_h,\ \bar\mu_h \in U_h \times U_h$, and the saddle point are looking under constraints on direct variables $\bar p_h \in P_{ad}^h, u_h \in U_{ad}^h$.

For further formulation the saddle point problem in algebraic form we assign to the functions of the finite element spaces H_h and U_h the vectors of their nodal parameters. Let $\omega_h = \{t_i\}_{i=1}^m$ be the set of vertices of triangles $e \in T_h$, lying in Ω, $m = \operatorname{card}\omega_h$, $\xi_h = \{t_i\}_{i=1}^s$ be the set of barycenters of the triangles $e \in T_h$. Put in correspondence function $y_h \in H_h$ and vector $y \in \mathbb{R}^m$ with coordinates $y_i = y_h(t_i)$, $t_i \in \omega_h$ (with any node numbering t_i), and the functions $u_h \in U_h$ — vector $u \in \mathbb{R}^s$ with coordinates $u_i = u_h(t_i)$, $t_i \in \xi_h$. We will use the notation $y \Leftrightarrow y_h$, $u \Leftrightarrow u_h$.

Further through $y_d, u_d, y^*, u_1^*, u_2^*$ we denote vectors of nodal parameters of the corresponding mesh functions.

Define the matrices $L \in \mathbb{R}^{m\times m}$, $M_u \in \mathbb{R}^{s\times s}$, $M_y \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{s\times m}$, $R_i \in \mathbb{R}^{m\times s}$ $(i = 1,2)$ by the equalities:

$$(Ly,z) = \int\limits_{\Omega} \nabla y_h \cdot \nabla z_h \, dx, \quad (M_u u, v) = \int\limits_{\Omega} u_h(x) v_h(x) \, dx, \quad (M_y y, z) = \int\limits_{\Omega_1} y_h z_h \, dx,$$

$$(R_i y, v) = \int\limits_{\Omega} \frac{\partial y_h}{\partial x_i}(x) v_h(x) \, dx, \quad (Su, y) = \int\limits_{\Omega} u_h(x) y_h(x) \, dx, \quad (S_1 u, y) = \int\limits_{\Omega_1} u_h(x) y_h(x) \, dx.$$

These equalities must be satisfied for all $y, z \in \mathbb{R}^m$ and $u, v \in \mathbb{R}^s$. Here $y_h, z_h \in H_h$, $y_h \Leftrightarrow u, z_h \Leftrightarrow z$ and, respectively, $u_h, v_h \in U_h, u_h \Leftrightarrow u, v_h \Leftrightarrow v$. By construction, M_u is a diagonal positive definite matrix.

We use the notations $\overline{M}_u = \operatorname{diag}(M_u, M_u)$, $R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$. From the definitions of the matrices and the Friedrichs inequality (11) follows:

$$\begin{gathered} (M_y y, y) \leqslant c_f^2 (Ly, y), \quad (Su, y) \leqslant c_f (M_u u, u)^{1/2} (Ly, y)^{1/2}, \\ (Ry, \bar{p}) \leqslant (Ly, y)^{1/2} (\overline{M}_u \bar{p}, \bar{p})^{1/2}. \end{gathered} \tag{14}$$

Lagrange function (13) and a sets of constraints in terms of vectors of nodal parameters of mesh functions take the form:

$$\mathcal{L}(y, u, \bar{p}, \lambda, \bar{\mu}) = \frac{1}{2}\,(M_y y, y) + (S_1 y_d, y) + \frac{\alpha}{2}\,(M_u(u - u_d), u - u_d) + \\ + (Ly - Su, \lambda) + (Ry - \overline{M}_u \bar{p}, \bar{\mu}),$$

$$P_{ad} = \{\bar{p} \in \mathbb{R}^s \times \mathbb{R}^s : \, p_{1j}^2 + p_{2j}^2 \leqslant y_j^{*2} \text{ for all } j = 1, 2, \ldots, s\},$$

$$U_{ad} = \{u \in \mathbb{R}^s : \, u_i \in [u_{1i}^*, u_{2i}^*] \text{ for all } i = 1, 2, \ldots, m\}.$$

