ON ONE 2-VALUED TRANSFORMATION, ITS INVARIANT MEASURE AND APPLICATION TO MASKED DYNAMICAL SYSTEMS

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ABSTRACT. We consider one family S of 2-valued transformations on the interval $[0,1]$ with measure μ, endowed with a set of weight functions. We construct invariant measure $\mu_S = \mu$ for this multi-valued dynamical system with weights and show the interplay between such systems and masked dynamical systems, which leads to image processing.

1. INTRODUCTION

Let X be a space with finite measure μ on σ-field $\mathfrak{B}$ of subsets of X, $N \in \mathbb{N}$ be an integer, $I = \{1, \dots, N\}$, and $S_i\colon X \to X$ — some measurable transformations. Consider a set of measurable functions (*endowment*)

$$\left\{ \alpha_i\colon X \to [0,1], \quad i \in I \quad \middle| \quad \sum_{i \in I} \alpha_i \equiv 1 \right\}.$$

A collection

$$(X; \mathfrak{B}; \mu; S_1, \dots, S_N; \alpha_1, \dots, \alpha_N) \tag{1}$$

is called *multi-valued dynamical system with weights*, and the map $S = \cup_{i \in I} S_i$ with fixed pairs $\{(S_i, \alpha_i)\}_{i \in I}$ — *endowed N-transformation* (see [1]). Regarding this, we can establish a new measure on $\mathfrak{B}$:

$$\mu_S(B) = \sum_{i \in I} \int_{S_i^{-1}(B)} \alpha_i(x)\, d\mu.$$

One of the important questions of dynamical system theory is finding an invariant measure $\mu_S = \mu$.

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The endowment α plays a rôle of a parameter which controls measure μ_S. On the other hand, $\alpha_i(x)$ could be considered as a probability of choosing and applying the transformation S_i (out of S) to a point $x \in X$ in stochastic dynamical system. Finally, as we show further, this parameter can uniquely define some single-valued dynamical system connected to S.

In this paper we continue (after [2]) studying the following 2-transformation $S = S_1 \cup S_2$ of the interval $[0,1]$ (see Fig. 1):

$$S_1(x) = \begin{cases} \frac{1}{1-a}x, & x \in [0, 1-a); \\ \frac{1}{1-a}x - \frac{a}{1-a}, & x \in [1-a, 1], \end{cases}$$

$$S_2(x) = \begin{cases} \frac{1}{1-a}x, & x \in [0, a); \\ \frac{1}{1-a}x - \frac{a}{1-a}, & x \in [a, 1], \end{cases}$$

with a shift $a \in \left(0, \frac{1}{2}\right]$ as its parameter. Dynamical system $([0,1], S)$ is tightly connected to the theory of β-decompositions (see [3, 4, 5, 6]).

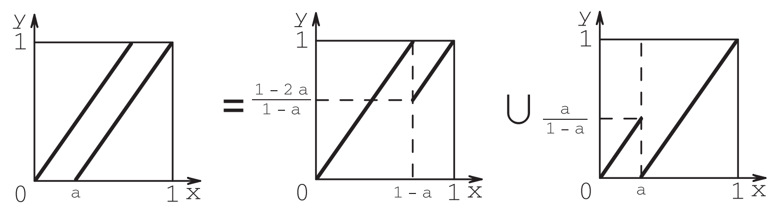

FIGURE 1. The design of 2-transformation S.

As a motivation for this paper in introduction we examine two points: invariance of measure for this endowed 2-transformation and masked dynamical system associated with it.

1.1. **Invariance of measure.** Let λ be the Lebesgue measure on $[0,1]$, $\mathfrak{B}$ — the Borel σ-field on $[0,1]$. Let also $\mu(B) = \int_B p(x)\, d\lambda$ be a measure, absolutely continuous with respect to the Lebesgue measure ($\mu \ll \lambda$), with density $p(x) \in L^1([0,1], \mathfrak{B}, \lambda)$ and $p(x) \geq 0$.

According to [1], we endow 2-transformation S with a set of weight functions $\alpha = \{\alpha_1(x), \alpha_2(x)\}$, $\alpha_1(x), \alpha_2(x) \in L^1([0,1], \mathfrak{B}, \lambda)$ such that $\alpha_1(x) + \alpha_2(x) = 1$ and $\alpha_1(x), \alpha_2(x) \geq 0$. Then we can introduce a new measure μ_S on $\mathfrak{B}$:

$$\mu_S(B) = \int\limits_{S_1^{-1}(B)} \alpha_1(x) p(x)\, d\lambda + \int\limits_{S_2^{-1}(B)} \alpha_2(x) p(x)\, d\lambda.$$

There are three independent parameters in the abovementioned construction: density function $p(x)$, shift number a and endowment $\alpha = \{\alpha_1(x), \alpha_2(x)\}$.

Whether we search for endowed transformation for a given measure μ or a measure $\mu_S = \mu$ for a given transformation S — there is a certain relation between these parameters, defined by equality $\mu_S = \mu$.

Further on, we fix three parameters: $a \in \left(0, \frac{1}{2}\right]$, $\{\alpha_1(x), \alpha_2(x)\}$, $p(x)$, and let $n \in \mathbb{N}$ be such that

$$\frac{1}{n+1} < a \leq \frac{1}{n} \qquad (n \geq 2).$$

Here we cite the following criterion of existence of invariant measure.

Theorem 1 (see [2]). *$\mu_S = \mu$ if and only if the following conditions hold true:*

$$\sum_{k=-1}^{n-1} p(x+ka) = \frac{1}{1-a} \sum_{k=-1}^{n-2} p\left(\frac{x+ka}{1-a}\right), \quad \forall x \in [a, 1-(n-1)a]; \tag{2}$$

$$\sum_{k=-1}^{n-2} p(\tilde{x}+ka) = \frac{1}{1-a} \sum_{k=-1}^{n-3} p\left(\frac{\tilde{x}+ka}{1-a}\right), \quad \forall \tilde{x} \in [1-(n-1)a, 2a); \tag{3}$$

$$\alpha_1(x+ma)p(x+ma) = \left(\sum_{k=-1}^{m} p(x+ka) - \frac{1}{1-a} \sum_{k=-1}^{m-1} p\left(\frac{x+ka}{1-a}\right)\right), \tag{4}$$

where for $n = 2$, $m = 0$, $x \in [a, 1-a)$,
for $n \geq 3$, $m = 0$, $x \in [a, 2a)$,
for $n \geq 3$, $m = 1, 2, \ldots$, $x \in [a, 2a)$ and $x + ma \in [2a, 1-a)$.

There is no restriction on function $\alpha_1(x)$ on the sets $[0, a)$ and $[1-a, 1]$.

Equations (2)–(3) define function $p(x)$ on the interval $[0, 1]$, and equation (4) defines endowment α. We can revise (4) into more compact and constructive formula:

$$\alpha_1(x)p(x) = \sum_{k=0}^{s} p(x-ka) - \frac{1}{1-a} \sum_{k=1}^{s} p\left(\tfrac{x-ka}{1-a}\right), \quad x \in [a, 1-a),$$

where $s = [\frac{x}{a}]$ ($[x]$ is an integer part of x).

To clarify the meaning of the theorem we give two corollaries from it.

Corollary 1 (see [2]). *Given measure $\mu \ll \lambda$ there exists endowed 2-transformation $S(a)$ preserving measure μ if and only if $p(x)$, a and $\{\alpha_1(x), \alpha_2(x)\}$ satisfy conditions (2)–(4).*

Corollary 2 (see [2]). *Given endowed 2-transformation $S(a)$ there exists measure $\mu \ll \lambda$ which is preserved by transformation S if and only if $p(x)$, a and $\{\alpha_1(x), \alpha_2(x)\}$ satisfy conditions (2)–(4).*

There is a convenient graphical scheme of summation intervals placement on the interval $[0,1]$ for the equations (2)–(3), see Fig. 2–3.

