УДК 517.925

A RELATIONSHIP BETWEEN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF A SPECIAL TYPE AND MODIFIED BESSEL FUNCTIONS

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СВЯЗЬ ОБОБЩЕННЫХ ГИПЕРГЕОМЕТРИЧЕСКИХ ФУНКЦИЙ СПЕЦИАЛЬНОГО ВИДА И МОДИФИЦИРОВАННЫХ ФУНКЦИЙ БЕССЕЛЯ

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Summary

A relationship between generalized hypergeometric functions of a special type and modified Bessel functions has been established. Using this relationship the solution of inhomogeneous differential equations of Bessel type containing even degrees of an independent variable in the right-hand part can be expressed in a form convenient for engineering and technical applications.

Key words: hypergeometric functions, modified Bessel functions, differential equations of Bessel type.

Аннотация

Установлена связь обобщенных гипергеометрических функций специального вида и модифицированных функций Бесселя, которая позволяет записать решение неоднородных дифференциальных уравнений бесселевского типа с правой частью, содержащей чётные степени независимой переменной, в удобном для инженерных и технических приложений виде.

Ключевые слова: гипергеометрические функции, модифицированные функции Бесселя, дифференциальные уравнения типа Бесселя.

Introduction

Within the frames of a relatively large class of problems [1-5] on restoration of the structure of quasi-stationary electromagnetic and temperature fields of a high-frequency induction discharge from measured values of one of its components, a necessity arises to obtain exact analytical solutions of differential equations of the form

$$y''(x) + \frac{y'(x)}{x} - Cy(x) = x^{2n} \tag{1}$$

(here $C = b^2 = \text{const}$ is a nonnegative constant depending on boundary conditions), to which MaxwellтАЩs equations for electromagnetic field are reduced in the general case. In the case of $C = 0$ (corresponding to one-dimensional statement of the problem from physical point of view), a particular solution of Eq. (1), apparently, reduces to a parabolic solution

$$y(x) = \frac{1}{4(n+1)^2}\, x^{2n+2}. \tag{2}$$

Let us introduce new variables $\tilde{x} = \sqrt{C}x = bx$, in which the initial equation will be rewritten as

$$y''(\tilde{x}) + \frac{y'(\tilde{x})}{\tilde{x}} - y(\tilde{x}) = \frac{\tilde{x}^{2n}}{b^{2(n+1)}}. \tag{3}$$

It can be easily seen that written in such a form equation (3) is one of the variants of inhomogeneous Bessel equations for unknown function $\Psi(\tilde{x})$, the solution of which is given, for instance, in Ref. [6].

$$\Psi(\tilde{x}) = \frac{(-1)^{n+1}}{b^{2(n+1)}}\, s_{2n+1,0}\left(\tilde{x}e^{i\frac{\pi}{2}}\right). \tag{4}$$

where $s_{\mu,\nu}(x)$ is a so-called Lommel function usually defined as [7]

$$s_{\mu,\nu}(x) = \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1;\frac{\mu-\nu+3}{2};\frac{\mu+\nu+3}{2};-\frac{x^2}{4}\right) \tag{5}$$

where ${}_1F_2$ is a generalized hypergeometric function [8]. So taking account for Eq. (6), Eq. (4) can be rewritten as follows

$$\Psi(\tilde{x}) = \frac{1}{\left[2(n+1)b^{n+1}\right]^2}\,\tilde{x}^{2(n+1)}\,{}_1F_2\left(1;n+2;n+2;\frac{\tilde{x}^2}{4}\right). \tag{6}$$

Solutions in the form of Eqs. (4) and (6) are inconvenient for technical and engineering applications since Lommel functions as well as hypergeometric functions, are not included into a standard tool kit of special functions used in reference tables and routine libraries. However, these expressions can be reduced to the form, which is more familiar for experts in the field of mathematical and engineering physics.

1. Method of solution

Let us examine expansion of generalized hypergeometric functions ${}_pF_q$ into their Maclaurin series. In particular, for ${}_1F_2$ function the series reads as

$${}_1F_2(a;b_1;b_2;z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b_1)_k(b_2)_k}\frac{z^k}{k!}$$

where $(a)_k$, $(b_{1,2})_k$ are the corresponding РохгаммегтАШs symbols

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

and

$$(b_{1,2})_k = b_{1,2}(b_{1,2}+1)\dots(b_{1,2}+k-1) = \frac{\Gamma(b_{1,2}+k)}{\Gamma(b_{1,2})}$$

For natural values of a and $b_{1,2}$, we obtain

$$(n)_k = n(n+1)\dots(n+k-1) = \frac{(n+k-1)!}{(n-1)!}$$

and then

$${}_1F_2\left(1;n+2;n+2;\frac{\tilde{x}}{4}\right) = [(n+1)!]^2\sum_{k=0}^{\infty}\frac{(1)_k}{(n+2)_k(n+2)_k}\frac{\tilde{x}^{2k}}{4^k k!}$$

$$= [(n+1)!]^2\sum_{k=0}^{\infty}\frac{k!}{(n+k+1)!(n+k+1)!}\frac{\tilde{x}^{2k}}{4^k k!} =$$

$$= [(n+1)!]^2\sum_{k=0}^{\infty}\frac{\tilde{x}^{2k}}{4^k[(n+k+1)!]^2} =$$

$$= [(n+1)!]^2\sum_{k=n+1}^{\infty}\frac{\tilde{x}^{2(k-n-1)}}{4^{k-n-1}(k!)^2} =$$

$$= [(n+1)!]^2\frac{4^{n+1}}{\tilde{x}^{2(n+1)}}\sum_{k=n+1}^{\infty}\frac{\tilde{x}^{2k}}{4^k(k!)^2} =$$

$$= [(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}} \sum_{k=n+1}^{\infty} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 .$$