Let $\varphi_p(\bar{p})$ and $\varphi_u(u)$ be the indicator functions of the sets P_{ad} and U_{ad}. Then the corresponding saddle point problem is

$$\begin{pmatrix} M_y & 0 & 0 & L & R^T \\ 0 & 0 & 0 & 0 & -\overline{M}_u \\ 0 & 0 & \alpha M_u & -S^T & 0 \\ L & 0 & -S & 0 & 0 \\ R & -\overline{M}_u & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \bar{p} \\ u \\ \lambda \\ \bar{\mu} \end{pmatrix} + \begin{pmatrix} -S_1 y_d \\ \partial\varphi_p(\bar{p}) \\ \partial\varphi_u(u) - M_u u_d \\ 0 \\ 0 \end{pmatrix} \ni 0. \tag{15}$$

The submatrix

$$\begin{pmatrix} M_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha M_u \end{pmatrix}$$

of this problem is only positive semidefinite. In order to convert (15) to an equivalent saddle point problem with a positive definite submatrix we use both equation $Ly = Su$

and $Ry = \overline{M}_u \bar{p}$. Obvious transformations of the first two relations in (15) lead to the system

$$\begin{pmatrix} M_y + rL & 0 & -rS & L & R^T \\ -rR & r\overline{M}_u & 0 & 0 & -\overline{M}_u \\ 0 & 0 & \alpha M_u & -S^T & 0 \\ L & 0 & -S & 0 & 0 \\ R & -\overline{M}_u & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \bar{p} \\ u \\ \lambda \\ \bar{\mu} \end{pmatrix} + \begin{pmatrix} -S_1 y_d \\ \partial\varphi_p(\bar{p}) \\ \partial\varphi_u(u) - M_u u_d \\ 0 \\ 0 \end{pmatrix} \ni 0. \tag{16}$$

Further we consider following scalar matrices:

$$K_r = \begin{pmatrix} r & -0.5r & -0.5rc_f \\ -0.5r & r & 0 \\ -0.5rc_f & 0 & \alpha \end{pmatrix}, \quad \overline{K}_r = \begin{pmatrix} r + c_f^2 & 0.5r & 0.5rc_f \\ 0.5r & r & 0 \\ 0.5rc_f & 0 & \alpha \end{pmatrix}.$$

Lemma 4. *Let $0 < r < 3\alpha/c_f^2$. Then the matrices*

$$A = \begin{pmatrix} M_y + rL & 0 & -rS \\ -rR & r\overline{M}_u & 0 \\ 0 & 0 & \alpha M_u \end{pmatrix} \quad \textit{and} \quad A_0 = \begin{pmatrix} L & 0 & 0 \\ 0 & \overline{M}_u & 0 \\ 0 & 0 & M_u \end{pmatrix}$$

are spectrally equivalent, i.e.

$$m(r)(A_0 x, x) \leqslant (Ax, x) \leqslant M(r)(A_0 x, x) \quad \forall\, x = (y, \bar{p}, u)^T.$$

Here $m(r) > 0$ is the minimum eigenvalue of K_r, and $M(r)$ is the maximum eigenvalue of $\overline{K}_r$.

Proof. First, we note that due to Sylvester criterion condition $0 < r < 3\alpha/c_f^2$ provides positive definiteness of the matrix K_r. Next, using the estimates (14), for any vector $x = (y, \bar{p}, u)^T$ we obtain:

$$\begin{aligned}(Ax, x) = r(Ly, y) + (M_y y, y) + r(\overline{M}_u \bar{p}, \bar{p}) + \alpha(M_u u, u) - r(Su, y) - r(Ry, \bar{p}) \geqslant \\ \geqslant r(Ly, y) + r(\overline{M}_u \bar{p}, \bar{p}) + \alpha(M_u u, u) - r\, c_f (M_u u, u)^{1/2} (Ly, y)^{1/2} - \\ - r(Ly, y)^{1/2} (\overline{M}_u \bar{p}, \bar{p})^{1/2} \geqslant m(r)\big((Lu, u) + (\overline{M}_u \bar{p}, \bar{p}) + (M_u u, u)\big) = m(r)(A_0 x, x).\end{aligned}$$

Similarly we can prove the second inequality.