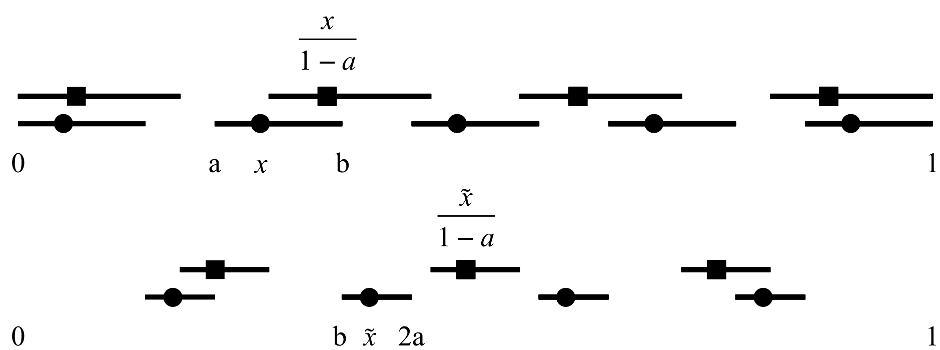

FIGURE 2. Scheme of summation intervals placement for equations (2) (upper) and (3) (lower), here $b = 1-(n-1)a$, even n.

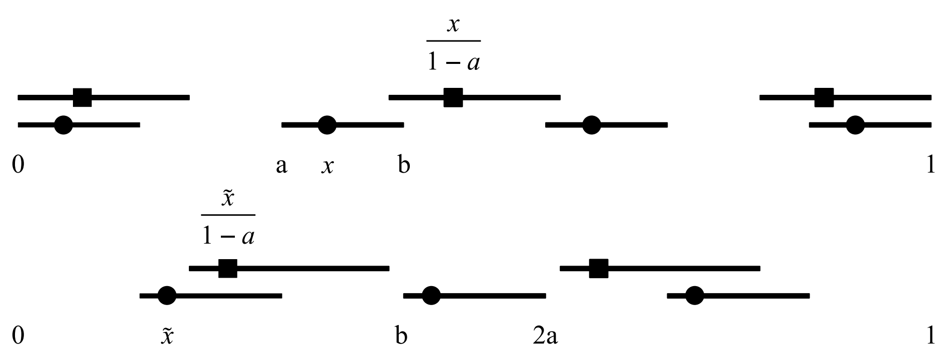

FIGURE 3. Scheme of summation intervals placement for equations (2) (upper) and (3) (lower), here $b = 1-(n-1)a$, odd n.

Informally, we can depict these equations (2)–(3) as follows:

$$\sum p(\bullet) = \frac{1}{1-a} \sum p(\blacksquare).$$

Regarding Theorem 1, the following question arises.

Question 1. *Are there functions satisfying equations* (2)–(3)*?*

One trivial solution is $p \equiv 0$.

Slightly less trivial example of constant density $p \equiv c$, $c \in \mathbb{R}$, $c > 0$, is presented in the following corollary.

Corollary 3 (see [2])**.** $(c \cdot \lambda)_S = c \cdot \lambda$ *if and only if* $a = \frac{1}{n}$, $n = 2, 3, \ldots$.

However, this 2-valued dynamical system allows even more sophisticated density: the equations (2)–(4) hold true for some non-constant $p(x)$, as shown in the next theorem.

Let $\chi_B(x) = \begin{cases} 0, x \notin B, \\ 1, x \in B, \end{cases}$ be a characteristic function for a subset $B \subset [0,1]$.

Theorem 2 (see [2])**.** *Given $n = 2, 3, \ldots$ there exist a shift a $(\frac{1}{n+1} < a < \frac{1}{n})$, piecewise constant density $p(x)$ and endowment $\{\alpha_1(x), \alpha_2(x)\}$, such that $\mu_S = \mu$. Namely,*

$$p(x) = \beta\chi_{[0,\delta)}(x) + (\beta+\gamma)(1-\delta)\chi_{[\delta,1-\delta)}(x) + \gamma\chi_{[1-\delta,1]}(x),$$
$$a = \frac{n+1-\sqrt{n^2+1}}{n}, \quad \delta = \frac{an}{2}, \quad \beta, \gamma > 0, \quad n \textit{ is even},$$

$$p(x) = \beta\chi_{[0,1-\delta)}(x) + (\beta+\gamma)(1-\delta)\chi_{[1-\delta,\delta)}(x) + \gamma\chi_{[\delta,1]}(x),$$
$$a = \frac{n+1-\sqrt{n^2-1}}{n+1}, \quad \delta = \frac{a(n+1)}{2}, \quad \beta, \gamma > 0, \quad n \textit{ is odd}.$$

Remark. *Theorem 2 yields a family of densities with two parameters $\beta, \gamma > 0$.*

For computational simplicity in this theorem a is chosen in such a way that the middle intervals in the graphical scheme touch each other, see Fig. 4 for even n.

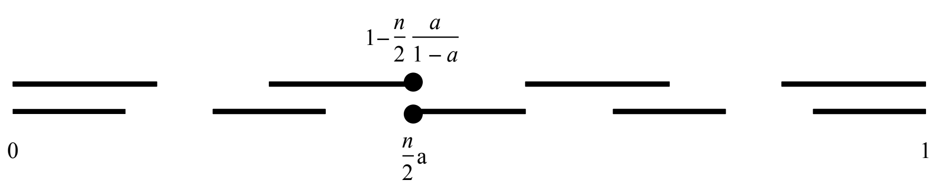

FIGURE 4. Special choice of a shift a: $\frac{n}{2}a = 1 - \frac{n}{2}\frac{a}{1-a}$, even n.

The resulting piecewise density consists of three domains, see Fig. 5.

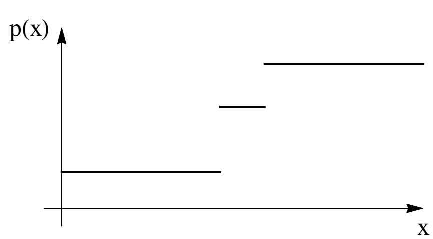

FIGURE 5. Typical view of a piecewise constant density from Theorem 2.

However, the same question arises again: are there another non-trivial (non-constant) densities satisfying equations (2)–(3)?

In Section 2 we present a scheme to construct non-trivial densities in case of $a = \frac{3-\sqrt{5}}{2}$ $(n = 2)$ and study some properties of the functions we obtain there. In Section 3 there is a scheme to construct such densities for arbitrary $a \in (0, \frac{1}{2}]$ $(n \geq 2)$.

Finally, in this subsection we cite the following lemma which implies "mirrow twoness" of invariant measures densities (see Corollary 4): if $p(x)$ is such a density, then the function $g(x) = p(1-x)$ is again a density of invariant measure.

Lemma 1 (see [2]). *Let $A_i(x)=\alpha_i(x)p(x)$, $i=1,2$. Then $\mu_S=\mu$ if and only if*

$$A_1((1-a)x) + \chi_{[\frac{1-2a}{1-a},1]}(x)A_1((1-a)x+a) + A_2((1-a)x+a)+ \\ + \chi_{[0,\frac{a}{1-a})}(x)A_2((1-a)x) = \frac{p(x)}{1-a} \quad (5)$$

λ-almost everywhere on $[0,1]$.

Corollary 4. *If $p(x)$ is invariant measure density, then the function $g(x) = p(1-x)$ with endowment $\beta_i(x) = \alpha_{3-i}(1-x)$, $i = 1,2$, is also invariant measure density.*

Proof. Let $p(x)$ be invariant measure density, $g(x) = p(1-x)$, $B_i(x) = \beta_i(x)g(x) = \alpha_{3-i}(1-x)p(1-x) = A_{3-i}(1-x)$. Substituting $1-x$ instead of x in equality (5) yields

$$\frac{g(x)}{1-a} = \frac{p(1-x)}{1-a} = A_1((1-a)(1-x)) + \chi_{[0,\frac{a}{1-a}]}(x)A_1((1-a)(1-x)+a)+ \\ + A_2((1-a)(1-x)+a) + \chi_{(\frac{1-2a}{1-a},1]}(x)A_2((1-a)(1-x)) = \\ = A_1(1-((1-a)x+a)) + \chi_{[0,\frac{a}{1-a}]}(x)A_1(1-(1-a)x) + A_2(1-(1-a)x)+ \\ + \chi_{(\frac{1-2a}{1-a},1]}(x)A_2(1-((1-a)x+a)) = \\ = B_2((1-a)x+a) + \chi_{[0,\frac{a}{1-a}]}(x)B_2((1-a)x) + B_1((1-a)x)+ \\ + \chi_{(\frac{1-2a}{1-a},1]}(x)B_1((1-a)x+a).$$

Thus equality (5) holds true for $g(x)$ almost everywhere. □

1.2. **Masked dynamical system.** As an extra motivation we consider here the following argument: endowment α of dynamical system S can be connected with mask endowment of some iterated functions system $\mathcal{F}$ (see below).