The last expression with an accuracy to the factor $[(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}}$, preceding the sum, coincides with a well-known expansion [9, 10] of a modified Bessel function of the zeroth order I_0 into its Maclaurin series and, therefore,

$$ {}_1F_2 \left(1; n+2; n+2; \frac{\tilde{x}^2}{4} \right) = [(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}} \sum_{k=n+1}^{\infty} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 =$$

$$= [(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}} \left[\sum_{k=0}^{\infty} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 - 1 - \frac{\tilde{x}^2}{4} - \ldots - \frac{\tilde{x}^{2n}}{2^n n!} \right] =$$

$$= [(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}} \left[\sum_{k=0}^{\infty} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 - \sum_{k=0}^{n} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 \right] =$$

$$= [(n+1)!]^2 \frac{4^{n+1}}{\tilde{x}^{2(n+1)}} \left[I_0(\tilde{x}) - \sum_{k=0}^{n} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 \right]$$

(the relationship which first establishes a link between generalized hypergeometric functions of the special kind and modified Bessel functions) and

$$s_{2n+1,0} \left(\tilde{x} e^{i\frac{\pi}{2}} \right) = \frac{\tilde{x}^{2(n+1)}}{4(n+1)^2} {}_1F_2 \left(1; n+2; n+2; \frac{\tilde{x}^2}{4} \right) =$$

$$= 4^n (n!)^2 \left[I_0(\tilde{x}) - \sum_{k=0}^{n} \left(\frac{\tilde{x}^k}{2^k k!} \right)^2 \right] \tag{7}$$

Finally we end up with

$$\Psi(\tilde{x}) = \left(\frac{2^n n!}{b^{n+1}} \right)^2 \left[I_0(\tilde{x}) - \sum_{k=0}^{n} \left(\frac{\tilde{x}^p}{2^p p!} \right)^2 \right]$$

$$y(x) = \left(\frac{2^n n!}{b^{n+1}} \right)^2 \left[I_0(bx) - \sum_{p=0}^{n} \left(\frac{b^p x^p}{2^p p!} \right)^2 \right] =$$

$$= \frac{(2^n n!)^2}{C^{n+1}}) \left[I_0(\sqrt{C}x) - \sum_{p=0}^{n} C^p \left(\frac{x^p}{2^p p!} \right)^2 \right] \tag{8}$$

2. Results and conclusions

The obtained relationship is a final formula for unknown function $y(x)$, written in a rather simple and physically transparent form. In particular, the expression in square brackets is nothing but a difference of the modified Bessel functions and the sums of the first (n+1) terms of the function expansion into its Taylors series near the point x=0. It is easy to verify that in the limit $C \to 0$, Eq. (8) gives us exactly Eq. (2), and at $b = 1$, we obtain an especially simple solution

$$y(x) = (2^n n!)^2 \left[I_0(x) - \sum_{p=0}^{n} \left(\frac{x^p}{2^p p!} \right)^2 \right]$$

Let us note that the results obtained in this work can be useful both for experts in the field of mathematical simulation of various plasma devices based on the principle of inductive gas heating and, perhaps, in several adjacent regions of engineering and mathematical physics.

REFERENCES

1. **Kirpichnikov A.P.** On the structure of quasi-stationary electromagnetic field of a high-frequency induction discharge under atmospheric pressure // Izvestia vuzov. Fizika. – 1994. – № 2. – P. 77-81.
2. **Kirpichnikov A.P.** The structure of quasi-stationary electromagnetic field of a high-frequency induction discharge in the vicinity of the axis of a plasmoid // TVT. – 1995. – V. 33, № 1. – P. 139-140.
3. **Kirpichnikov A.P., Gerasimov A.V.** The structure of a high-frequency induction discharge in the vicinity of plasmoid axis in the case of a work coil of finite dimensions // Plasma technology -95. Proceedings. – Zaporozhye, 1995. – P. 28-29.
4. **Gerasimov A.V., Kirpichnikov A.P.** On the structure of temperature field of a high-frequency induction discharge of atmospheric pressure in the vicinity of the axis of a plasmoid// TVT. – 1998. – V. 36, № 2. – P. 342-343.
5. **Gerasimov A.V., Kirpichnikov A.P.** Two-temperature model of the energy balance for plasma of a high-frequency induction discharge in the vicinity of the axis of a plasmoid// TVT. – 2000. – V. 38, № 5. – P. 710-715.
6. Higher Transcendental Functions (A. Erde'lyi, Ed.), Vol. II. – McGraw-Hill, New York, 1953. – 396 p.
7. **Prudnikov A.P., Brychkov Yu.A., Marichev O.I.** Integrals and series. Special functions [Integraly i ryady. Spetcial'nye fynktcii]. – Moscow: Nauka, 1983. – 752 p. (in Russian)
8. **Prudnikov A.P., Brychkov Yu.A., Marichev O.I.** Integrals and series. Supplementary chapters [Integraly i ryady. Dopolnitel'nye glavy]. – Moscow: Nauka, 1983. – 800 p. (in Russian)
9. **Gradstein I.S., Ryzhik I.M.** Tables of Integrals, Sums, Series, and Products [Tablitcy integralov, summ, ryadov i proizvedenii]. – Moscow: GIFML, 1962. – 1100 p. (in Russian)
10. **Dwight G.B.** Tables of Integrals and other mathematical data. – New York: The Macmillan Company, 1957. – 288 p.