Introduce the notations:

$$A = \begin{pmatrix} M_y + rL & 0 & -rS \\ -rR & r\overline{M}_u & 0 \\ 0 & 0 & \alpha M_u \end{pmatrix}, \quad B = \begin{pmatrix} L & 0 & -S \\ R & -\overline{M}_u & 0 \end{pmatrix},$$

$$x = (y, \bar{p}, u)^T, \quad \eta = (\lambda, \quad \bar{\mu})^T, \quad f = (M_y y_d, 0, M_u u_d)^T, \quad \varphi(x) = \varphi_u(u) + \varphi_p(\bar{p}).$$

Then the problem (16) can be written as

$$\begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} \partial\varphi(x) \\ 0 \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{17}$$

We assume that the parameter r is chosen so that $0 < r < 3\alpha/c_f^2$. Then the matrix A is positive definite. In its turn, the matrix B has full column rank, since its block $\begin{pmatrix} L & 0 \\ R & -\overline{M}_u \end{pmatrix}$ is a nonsingular matrix. Vector with coordinates $u = 0, \bar{p} = 0, y = 0$ belongs to interior of the constraint sets, as well as to the kernel of matrix B. Thus, all the assumptions of Lemma 1 from [18] are fulfilled, and this implies the existence of a solution $(y, \bar{p}, u, \lambda, \bar{\mu})$ to problem (16) with unique $(y, \bar{p}, u)$ (the components $\eta = (\lambda, \mu)$ of the solution are not uniquely defined). Corresponding to the vector $(y, \bar{p}, u)$ mesh function $(y_h, \bar{p}_h, u_h)$ coincides with the solution of the discrete optimal control problem (12).

4 Preconditioned Uzawa-type iterative method

From the system (17) we obtain the equation for $\eta = (\lambda, \bar{\mu})^T$:

$$B(A + \partial\varphi)^{-1}(B^T\eta + f) = 0$$

To solve it we apply one-step iterative method

$$\frac{1}{\tau} D(\eta^{k+1} - \eta^k) + B(A + \partial\varphi)^{-1}(B^T\eta^k + f) = 0 \tag{18}$$

with a symmetric and positive definite matrix D. This is preconditioned Uzawa method for solving (17). By Theorem 1 from [18], it converges from any initial approximation η^0, if the following condition holds for the pair preconditioner D — iteration parameter τ:

$$(D\eta, \eta) > \frac{\tau}{2} (BA_s^{-1}B^T\eta, \eta) \quad \forall\, \eta \neq 0, \tag{19}$$

where $A_s = 0.5(A + A^T)$ is symmetric part of the matrix A. Moreover the sequence $\{\eta^k\}_k$ converges to some vector η^* from the set of solutions in the energy norm of the matrix D:

$$\|\eta^k - \eta^*\|_D \to 0 \ \text{ при } \ k \to \infty.$$

Since, in general there are no estimates of the rate of convergence of the method (18), then it makes sense to choose a preconditioner assuming that the problem is solved without constraints, i.e. $\partial\varphi = 0$. In this case, the optimal preconditioner matrix is spectrally equivalent to matrix $(BA^{-1}B^T)_s = BA_s^{-1}B^T$.

By Lemma 4 matrix A_s is spectrally equivalent to the matrix

$$A_0 = \begin{pmatrix} L & 0 & 0 \\ 0 & \overline{M}_u & 0 \\ 0 & 0 & M_u \end{pmatrix}.$$

Direct calculations yield

$$BA_0^{-1}B^T = \begin{pmatrix} L + SM_u^{-1}S^T & R^T \\ R & RL^{-1}R^T + \overline{M}_u \end{pmatrix}.$$

Matrix $BA_0^{-1}B^T$ is spectrally equivalent to $BA_s^{-1}B^T$, therefore it can be taken as a preconditioner in the Uzawa method. However, in this case, on each step of the iterative method it is necessary to solve the coupled system of equations for λ and $\bar{\mu}$. A more efficient method is to implement a block-diagonal preconditioner, which can be taken thanks to the following lemma.