Consider some disjoint cover $\mathcal{M} = \{M_i\}_{i\in I}$ of the set X: $M_i \in \mathfrak{B}$, $i \in I$, $M_i \cap M_j = \emptyset$, $i,j \in I$, $i \neq j$, $\cup_{i\in I} M_i = X$. Let $\alpha_i = \chi_{M_i}$, $i \in I$, be characteristic functions of the subsets $M_i \subset X$.

We may say that, regarding the contribution of $S_i^{-1}(B)\cap M_i$ to the measure

$$\mu_S(B) = \sum_{i\in I} \mu(S_i^{-1}(B)\cap M_i) = \mu\left(\cup_{i\in I}(S_i^{-1}(B)\cap M_i)\right),$$

N-valued transformation turns into the following single-valued one:

$$\tilde{S}(x) = \begin{cases} S_1(x), & x \in M_1, \\ \vdots & \\ S_N(x), & x \in M_N. \end{cases}$$

In the case of arbitrary endowment α we may consider single-valued stochastic dynamical system:

$$\tilde{\tilde{S}}(x) = \begin{cases} S_1(x) & \text{with probability } \alpha_1(x), \\ \vdots & \\ S_N(x) & \text{with probability } \alpha_N(x). \end{cases}$$

Such an approach that turns multi-valued dynamical system into single-valued one is implemented in [7] for mappings S, connected with iterated function systems (IFS). It lets us establish and control fractal transformations between IFS attractors. Such transformations have direct practical value (see below). Here we introduce main points from [7] (relevant to this paper).

Let $X \neq \emptyset$ be a compact Hausdorff space, $K(X)$ — a set of nonempty compact subsets of X. Let $I = \{1, \dots, N\}$ be a finite set of positive integers, I^∞ — a set of infinite sequences of numbers from I, $f_i\colon X \to X, i \in I$, — continuous mappings. Then $\mathcal{F} = (X; f_1, \dots, f_N)$ is called *iterated function system (IFS).*

Due to decreasing monotone inclusion of corresponding compact subsets one can correctly define the mapping

$$\Pi\colon I^\infty \to K(X), \quad \sigma = \sigma_1\sigma_2\ldots \mapsto \bigcap_{k=1}^{\infty} f_{\sigma_1}\circ f_{\sigma_2}\circ\cdots\circ f_{\sigma_k}(X).$$

If for all $\sigma \in I^\infty$, $\Pi(\sigma)$ is a singleton, then the IFS is called *point-fibred.* In this case a mapping

$$\pi\colon I^\infty \to A = \pi(I^\infty) \subset X, \quad \{\pi(\sigma)\} = \Pi(\sigma),$$

is called the *coding map* of $\mathcal{F}$, I^∞ — the *code space* of $\mathcal{F}$, and $\sigma \in I^\infty$ — the *address* of the point $\pi(\sigma) \in A$.

For point-fibred IFS on a compact Hausdorff space there exists a unique set $A \in K(X)$ such that

$$A = \bigcup_{i\in I} f_i(A),$$

and $A = \pi(I^\infty)$ (see [7]). This set is called the *attractor* of the given IFS.

IFS attractor often happens to be a fractal set or even self-similar one, which is usually of huge interest.

Henceforth, we constrain ourselves to point-fibred IFS on some compact Hausdorff space only (however, this is rather typical, cf. Remark 2.5 in [7]).

A point $x \in A$ may have more than one address (even uncountably many). The following definition will be useful to make the choice of address unique. A subset $\Omega \subset I^\infty$ is called the *address space* of the IFS $\mathcal{F}$ if $\pi|_\Omega \colon \Omega \to A$ is bijective. Then the inverse mapping

$$\tau \colon A \to \Omega, \quad x \mapsto (\pi|_\Omega)^{-1}(x),$$

is called the *section of* π.

If there are two point-fibred IFS $\mathcal{F} = \{X; f_1, \dots, f_N\}$ and $\mathcal{G} = \{Y; g_1, \dots, g_N\}$ (with common I^∞) on compact Hausdorff spaces X and Y, $A_\mathcal{F}$ and $A_\mathcal{G}$ are their attractors, $\pi_\mathcal{G}$ — the coding mapping of $\mathcal{G}$, $\tau_\mathcal{F}$ — the section of $\pi_\mathcal{F}$, then we can define the *fractal transformation*[1] between attractors of $\mathcal{F}$ and $\mathcal{G}$:

$$T_{\mathcal{F}\mathcal{G}} \colon A_\mathcal{F} \to A_\mathcal{F}, \quad x \mapsto \pi_\mathcal{G} \circ \tau_\mathcal{F}(x).$$

The paper [7] gives a continuity criteria for $T_{\mathcal{F}\mathcal{G}}$, and also describes some applications of fractal transformations for conversion and filtering images and steganography (hidden data transmission, for example, packing several images into one).

The choice of the address space $\Omega_\mathcal{F}$ of $\mathcal{F}$ defines a fractal transformation. In [7] two methods for construction of $\Omega_\mathcal{F}$ are proposed, they lead to sections $\tau_\mathcal{F}$ with good properties.

One of the methods is to use *top addresses*: sequences from I^∞ may be put in lexicographic order, which lets us choose a unique ("top") element from $\pi^{-1}(x)$ for all $x \in A$ (see [5, 8]). This method is computationally simple and can be easily implemented on computer. However, only a few certain sections can be obtained in this way.

Let us consider the second method in more detail. Let $\mathcal{F}$ be a point-fibred IFS with injective maps $f_i, i \in I$. A collection of subsets $\mathcal{M} = \{M_i \subset A, i \in I\}$ is called the *mask* of $\mathcal{F}$ if

(1) $M_i \subset f_i(A), i \in I$;
(2) $M_i \cap M_j = \emptyset, i, j \in I, i \neq j$;
(3) $\cup_{i \in I} M_i = A$.

[1]Under this transformation the fractal dimension of a set could be changed.

For all $x \in A$, there exists a unique $i \in A$ such that $x \in M_i \subset f_i(A)$. The mapping

$$T\colon A \to A, \qquad x \mapsto \begin{cases} f_1^{-1}(x), & x \in M_1 \\ \vdots \\ f_N^{-1}(x), & x \in M_N, \end{cases}$$

is called the *masked dynamical system* for $\mathcal{F}$.

This system is used to construct a section $\tau\colon A \to \tau(A) \subset I^\infty$ by following the orbit $T^n(x) = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}(x)$ of point x, namely,

$$\tau(x) = \sigma(x) = \sigma_1(x)\sigma_2(x)\ldots, \quad \text{where } x \in (T^{k-1})^{-1}(M_{\sigma_k(x)}),\ k = 1, 2, \ldots.$$

In this case $\pi(\sigma(x)) = x$ (see [7]).

Thus the mask $\mathcal{M}$ of dynamical system connected with IFS is a special case of endowment α, when $\alpha_i = \chi_{M_i}$, $i \in I$. We can also consider stochastic mask defined by endowment weight functions: if $\operatorname{supp}\alpha_i \subset f_i(A)$, $i \in I$, then

$$\tilde{\tilde{T}}(x) = \begin{cases} f_1^{-1}(x) & \text{with probability } \alpha_1(x), \\ \vdots \\ f_N^{-1}(x) & \text{with probability } \alpha_N(x). \end{cases}$$

Let us describe the connection between this mask construction and 2-transformation S. Consider the following IFS (see Fig. 6):

$$(X = [0,1]; \quad f_1(x) = (1-a)x, \quad f_2(x) = (1-a)x + a). \tag{6}$$

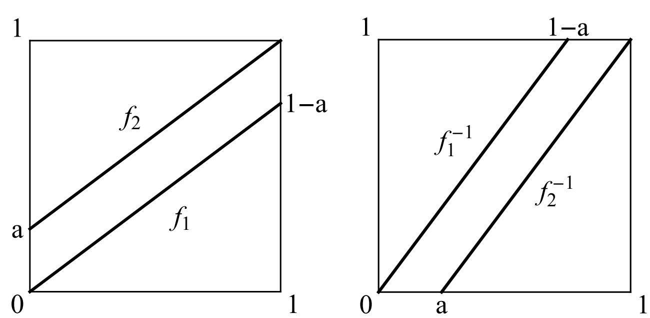

FIGURE 6. IFS (6) (left) and multi-valued (without mask) dynamical system (coincides with S) connected with it (right).