Lemma 5. *Let* $D = \begin{pmatrix} L & 0 \\ 0 & \overline{M}_u \end{pmatrix}$. *Then*

$$\frac{3-\sqrt{5}}{2}\,(D\eta,\eta) \leqslant (BA_0^{-1}B^T\eta,\eta) \leqslant \max\{2+c_f^2, 3\}\,(D\eta,\eta) \quad \forall\, \eta = (\lambda,\bar{\mu})^T.$$

Proof. Straightforward calculations lead to the equality

$$(BA_0^{-1}B^T\eta,\eta) = (L\lambda,\lambda) + (M_u^{-1}S^T\lambda, S^T\lambda) + (\overline{M}_u\bar{\mu},\bar{\mu}) + (RL^{-1}R^T\bar{\mu},\bar{\mu}) + 2(R\lambda,\bar{\mu}).$$

To estimate the right-hand side we use the following inequalities (hereinafter $y \Leftrightarrow y_h$, $u \Leftrightarrow u_h, \bar{p} \Leftrightarrow \bar{p}_h$):

$$(L^{-1}R^T\bar{p}, R^T\bar{p})^{1/2} = \sup_{y_h\in H_h} \frac{\int\limits_\Omega \nabla y_h\cdot\bar{p}_h\,dx}{\Big(\int\limits_\Omega |\nabla y_h|^2\,dx\Big)^{1/2}} \leqslant \Big(\int\limits_\Omega |\bar{p}_h|^2\,dx\Big)^{1/2} = (\overline{M}_u\bar{p},\bar{p})^{1/2},$$

$$(M_u^{-1}S^Ty, S^Ty)^{1/2} = \sup_{u_h\in U_h} \frac{\int\limits_\Omega y_h u_h\,dx}{\Big(\int\limits_\Omega u_h^2\,dx\Big)^{1/2}} \leqslant \Big(\int\limits_\Omega y_h^2\,dx\Big)^{1/2} \leqslant c_f(Ly,y)^{1/2}.$$

Using the first auxiliary inequality, we obtain the lower bound

$$\begin{aligned}(BA_0^{-1}B^T\eta,\eta) \geqslant (L\lambda,\lambda) + (\overline{M}_u\bar{\mu},\bar{\mu}) + (L^{-1}R^T\bar{\mu}, R^T\bar{\mu}) - 2(L^{-1}R^T\bar{\mu}, R^T\bar{\mu})^{1/2}(L\lambda,\lambda)^{1/2} \geqslant \\ \geqslant (1-\varepsilon)(L\lambda,\lambda) + (1-\varepsilon)(\overline{M}_u\bar{\mu},\bar{\mu}) + (1+\varepsilon-1/\varepsilon)(L^{-1}R^T\bar{\mu}, R^T\bar{\mu}).\end{aligned}$$

Let now ε be the positive solution of the equation $1+\varepsilon-1/\varepsilon = 0$, then

$$(BA_0^{-1}B^T\eta,\eta) \geqslant \frac{3-\sqrt{5}}{2}\,\big((L\lambda,\lambda) + (\overline{M}_u\bar{\mu},\bar{\mu})\big).$$

To obtain upper bound we use both auxiliary inequalities:

$$\begin{aligned}(BA_0^{-1}B^T\eta,\eta) \leqslant (1+c_f^2)(L\lambda,\lambda) + 2(\overline{M}_u\bar{\mu},\bar{\mu}) + 2(L\lambda,\lambda)^{1/2}(\overline{M}_u\bar{\mu},\bar{\mu})^{1/2} \leqslant \\ \leqslant (2+c_f^2)(L\lambda,\lambda) + 3(\overline{M}_u\bar{\mu},\bar{\mu}).\end{aligned}$$

The results of Lemmas 4 and 5 ensure the spectral equivalence of matrices $BA_s^{-1}B^T$ и D:

$$c_{\min} D \leqslant BA_s^{-1}B^T \leqslant c_{\max} D,$$

where $c_{\min} = (3-\sqrt{5})/(2M(r))$ and $c_{\max} = \max\{2+c_f^2, 3\}/m(r)$, and the constants $m(r)$ and $M(r)$ are defined in Lemma 4.

Theorem 2. *Let* $0 < r < 3\alpha/c_f^2$ *and* $m(r) > 0$ *be minimal eigenvalue of* K_r*, defined in Lemma 4. Then Uzawa method* (18) *for problem* (16) *converges if*

$$0 < \tau < \frac{2m(r)}{\max\{2+c_f^2, 3\}}. \tag{20}$$

Proof. As noted above, it suffices to prove the inequality (19). But from Lemma 5 it follows that

$$BA_s^{-1}B^T \leqslant m(r)^{-1} BA_0^{-1}B^T \leqslant \max\{2+c_f^2, 3\} m(r)^{-1} D,$$

so $D > \tau/2 \, BA_s^{-1}B^T$ due to (20).

4.1 Implementation of the Preconditioned Uzawa method

It is easy to see that one iteration of method (18) reduces to implementation of the following calculations for the known λ^k и $\bar{\mu}^k$:

1. $u^{k+1} = (\alpha M_u + \partial\varphi_u)^{-1}(S^T\lambda^k + M_u u_d) = Pr_{U_{ad}}(\alpha^{-1}M_u^{-1}(S^T\lambda^k + M_u u_d)$;
2. $y^{k+1} = (M_y + rL)^{-1}(S_1 y_d + rSu^{k+1} - L\lambda^k - R^T\bar{\mu}^k)$;
3. $\bar{p}^{k+1} = (r\overline{M}_u + \partial\varphi_p)^{-1}(\overline{M}_u\bar{\mu}^k + rRy^{k+1}) = Pr_{P_{ad}}(r^{-1}\bar{\mu}^k + \overline{M}_u^{-1}Ry^{k+1})$;
4. $\lambda^{k+1} = \lambda^k + \tau(y^{k+1} - L^{-1}Su^{k+1})$;
5. $\bar{\mu}^{k+1} = \bar{\mu}^k + \tau(\overline{M}_u^{-1}Ry^{k+1} - \bar{p}^{k+1})$.

By virtue of diagonality of the matrices M_u and $\overline{M}_u = \operatorname{diag}(M_u, M_u)$ and pointwise constraints for $u \in U_{ad}$ and $\bar{p} \in P_{ad}$ the determination of u^{k+1} and $\bar{p}^{k+1}$ reduces to the pointwise projections of known vectors to the corresponding sets of constraints. More precisely, for a fixed i:

$$u_i^{k+1} = Pr_{[-u_{1i}^*, u_{2i}^*]}\Big(\frac{1}{\alpha m_{ii}}\,(S^T\lambda^k + M_u u_d)_i\Big),$$

where m_{ii} is a diagonal element M_u, and

$$|\bar{p}_i^{k+1}| = Pr_{[0,y_i^*]}|\bar{F}|, \;\; p_{i1}^{k+1} = |\bar{p}_i^{k+1}|^{-1}F_1, \; p_{i2}^{k+1} = |\bar{p}_i^{k+1}|^{-1}F_2,$$

where $\bar{F} = (F_1, F_2) = (r^{-1}\bar{\mu}^k + \overline{M}_u^{-1}Ry^{k+1})_i$.

4.2 Control of accuracy and stopping criterion

When the saddle point problem (17) is solved by any iterative method, we find not only an approximation of (x^k, η^k) to the exact solution (x, η), but also the vector $\gamma^k \in \partial\varphi(x^k)$ — the unique selection from the set $\partial\varphi(x^k)$. We define the components of the residual vector by the equalities

$$r_x^k = f - Ax^k - \gamma^k + B^T\eta^k, \quad r_\eta^k = -Bx^k.$$

Then the error vector $(x - x^k, \eta - \eta^k)^T$ satisfies the system

$$\begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x - x^k \\ \eta - \eta^k \end{pmatrix} + \begin{pmatrix} \partial\varphi(x) - \gamma^k \\ 0 \end{pmatrix} \ni \begin{pmatrix} r_x^k \\ r_\eta^k \end{pmatrix}.$$

Multiplying this system scalarly by the vector $(x - x^k, \eta - \eta^k)^T$ and applying the inequality $(\partial\varphi(x) - \partial\varphi(x^k), x - x^k) \geqslant 0$, we get

$$(A(x - x^k), x - x^k) \leqslant (r_x^k, x - x^k) + (r_\eta^k, \eta - \eta^k).$$

Hence

$$\|x - x^k\|_{A_s}^2 \leqslant \|r_x^k\|_{A_s^{-1}} \|x - x^k\|_{A_s} + |(r_\eta^k, \eta - \eta^k)|. \tag{21}$$