This is point-fibred IFS with injective functions f_1, f_2, its attractor is the interval $A = X = [0,1]$. Consider f_1^{-1}, f_2^{-1} for construction of masked dynamical system T. As might be seen on Fig. 6, this dynamical system is the object of this paper. Let $\mathcal{M} = \{M_1, M_2\}$ be a mask of this IFS. Then obviously,

$[0, a) \subset M_1$ and $(1-a, 1] \subset M_2$. Define $M_1 \cap [a, 1-a]$ and $M_2 \cap [a, 1-a]$ arbitrary ($M_1 \cap M_2 = \emptyset$, $M_1, M_2 \in \mathfrak{B}$). The example of a mask and the process of finding masked address of a point $x \in A$ are illustrated on Fig. 7.

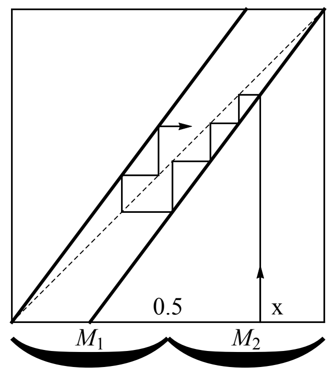

FIGURE 7. Example of masked dynamical system for IFS (6), $M_1 = [0, 0.5)$, $M_2 = [0.5, 1]$, $\tau(x) = 222211\ldots$.

As we have already mentioned, mask endowment $\mathcal{M}$ of $\mathcal{F}$ in this case coincides with endowment $\alpha = \{\alpha_1(x) = \chi_{M_1}(x), \alpha_2(x) = \chi_{M_2}(x)\}$ of S.

Then the following question arises.

Question 2. *Is there an invariant measure for this masked dynamical system?*

We give an example of such a measure in Section 2.

2. The case of $\mathbf{n = 2}$

Here we consider the case of $n = 2$ in detail. The main ideas of this section can be used further for other values of n. The conditions (2)–(3) now can be written as:

$$p(x-a) + p(x) + p(x+a) = \tfrac{1}{1-a}\left(p\left(\tfrac{x-a}{1-a}\right) + p\left(\tfrac{x}{1-a}\right)\right), x \in [a, 1-a); \quad (7)$$

$$p(\tilde{x}-a) + p(\tilde{x}) = \tfrac{1}{1-a} p\left(\tfrac{\tilde{x}-a}{1-a}\right), \tilde{x} \in [1-a, 2a). \quad (8)$$

Or in equivalent way:

$$p(x) + p(x+a) + p(x+2a) = \tfrac{1}{1-a}\left(p\left(\tfrac{x}{1-a}\right) + p\left(\tfrac{x+a}{1-a}\right)\right), x \in [0, 1-2a); \quad (9)$$

$$p(\tilde{x}) + p(\tilde{x}+a) = \tfrac{1}{1-a} p\left(\tfrac{\tilde{x}}{1-a}\right), \tilde{x} \in [1-2a, a). \quad (10)$$

To make it simple, we consider special shift, according to the scheme on Fig. 4. In our case $n = 2$, $a = \frac{3-\sqrt{5}}{2}$, see Fig. 8.

FIGURE 8. Interval placement, $a = \frac{3-\sqrt{5}}{2}$ $(n = 2)$.

Here we introduce a scheme to construct a density $p(x)$ satisfying equations (9)–(10), see Fig. 9. Consider the following marks on the X-axis: a, $1-a$, $2a$, $2-3a$, $x_k = a(1-a)^k$, $k \geq 1$, $(x_1 = 1-2a)$.

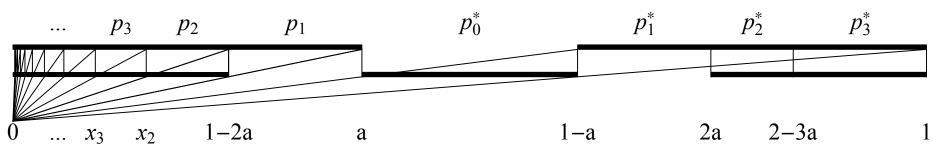

FIGURE 9. Scheme to construct a density $p(x)$, with auxiliary intervals marked, $a = \frac{3-\sqrt{5}}{2}$ $(n = 2)$.

- Fix functions $p_0^*, p_1^* \in L^1$, $p_0^*, p_1^* \geq 0$, arbitrarily, and define

$$p(x) = \begin{cases} p_1(x) = \frac{1}{1-a} p_0^*(\frac{x}{1-a}) - p_1^*(x+a), & x \in (1-2a, a], \\ p_0^*(x), & x \in (a, 1-a], \\ p_1^*(x), & x \in (1-a, 2a]. \end{cases}$$

- Fix function $p_3^* \in L^1$, $p_3^* \geq 0$, arbitrarily, and define

$$p(x) = \begin{cases} p_3^*(x), & x \in (2-3a, 1], \\ p_2(x) = \frac{1}{1-a}(p_1(\frac{x}{1-a}) + p_3^*(\frac{x+a}{1-a})) - & \\ \qquad - p_0^*(x+a) - p_3^*(x+2a), & x \in (x_2, 1-2a]. \end{cases}$$

- Fix function $p_2^* \in L^1$, $p_2^* \geq 0$, arbitrarily, and define

$$p(x) = \begin{cases} p_2^*(x), & x \in (2a, 2-3a], \\ p_3(x) = \frac{1}{1-a}(p_2(\frac{x}{1-a}) + p_2^*(\frac{x+a}{1-a})) - & \\ \qquad - p_0^*(x+a) - p_3^*(x+2a), & x \in (x_3, x_2]. \end{cases}$$

- Define for each $k \geq 4$

$$p(x) = p_k(x) = \frac{1}{1-a}(p_{k-1}(\frac{x}{1-a}) + p_1^*(\frac{x+1}{1-a})) - \\ - p_0^*(x+a) - p_2^*(x+2a), \qquad \text{where} \quad x \in (x_k, x_{k-1}].$$

- Fix the value $p(0) \geq 0$ arbitrarily.

By construction, $p(x)$ satisfies the conditions (9)–(10) (perhaps except at most countable number of points on intervals boundaries). Notice, the function $p(x)$ is defined arbitrarily on $(a,1]$ and is restored on $[0,a]$ after that. We need the partition $p_0^*, p_1^*, p_2^*, p_3^*$ of the function $p(x)$ to study its properties in more detail.

Proposition 1. *If $p_0^*, p_1^*, p_2^*, p_3^*$ are constants, then p_2 is a constant, and $p_3 = p_4 = \ldots$ are constants.*

Proof. We denote $A = \frac{1}{1-a}$, then

$$\begin{aligned} p_1 &= Ap_0^* - p_1^*, & p_3 &= A(p_2 + p_2^*) - p_0^* - p_3^*,\\ p_2 &= A(p_1 + p_3^*) - p_0^* - p_3^*, & p_k &= A(p_{k-1} + p_1^*) - p_0^* - p_2^*, \quad k \geq 4. \end{aligned}$$

Consider the following difference:

$$\begin{gathered} p_4 - p_3 = A(p_3 - p_2 + p_1^* - p_2^*) - p_2^* + p_3^* = A(A(p_2 - p_1 + p_2^* - p_3^*) + p_1^* - p_2^*) - p_2^* + p_3^* = \\ = A(A(A(p_1 + p_3^*) - p_0^* - p_3^* - p_1 + p_2^* - p_3^*) + p_1^* - p_2^*) - p_2^* + p_3^* = \\ = A(A(A(p_1 + p_3^*) - \underline{p_0^*} - p_3^* - p_1 + p_2^* - p_3^*) + \underline{Ap_0^*} - p_1 - p_2^*) - p_2^* + p_3^* = \\ = A(A(A(p_1 + p_3^*) + (p_2^* - p_3^*) - (p_1 + p_3^*)) - (p_1 + p_3^*) - (p_2^* - p_3^*)) - (p_2^* - p_3^*) = \\ = (A^2 - A - 1)(A(p_1 + p_3^*) + (p_2^* - p_3^*)). \quad (11) \end{gathered}$$

To simplify the calculations henceforth, we need the following equalities:

$$\begin{cases} (1-a)^2 = a, \\ a^2 = 3a - 1, \\ \frac{1-2a}{1-a} = a, \\ A^2 - A - 1 = \frac{1}{(1-a)^2}\left(1 - (1-a) - (1-a)^2\right) = 0. \end{cases} \quad (12)$$

Thus the last expression in equalities (11) equals zero.