Since the inclusion $Ax - B^T\eta + \partial\varphi(x) \ni f$ is solved exactly at each iteration of Uzawa method (18), therefore $r_x^k = 0 \; \forall\, k$, and estimate (21) takes the form

$$\|x - x^k\|_{A_s} \leqslant |(r_\eta^k, \eta - \eta^k)| \leqslant \|\eta - \eta^k\|_D^{1/2} \|r_\eta^k\|_{D^{-1}}^{1/2} \quad \forall\, k, \tag{22}$$

where D is the preconditioner of this method. Since $\|\eta - \eta^k\|_D \to 0$ for $k \to \infty$, inequality (22) gives the information about error $\|x - x^k\|_{A_s}$ through the estimate of the norm of the residual component $\|r_\eta^k\|_{D^{-1}}$, namely,

$$\|x - x^k\|_{A_s} = o(\|r_\eta^k\|_{D^{-1}}^{1/2}) \quad \text{when} \quad k \to \infty.$$

In the problem (16) vector $r_\eta^k = (Ly^k - Su^k, Ry^k - \overline{M}_u \bar{p}^k)$, so the upper bound for number of iterations is the value

$$\delta^k = \|r_\eta^k\|_{D^{-1}}^{1/2} = \big((Ly^k - Su^k, y^k - L^{-1}Su^k) + (Ry^k - \overline{M}_u\bar{p}^k, \overline{M}_u^{-1} Ry^k - \bar{p}^k)\big)^{1/2}.$$

Note that the vectors

$$Ly^k - Su^k, \quad y^k - L^{-1}Su^k = (\lambda^k - \lambda^{k-1})/\tau, \quad Ry^k - \overline{M}_u\bar{p}^k, \quad M_u^{-1}Ry^k - \bar{p}^k = (\bar{\mu}^{k+1} - \bar{\mu}^k)/\tau,$$

are computed when implementing the algorithm, thus, control of the value δ^k does not lead to additional computational cost.

Remark 1. *Discrete objective function can be constructed by using approximations of the integrals by composite quadrature formulas, for example, on the basis of one-point quadrature formulae with the node coinciding with the barycenter a_e of the triangle $e \in \mathcal{T}_h$:*

$$\int_e g(x)\,dx \approx S_e(g) = \mathrm{meas}\,e\, g(a_e), \quad \int_{\Omega_1} g(x)\,dx \approx S_{\Omega_1}(g) = \sum_{e \in \bar{\Omega}_1 \cap \mathcal{T}_h} S_e(g).$$

In this case, the objective function $J_h : H_h \times U_h \to \mathbb{R}$ defined by

$$J_h(y_h, u_h) = \frac{1}{2}\, S_{\Omega_1}((y_h - y_{dh})^2) + \frac{\alpha}{2} \int_\Omega (u_h - u_{dh})^2\,dx.$$

Discrete optimal control problem

$$\min_{(y_h,u_h)\in K_h} J_h(y_h, u_h),$$
$$K_h = \{(y_h, u_h) : \ y_h \text{ is a solution of } (5) \text{ and } y_h \in Y^h_{ad},\ u_h \in U^h_{ad}\}$$

has a unique solution (y_h, u_h). Results about the convergence of discrete scheme and Uzawa iterative method remain in force. Moreover bounds on parameters r and τ are the same as in preceding case.

References

[1] *R. Dautov, R. Kadyrov, E. Laitinen, A. Lapin, J. Pieskä and V. Toivonen.* On 3D dynamic control of secondary cooling in continuous casting process// Lobachevskii J. Math. - 2003. - V13. - P. 3-13.

[2] *E. Laitinen, A. Lapin, J. Pieskä.* Asynchronous domain decomposition methods for continuous casting problem// J. Comp. Appl. Math. - 2003. - V.154. - P. 393-413.

[3] *J. Pieskä, E. Laitinen, A. Lapin.* Predictor-corrector methods for solving continuous casting problem// Lecture Notes in Computational Science and Engineering 40, Berlin: Springer - 2005. - P. 677-684.

[4] *M. Bergounioux, V. Haddou, M. Hintermuller, K. Kunisch* A comparison of a Moreau-Yosida-based active set strategy and interior point methods for constrained optimal control problems // SIAM J. Optim. - V. 11, No.2. - 2000. - P. 495-521.

[5] *M. Bergounioux, K. Kunisch* Primal-dual active set strategy for state-constrained optimal control problems // Comput. Optim. Appl. - V. 22, No.2. - 2002. - P. 193-224.