Then $p_{k+1} - p_k = A(p_k - p_{k-1}) = \ldots = A^{k-3}(p_4 - p_3) = 0$, $k \geq 4$, q.e.d. $\square$

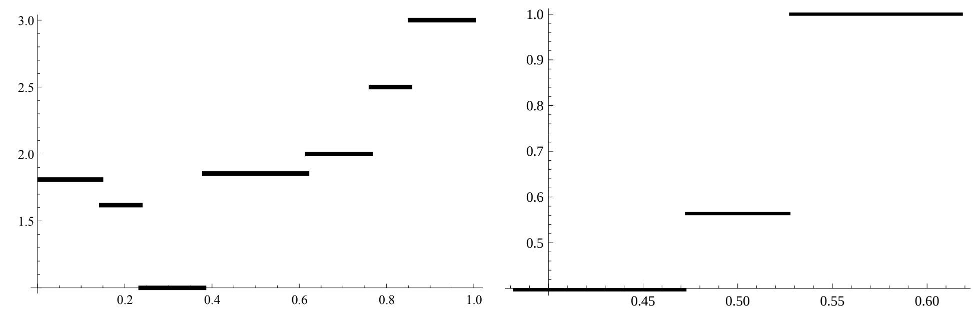

FIGURE 10. Example of density $p(x)$ from Proposition 1 ($p_1 = 1, p_1^* = 2, p_3^* = 3, p_2^* = 2.5$) (left) and corresponding function $\alpha_1(x)$ (on $[a, 1-a]$) (right).

However, the values of function $p(x)$ we obtain can be negative. In the case of piecewise constant density we give the following criterion for $p(x)$ to be non-negative.

Proposition 2. *Let $p_0^*, p_1^*, p_2^*, p_3^* \in \mathbb{R}$ be constants, function $p(x)$ is obtained according to the scheme above. Then $p(x) \geq 0$ for all $x \in [0,1]$ if and only if*

$$\begin{cases} p_0^*, p_1^*, p_2^*, p_3^* \geq 0, \\ \frac{1}{1-a} p_0^* - p_1^* \geq 0, \\ \frac{1}{1-a}(p_3^* + p_0^* - p_1^*) - p_3^* \geq 0, \\ \frac{1}{1-a}(p_2^* + p_0^* - p_1^*) - p_1^* \geq 0. \end{cases}$$

These inequalities define unbounded convex set in $\mathbb{R}^4 \ni \{p_0^*, p_1^*, p_2^*, p_3^*\}$.

Proof. In view of (12), it is sufficient to notice that

$$p_2 = A(p_1 + p_3^*) - p_0^* - p_3^* = A(Ap_0^* - p_1^* + p_3^*) - p_0^* - p_3^* = \\ = (A^2 - 1)p_0^* + A(p_3^* - p_1^*) - p_3^* = A(p_3^* + p_0^* - p_1^*) - p_3^*,$$

$$p_3 = A(p_2 + p_2^*) - p_0^* - p_3^* = A(A(p_3^* + p_0^* - p_1^*) - p_3^* + p_2^*) - p_0^* - p_3^* = \\ = (A^2-1)(p_3^*+p_0^*) + A(p_2^*-p_3^*) - A^2 p_1^* = A(p_2^*+p_0^*) - (A+1)p_1^* = A(p_2^*+p_0^*-p_1^*) - p_1^*.$$

□

Now consider obtaining a function $p(x)$ with the property of continuity. This is discussed in Propositions 3–6.

Proposition 3. *Given function $p(x)$ obtained by the scheme above. Then $p(x)$ is continuous on $(0,1]$ if and only if*

- *parameters $y_2, y_3, y_4 \geq 0$ satisfy the equation*

$$y_2 + y_4 = \tfrac{1}{1-a} y_3, \tag{13}$$

- *$y_5 \geq 0$ is arbitrary,*
- *graphs of continuous functions $p_0^*(x), p_1^*(x), p_2^*(x), p_3^*(x)$ connect points (a, y_2), $(1-a, y_3)$,$(2a, y_4)$, $(2-3a, y_5)$ and $(1, 0)$, see Fig. 11.*

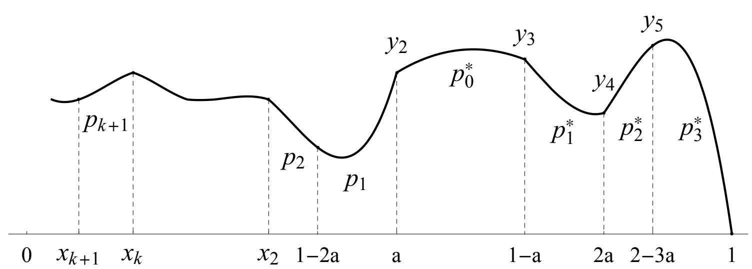

FIGURE 11. Illustration for Proposition 3.

Proof. Let $p(x)$ be continuous, then we substitute $x = \tilde{x} = 1-2a$ into (9)–(10) and obtain

$$p(1-2a) + p(1-a) + p(1) = \tfrac{1}{1-a}(p(\tfrac{1-2a}{1-a}) + p(1)),$$
$$p(1-2a) + p(1-a) = \tfrac{1}{1-a} p(\tfrac{1-2a}{1-a}),$$

wherefrom $p(1) = 0$. By substitution $\tilde{x} = a$ into (10) and taking into account $\frac{a}{1-a} = 1-a$, we have

$$p(a) + p(2a) = \tfrac{1}{1-a} p(1-a), \tag{14}$$

which is equal to (13).

To prove the backward implication, let $g(x)$ be a piecewise function, made of functions $p_0^*, p_1^*, p_2^*, p_3^*$ "glued together".

By construction, $p_1(x) = \frac{1}{1-a} g(\frac{x}{1-a}) - g(x+a)$ on $(1-2a, a]$. Then $p_1(x)$ is continuous, because $g(x)$ is continuous, and

$$p_1(a) = \tfrac{1}{1-a} g(\tfrac{a}{1-a}) - g(2a) = \tfrac{1}{1-a} y_3 - y_4 = y_2 = g(a).$$

Now add function $p_1(x)$ leftside into the set of functions which define $g(x)$.

By construction, $p_2(x) = \frac{1}{1-a}(g(\frac{x}{1-a}) + g(\frac{x+a}{1-a})) - g(x+a) - g(x+2a)$ on $(x_2, 1-2a]$, wherefrom $p_2(x)$ is continuous, and (considering $g(1) = 0$)

$$p_2(1-2a) = \tfrac{1}{1-a}(g(\tfrac{1-2a}{1-a}) + g(1)) - g(1-a) - g(1) =$$
$$= \tfrac{1}{1-a} g((\tfrac{1-2a}{1-a})+) - g((1-a)+) = g((1-2a)+).$$

For $k \geq 2$, by construction, $p_{k+1}(x) = \frac{1}{1-a}(g(\frac{x}{1-a}) + g(\frac{x+a}{1-a})) - g(x+a) - g(x+2a)$ on $(x_{k+1}, x_k]$, where $g(x)$ made of functions $p_k, \dots, p_2, p_1, p_0, p_1^*, p_2^*, p_3^*$. Thus $p_{k+1}(x)$ is continuous, and

$$p_{k+1}(x_k) = \tfrac{1}{1-a}(g(\tfrac{x_k}{1-a}) + g(\tfrac{x_k+a}{1-a})) - g(x_k+a) - g(x_k+2a) =$$
$$= \tfrac{1}{1-a}(g((\tfrac{x_k}{1-a})+) + g((\tfrac{x_k+a}{1-a})+)) - g((x_k+a)+) - g((x_k+2a)+) = g(x_k+).$$

□

Proposition 4. *Let $p(x)$ satisfy the equations* (9)–(10). *If $p\colon [0,1] \to \mathbb{R}$ is continuous, then $p(0) = p(1) = 0$.*

Proof. It suffices to show $p(0) = 0$. We substitute $x = 0$ into (9) and obtain

$$p(0) + p(a) + p(2a) = \tfrac{1}{1-a}(p(0) + p(\tfrac{a}{1-a})),$$

wherefrom we have $p(0) = 0$ (considering (14)). □

However, the next question arises.

Question 3. *Under which conditions our construction yields $p(0+) = 0$?*

Notice that if by construction of density $p(x)$ the equality $p(0+) = 0$ is not fulfilled, then $p(x)$ is continuous on $(0,1]$ but does not have finite limit at 0. Its graph is unbounded and (or) oscillates greatly in neighborhood of 0.

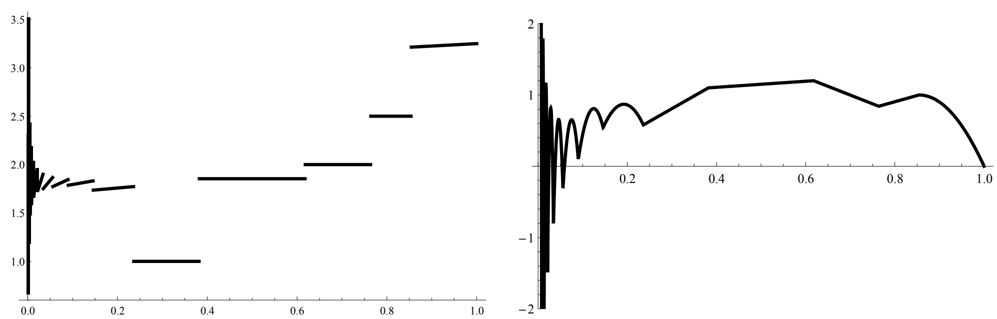

FIGURE 12. Oscillating and unbounded functions.

Such situation is quite typical while constructing $p(x)$, see Fig. 12. However, the following Proposition 5 gives an example of a density with good properties.

Proposition 5. *Let function $p(x)$ be obtained according to the scheme above. If functions $p_0^*, p_1^*, p_2^*, p_3^*$ form a spline of degree 1 with $p_3^*(1) = 0$, then function $p(x)$ is also a spline of degree 1, furthermore*

- *$p\colon [0,1] \to \mathbb{R}$ is continuous function;*
- *p_1, p_2 are linear functions;*

- *graphs of the functions $p_3, p_4, \dots$ make up one graph of linear function, which connects points $(0,0)$ and $(x_2, p_2(x_2))$, see Fig. 13.*

Proof. Again let $A = \frac{1}{1-a}$, and denote the slope of spline on corresponding intervals as $p_k' \in \mathbb{R}$ (for $p_k(x)$). Then from the construction scheme of $p(x)$ itself, we obtain formulae equal to those from the proof of Proposition 1:

$$p_1' = A^2 p_0^{*\prime} - p_1^{*\prime}, \qquad p_3' = A^2(p_2' + p_2^{*\prime}) - p_0^{*\prime} - p_3^{*\prime},$$
$$p_2' = A^2(p_1' + p_3^{*\prime}) - p_0^{*\prime} - p_3^{*\prime}, \qquad p_k' = A^2(p_{k-1}' + p_1^{*\prime}) - p_0^{*\prime} - p_2^{*\prime}, \quad k \geq 4.$$

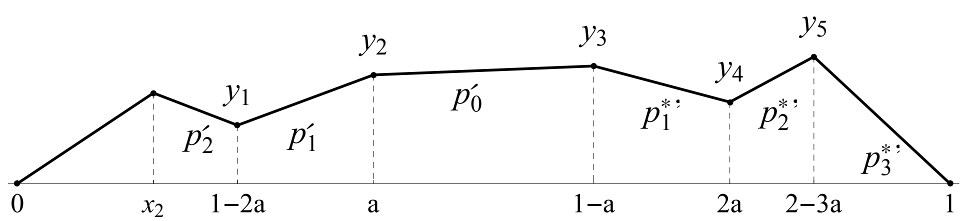

FIGURE 13. Illustration for Proposition 5.

Substituting A by A^2 in the equation (11), we get

$$p_4' - p_3' = (A^4 - A^2 - 1)(A^2(p_1' + p_3^{*\prime}) + (p_2^{*\prime} - p_3^{*\prime})). \tag{15}$$

We need to show that the last multiplier equals zero. Let $y_1 = p(1-2a)$, $y_2 = p(a)$, $y_3 = p(1-a)$, $y_4 = p(2a)$, $y_5 = p(2-3a)$. We use equalities (12) again:

$$p_2^{*\prime} - p_3^{*\prime} + A^2(p_1' + p_3^{*\prime}) = \tfrac{y_5 - y_4}{2-5a} - \tfrac{y_5}{1-3a} + A^2\left(\tfrac{y_2 - y_1}{3a-1} + \tfrac{y_5}{1-3a}\right) =$$
$$= \tfrac{1}{(2-5a)(1-3a)}\big((1-3a)(y_5 - y_4) - (2-5a)y_5 + A^2(2-5a)(y_1 - y_2 + y_5)\big) =$$
$$= \tfrac{1}{(2-5a)(1-3a)}\big(y_5(2a - 1 + A^2(2-5a)) - y_4(1-3a) + A^2(2-5a)(y_1 - y_2)\big) =$$
$$= \tfrac{1}{(2-5a)(1-3a)}\big(-A(y_2 - y_1)(1-3a) + A^2(2-5a)(y_1 - y_2)\big) =$$
$$= \tfrac{1}{(2-5a)(1-3a)}(y_1 - y_2)\big((1-3a)A + A^2(2-5a)\big) =$$
$$= \tfrac{1}{(2-5a)(1-3a)}(y_1 - y_2)\left((1-3a)A + (A+1)(2-5a)\right).$$

Taking into account (12), we have

$$(1-3a)A + (A+1)(2-5a) = A(1 - 3a + 2 - 5a) + 2 - 5a =$$
$$= A(3-8a+(1-a)(2-5a)) = A(3-8a+2-7a+5a^2) = 5A(a^2-3a+1) = 0.$$

Then $p_{k+1}' - p_k' = A(p_k' - p_{k-1}') = \ldots = A^{k-3}(p_4' - p_3') = 0$, $k \geq 4$, q.e.d.

Since the second statement of Proposition 3 holds true (by the construction scheme of the spline), function p is continuous on $(0,1]$. Since it is linear on $(0, x_2]$, then limit $p(0+)$ exists, and by Proposition 4, $p(0) = p(0+) = 0$. □

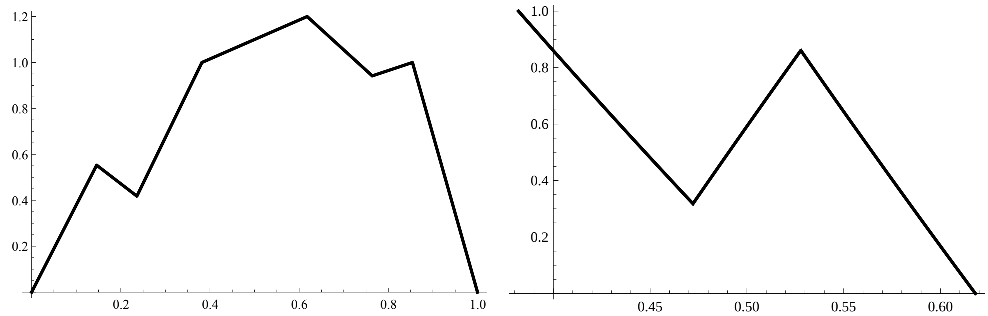

FIGURE 14. Example of spline density $p(x)$ from Proposition 5 (left). Here $y_2 = 1.1, y_3 = 1.2, y_5 = 1,\ y_4 = \frac{1}{1-a}y_3 - y_2$. Corresponding function α_1 (on $[a, 1-a]$) (right).

Fig. 14 shows an example of non-trivial density discussed in Proposition 5. Obviously such function is integrable. To accomplish, we add non-negativity criterion.

Proposition 6. *Let $p(x)$ satisfy the conditions of Proposition 5. Denote $y_3 = p(1-a), y_4 = p(2a), y_5 = p(2-3a)$. Then $p(x) \geq 0$ for all $x \in [0,1]$ if and only if*

$$\begin{cases} y_3 \geq y_4 \geq 0, \\ y_5 \geq \frac{11a-4}{2-5a}y_4 - \frac{29a-11}{8a-3}y_3 \qquad (\frac{11a-4}{2-5a} \approx 2.24, \frac{29a-11}{8a-3} \approx 1.38), \\ y_5 \geq 0. \end{cases}$$

These inequalities define unbounded convex set in $\mathbb{R}^3 \ni \{y_3, y_4, y_5\}$.