[6] *F. Troltzsch, U. Pufert, M. Weiser* The convergence of an interior point method for an elliptic control problem with mixed control-state constraints// Comput. Optim. Appl. - V.39 - 2008. - P. 183-218.

[7] *F. Troltzsch, I. Yousept* A regularization method for the numerical solution of elliptic boundary control problems with pointwise state constraints // Comput. Optim. Appl. - V. 42, No.1. - 2009. - P. 43-66.

[8] *M. Hintermuller, M. Hinze* Moreau-Yosida regularization in state constrained elliptic control problems: error estimates and parameter adjustment.// SIAM J. Numer. Anal. - V. 47, No. 3. - 2009. - P. 1666-1683.

[9] *M. Hinze, A. Schiela* Discretization of interior point methods for state constrained elliptic optimal control problems: Optimal error estimates and parameter adjustment// Comput. Optim. Appl. - V. 48, No. 3. - 2011. - P. 581-600.

[10] *E. Casas, L. Fernandez* Optimal control of semilinear elliptic equations with pointwise constraints on the gradient of the state// Appl. Math. Optimization. - V. 27. - 1993. - P. 35-56.

[11] *K. Deckelnick, A. Gunther, M. Hinze* Finite element approximation of elliptic control problems with constraints on the gradient// Numer. Math. - V.111.- 2009. - P.335-350.

[12] *A. Gunther, M. Hinze* Elliptic Control Problems with Gradient Constraints - Variational Discrete Versus Piecewise Constant Controls// Comput. Optim. Appl., DOI: 10.1007/s10589-009-

[13] *C. Ortner, W. Wollner* A priori error estimates for optimal control problems with pointwise constraints on the gradient of the state// Numer. Math. - V.118. - 2011. - P. 587-600.

[14] *R. Griesse, K. Kunisch* A semi-smooth newton method for solving elliptic equations with gradient constraints// ESAIM Math. Model. Numer. Anal. - V.43. - 2009. - P. 209-238.

[15] *R. A. Schiela, W. Wollner* Barrier methods for optimal control problems with convex nonlinear gradient constraints// SIAM J. Optim. - V. 21, No 1. - 2011. - P. 269-286.

[16] *M. Hintermuller, M. Hinze, R. Hoppe* Weak-duality based adaptive finite element methods for PDE-constrained optimization with pointwise gradient state-constraints// J. of Computational Mathematics. - V.30, No.2. - 2012. - P. 101 - 123.

[17] *R. Glowinski, J.-L. Lions, R. Tremolieres* Analyse nume'rique des ine'quations variationnelles – Paris: Dunod. – 1976.

[18] *A. Lapin* Preconditioned Uzawa type methods for finite-dimensional constrained saddle point problems // Lobachevskii J. Math. - V. 31, No 4. - 2010. - P. 309-322.

[19] *Lapin A., Khasanov M.* State-constrained optimal control of an elliptic equation with its right-hand side used as control function// Lobachevskii J. Math. - 2011. - V. 32. No. 4. - P. 453-462.

[20] *Laitinen E., Lapin A., Lapin S.* Iterative solution methods for variational inequalities with nonlinear main operator and constraints to gradient of solution// Lobachevskii Journal of Mathematics. - 2012. - V.33, No 4 .- P. 364-371.

[21] *Laitinen E., Lapin A.* Iterative solution methods for large-scale constrained saddle-point problems// In: "Numerical Methods for Differential Equations, Optimization and Technological Problems", Comp. Meth. Appl. Sc., 27., Springer. - 2013. - P. 19-39.

[22] *Lapin A.* Iterative Methods for Solving Grid Variational Inequalities (Kazan. Gos. Univ., Kazan, 2008) [in Russian]

[23] *Ph. G. Ciarlet* The finite element method for elliptic problems. - North Holland, Amsterdam. - 1978.

[24] *R. Rannacher, R. Scott* Some Optimal Error Estimates for Piecewise Linear Finite Element Approximations // Mathematics of Computation. - V. 38, No. 158. - 1982. -P. 437-445.

[25] *R. Nochetto* Pointwise Accuracy of a Finite Element Method for Nonlinear Variational Inequalities// Numer. Math. - V. 54. -1989. - P. 601-618.