Proof. Let $y_0 = p(x_2), y_2 = p(a)$. In view of Proposition 5, this statement is equivalent to non-negativeness of spline values $p(x)$ at the vertices: $y_k \geq 0$, $k = 0, \dots, 5$.

Let $y_1 = p(1-2a)$, and substitute $\tilde{x} = 1-2a$ into (10). Then using $\frac{1-2a}{1-a} = a$, we have

$$y_1 + y_3 = \frac{1}{1-a}y_2.$$

We use expression (12) to simplify the quantities henceforth:

$$y_1 = \frac{1}{1-a}y_2 - y_3 = \frac{1}{1-a}(\frac{1}{1-a}y_3 - y_4) - y_3 = ((\frac{1}{1-a})^2 - 1)y_3 - \frac{1}{1-a}y_4 = \frac{1}{1-a}(y_3 - y_4) \geq 0,$$

wherefrom $y_1 \geq 0$ if and only if $y_3 \geq y_4$.

Further, $y_2 = \frac{1}{1-a}y_3 - y_4 \geq y_3 - y_4 \geq 0$ ($\frac{1}{1-a} \approx 1.6$), thus condition $y_2 \geq 0$ holds true if $y_3 \geq y_4$.

To get the last restriction of the proposition, we consider an equality

$$y_0 + p(x_2 + a) + p(x_2 + 2a) = \frac{1}{1-a}(y_1 + y_5).$$

Since $x_2 + a \in [a, 1-a]$, then

$$p(x_2 + a) = y_2 + (y_3 - y_2)\tfrac{x_2+a-a}{1-a-a} = y_2 + (1-a)(y_3 - y_2).$$

Since $x_2+2a = (1-2a)(1-a)+2a = 2a^2-a+1 = 5a-1 \approx 0.91 > 2-3a \approx 0.85$, then $x_2 + 2a \in [2-3a, 1]$, and

$$p(x_2 + 2a) = y_5 - y_5\tfrac{5a-1-2+3a}{1-2+3a} = \frac{2-5a}{3a-1}y_5.$$

Thus

$$\begin{aligned} y_0 = \tfrac{1}{1-a}(y_1 + y_5) - y_2 - (1-a)(y_3 - y_2) - \tfrac{2-5a}{3a-1}y_5 = \\ = \tfrac{1}{1-a}(\tfrac{1}{1-a}y_2 - y_3) - y_3(1-a) - ay_2 + (\tfrac{1}{1-a} - \tfrac{2-5a}{3a-1})y_5 = \\ = ((\tfrac{1}{1-a})^2 - a)y_2 - y_3(\tfrac{1}{1-a} + 1 - a) + \tfrac{2-5a}{(3a-1)(1-a)}y_5 = \\ = (\tfrac{1}{1-a} + 1 - a)(y_2 - y_3) + \tfrac{2-5a}{(3a-1)(1-a)}y_5. \end{aligned}$$

Finally, $y_0 \geq 0$ if and only if $-\frac{2-5a}{(3a-1)(1-a)}y_5 \leq (\frac{1}{1-a} + 1 - a)(y_2 - y_3)$, which leads to $y_5 \geq \frac{11a-4}{2-5a}(y_3 - y_2)$ $(2-5a, 3a-1, 1-a > 0)$, and equals to

$$y_5 \geq \tfrac{11a-4}{2-5a}(y_3 - y_2) = \tfrac{11a-4}{2-5a}(y_3 - \tfrac{1}{1-a}y_3 + y_4) = \tfrac{11a-4}{2-5a}y_4 - \tfrac{29a-11}{8a-3}y_3.$$

□

Notice that functions in Fig. 12 differ a little from those in Fig. 10 and 14: $p_3^*(x)$ is slightly changed in both cases. Such change leads to great oscillation and (or) unboundedness of $p(x)$. We formulate here the following questions.

Question 4. *Explain such "bad" behavior of function $p(x)$. Are there locally non-linear densities (for them we need to check conditions* (9)–(10)*, non-negativeness and integrability of $p(x)$)?*

Question 5. *How can we provide $\alpha_1(x) \in [0, 1]$ in cases above ($\alpha_1(x)$ is derived from equation* (4)*)?*

In conclusion, consider the case when α is an endowment by characteristic functions of sets of IFS (6) mask. Let $\mathcal{M} = \{M_1, M_2\}$ be the IFS mask: $[0, a) \subset M_1$ and $(1-a, 1] \subset M_2$, $M_1' = M_1 \cap [a, 1-a]$ and $M_2' = M_2 \cap [a, 1-a]$ are arbitrary ($M_1 \cap M_2 = \emptyset$, $M_1, M_2 \in \mathfrak{B}$).

Let $\alpha_i(x) = \chi_{M_i}(x)$, $i = 1, 2$. Condition (4) for $n = 2$ turns into

$$\alpha_1(x) = \tfrac{1}{p(x)}(p(x-a) + p(x) - \tfrac{1}{1-a}p(\tfrac{x-a}{1-a})), \qquad \forall x \in [a, 1-a). \tag{16}$$

On the set M_1, $\alpha_1(x) \equiv 1$, and (16) implies

$$p(x-a) = \tfrac{1}{1-a}p(\tfrac{x-a}{1-a}), \qquad x \in [a, 1-a) \cap M_1.$$

Similarly, on the set M_2, $\alpha_1(x) \equiv 0$, and (16) yields

$$p(x-a) + p(x) = \tfrac{1}{1-a} p(\tfrac{x-a}{1-a}), \qquad x \in [a, 1-a) \cap M_2.$$

One can see that condition (7) splits into two (as sketched in Fig. 15):

$$\text{on } [a, 1-a) \cap M_1 \quad \begin{cases} p(x-a) = \tfrac{1}{1-a} p(\tfrac{x-a}{1-a}), \\ p(x) + p(x+a) = \tfrac{1}{1-a} p(\tfrac{x}{1-a}); \end{cases} \tag{17}$$

$$\text{on } [a, 1-a) \cap M_2 \quad \begin{cases} p(x-a) + p(x) = \tfrac{1}{1-a} p(\tfrac{x-a}{1-a}), \\ p(x+a) = \tfrac{1}{1-a} p(\tfrac{x}{1-a}). \end{cases} \tag{18}$$

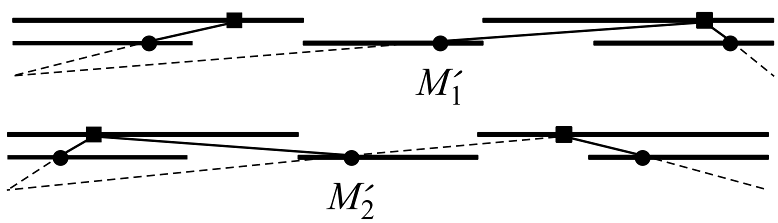

FIGURE 15. Scheme of conditions (17) (above) and (18) (below).

Thus we can introduce the following scheme of construction $p(x)$, which is slightly changed version of the one above.

Consider the following marks on the X-axis: $x_k = a(1-a)^k$, $x^*_k = 1-a(1-a)^k$, $k \geq 1$.

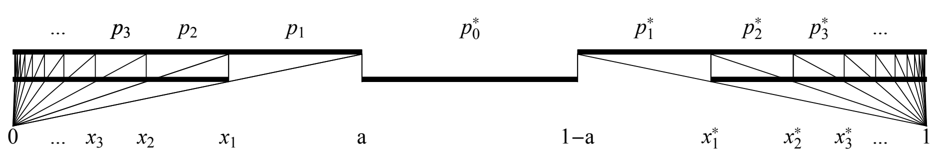

FIGURE 16. Construction scheme for density $p(x)$, extra intervals marked, $a = \frac{3-\sqrt{5}}{2}$, case of $n = 2$.

- Fix functions $p^*_0, p^*_1 \in L^1$, $p^*_0, p^*_1 \geq 0$, arbitrarily, and define

$$p(x) = \begin{cases} p_1(x) = \tfrac{1}{1-a} p^*_0(\tfrac{x}{1-a}) - p^*_1(x+a), & x \in (1-2a, a], \\ p^*_0(x), & x \in (a, 1-a], \\ p^*_1(x), & x \in (1-a, 2a]. \end{cases}$$

- By induction on $k \geq 2$, define

$$p(x) = p_k(x) = \begin{cases} \tfrac{1}{1-a} p_{k-1}(\tfrac{x}{1-a}), & x \in (x_k, x_{k-1}], x+a \in M'_1, \\ \tfrac{1}{1-a} p_{k-1}(\tfrac{x}{1-a}) - p^*_0(x+a), & x \in (x_k, x_{k-1}], x+a \in M'_2. \end{cases}$$

- By induction on $k \geq 2$, define

$$p(x) = p_k^*(x) = \begin{cases} \frac{1}{1-a} p_{k-1}^*(\frac{x-a}{1-a}), & x \in (x_{k-1}^*, x_k^*], x - a \in M_2', \\ \frac{1}{1-a} p_{k-1}^*(\frac{x-a}{1-a}) - p_0^*(x-a), & x \in (x_{k-1}^*, x_k^*], x - a \in M_1'. \end{cases}$$

- Fix values $p(0), p(1) \geq 0$ arbitrarily.

Thus function $p(x)$ is completely defined by its values on $(1-2a, 1-a]$, which are defined by functions p_0^* and p_1^*.

Here next question arises.

Question 6. *Under which conditions $p(x) \in L^1$, $p(x) \geq 0$? Is it possible to construct such function for any mask $\mathcal{M}$?*

The examples of two masks (see Fig. 17) are the partial answer to it. In these examples masks $\mathcal{M}$ are connected with partition structure of interval $[0,1]$ over iteration process of density construction. Namely, if $M_1 = [0, 4a-1)$, $M_2 = [4a-1, 1]$ $(4a - 1 = x_2 + a)$, $p_1 = c \geq 0$, $p_0 = \frac{1}{1-a}c$, then one can show that $p_1^* = \frac{1}{1-a}c$, $p(x) \equiv 0$ outside $[1-2a, x_2^*]$. If $M_1 = [0, a) \cup [4a-1, 1-a]$, $M_2 = [a, 4a-1) \cup (1-a, 1]$, $p_1 = p_0 = c$, then $p_1^* = (1-a)c$, $p_2^* = c$, $p(x) \equiv \frac{1}{1-a}c$ outside $[1-2a, x_2^*]$.

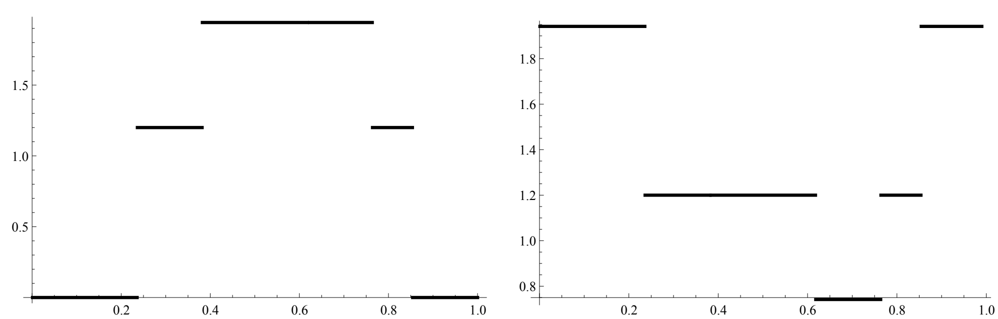

Figure 17. Graphs of invariant measure densities with mask endowment $(n = 2)$ (see text): $M_1 = [0, 4a-1)$, $M_2 = [4a-1, 1]$, $c = 1.2$ (left) and $M_1 = [0, a) \cup [4a-1, 1-a]$, $M_2 = [a, 4a-1) \cup (1-a, 1]$, $c = 1.2$ (right).

We haven't found an example of density $p(x)$ for arbitrary mask (for instance, that in Fig. 7): the function constructed had negative values, unbounded and (or) oscillated greatly.

3. The case of $\mathbf{n > 2}$

In conclusion of the paper, we introduce one of the possible construction schemes for density $p(x)$ for all $n \geq 2$ and any a,

$$\frac{1}{n+1} < a \leq \frac{1}{n}. \tag{19}$$

Consider the following variables (see Fig. 18):

$$x_k = (1-na)(1-a)^{k-1}, \quad k \in \mathbb{Z}.$$

Lemma 2. *There exists a unique number $K \in \mathbb{Z}$, $K \leq 1$, such that $x_K < a$, $x_{K-1} \geq a$.*

Proof. Since $x_k = (1-a)x_{k-1}$, it is sufficient to consider the chain of inequalities:

$$(1-a)a \leq (1-na)(1-a)^{k-1} < a,$$

$$\frac{(1-a)a}{1-na} \leq (1-a)^{k-1} < \frac{a}{1-na},$$

$$\log_{1-a}\frac{a}{1-na} + 1 \geq k-1 > \log_{1-a}\frac{a}{1-na}. \tag{20}$$

According to (19), we have $\frac{a}{1-na} > 1$, hence $\log_{1-a}\frac{a}{1-na} < 0$. Then (20) completes the proof. □

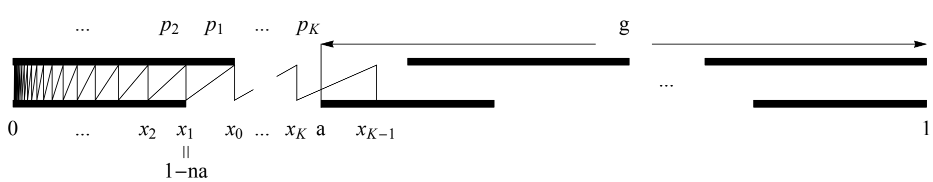

Figure 18. Construction scheme for density $p(x)$ in the case of arbitrary $n \geq 2$.

Now we introduce the following scheme to construct $p(x)$, with equations (2)–(3) satisfied.

- Fix $g \in L^1$, $g \geq 0$, arbitrarily, and define

$$p(x) = g(x) \text{ on } (a, 1].$$

- For $k = K$, define (see formula (3))

$$p(x) = p_K(x) = \tfrac{1}{1-a}\left(g(\tfrac{x}{1-a}) + \sum_{i=1}^{n-2} g(\tfrac{x+ia}{1-a})\right) - \sum_{i=1}^{n-1} g(x+ia), \quad x \in (x_K, a].$$

- By induction on $k = K+1, \dots, 1$, define (see (3))

$$p(x) = p_k(x) = \tfrac{1}{1-a}\left(p_{k-1}(\tfrac{x}{1-a}) + \sum_{i=1}^{n-2} g(\tfrac{x+ia}{1-a})\right) - \sum_{i=1}^{n-1} g(x+ia), \quad x \in (x_k, x_{k-1}].$$

- By induction on $k = 0, 1, \dots$, define (see (2))

$$p(x) = p_k(x) = \tfrac{1}{1-a}\left(p_{k-1}(\tfrac{x}{1-a}) + \sum_{i=1}^{n-1} g(\tfrac{x+ia}{1-a})\right) - \sum_{i=1}^{n} g(x+ia), \quad x \in (x_k, x_{k-1}].$$

- Fix value $p(0) \geq 0$ arbitrarily.

Here the following question appears.

Question 7. *Under which conditions $p(x) \in L^1$, $p(x) \geq 0$? Under which conditions $\alpha_1(x) \in [0,1]$ ($\alpha_1(x)$ is derived from (4))?*

Obvious "mirror" change of this scheme is shown in Fig. 19 (replacing x_k by $x_k^* = 1 - x_k$), compare with lemma 4.

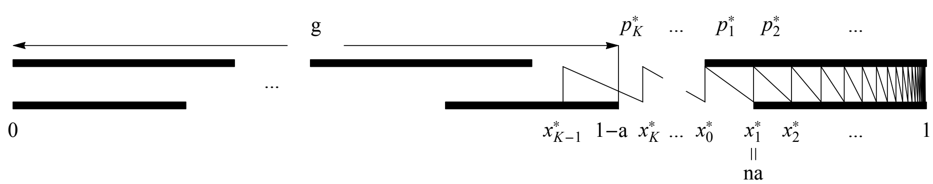

FIGURE 19. "Mirror" construction scheme of density $p(x)$ in the case of arbitrary $n \geq 2$.

4. Conclusion

Section 1 contains motivation part. It overviews previously derived criteria of measure invariance and some related results, as well as connection between endowment and mask. In Section 2 we consider the case of $a = \frac{3-\sqrt{5}}{2}$, and example of mask is given. Section 3 introduces construction scheme for densities with arbitrary $a \in (0, \frac{1}{2}]$.

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