KAZAN (VOLGA REGION) FEDERAL UNIVERSITY

N.I. LOBACHEVSKII INSTITUTE OF MATHEMATICS AND MECHANICS

Chair of mathematical analysis

BORIS A. KATS

CONTEMPORARY THEORY OF THE RIEMANN BOUNDARY VALUE PROBLEM ON NON-RECTIFIABLE CURVES AND RELATED QUESTIONS

Kazan 2015

UDK 517.544

The paper is confirmed by the chair of mathematical analysis May 14 2015, protocol no 8.
Reviewer doctor of science, professor L.A. Aksentiev

Boris A. Kats Contemporary theory of the Riemann boundary value problem on non-rectifiable curves and related questions / Boris A. Kats. - Kazan: Kazan Univ., - 2015. - 20 p.

The work is dealing with the Riemann boundary value problem for analytic functions. In the classical monographs the boundary in formulation of the problem is assumed piecewise-smooth, and technique of its solving is based on this assumption. But the Riemann problem keeps its sense for non-rectifiable curves, where customary methods lose their applicability. This obstacle was overcome by the author in early eighties. The present paper is a brief review of three decades of development of theory of the Riemann boundary value problem on non-rectifiable curves and related questions.

Keywords: fractal dimensions, analytic functions, boundary value problems.

AMS subject classification: Primary 30E20, Secondary 30E25.

1 Preliminaries.

We are dealing with the Riemann boundary value problem for analytic functions. In the well known monographs [18, 26, 49, 52] a reader can find the classical results on this problem for piecewise smooth curves and arcs. The technique for solving of this problem bases on the piecewise smoothness of the boundary. But the Riemann problem itself keeps the sense for non-rectifiable curves, where customary methods lose their applicability. This obstacle was overcome by the author of the present paper in early eighties. We will describe the three decades of development of this branch of complex analysis. But first we have to elucidate the basic results of classical researches.

Let D^+ be a finite domain on the complex plane $\mathbb{C}$ with Jordan boundary Γ, and $D^- = \overline{\mathbb{C}} \setminus D^+$. Let functions $G(t)$ and $g(t)$ be defined on Γ. We seek a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

Here $\Phi^+(t)$ and $\Phi^-(t)$ are boundary values of restrictions of desired function $\Phi(z)$ onto domains D^+ and D^- correspondingly; the formulation of the Riemann problem assumes that these boundary values exist. The jump problem

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma, \tag{2}$$

and the homogeneous Riemann problem

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma, \tag{3}$$

are important cases of the Riemann boundary value problem. The customary method for solving of the problems (see [18], [49]) bases on the properties of so called Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{g(t)dt}{t-z}, \quad z \notin \Gamma. \tag{4}$$

We describe these properties for the case where g satisfy so called Hölder condition

$$\sup\left\{\frac{|g(t') - g(t'')|}{|t'-t''|^\nu} : t', t'' \in \Gamma, t' \neq t''\right\} := h_\nu(g,\Gamma) < \infty \tag{5}$$

with exponent $\nu \in (0,1]$. Below we use notation $H_\nu(\Gamma)$ for set of all defined on Γ functions satisfying this condition. As known (see, for instance, [18, 49]), for piecewise smooth contour Γ and $g \in H_\nu(\Gamma)$, $\nu \in (0,1]$, the equality (4) gives holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$, which has boundary values from the left and from the right $\Phi^+(t)$ and $\Phi^-(t)$ at any point $t \in \Gamma$, and both these boundary functions satisfy the Hölder condition with the same exponent ν for $\nu \in (0,1)$ and with any exponent $\mu < 1$ for $\nu = 1$. Moreover, by virtue of so called Sokhotskii-Plemely formulas (see, for instance, [18, 49]) the function Φ satisfies (2), i.e., the Cauchy integral (4) is a solution of the jump problem (2) if $g \in H_\nu(\Gamma)$, $\nu \in (0,1]$. This is a reason for interest to properties of the Cauchy type integral in the theory of boundary value problems.

The problem on continuity the Cauchy integral over non-smooth rectifiable curve on this curve was investigated during several decades. Finally, for the Hölder densities it was solved in 1979 simultaneously by E.M. Dyn'kin [15, 16] and T. Salimov [50]. They proved that the Cauchy integral (4) is continuous in closures of domains D^+ and D^- if $g \in H_\nu(\Gamma)$ for $\nu > \frac{1}{2}$, and the boundary values $\Phi^\pm(t)$ satisfy the Hölder condition with any exponent lesser than $2\nu - 1$. Furthermore, the following propositions are valid:

– for arbitrarily fixed $\nu \in (0, \frac{1}{2})$ one can find a rectifiable curve and a function $f \in H_\nu(\Gamma)$ such that the Cauchy integral (4) loses its continuity on Γ;

– for any $\nu \in (\frac{1}{2}, 1]$ one can build a rectifiable curve Γ and a function $g \in H_\nu(\Gamma)$ such that $\Phi^\pm \notin H_\mu(\Gamma)$ for $\mu \geq 2\nu - 1$.

Thus, the condition $g \in H_\nu(\Gamma)$, $\nu > \frac{1}{2}$, is sharp sufficient condition for solvability of the jump problem (2) on a non-smooth rectifiable curve Γ.

The recent achievements in research of boundary properties of the Cauchy type integral over curves of various classes are described in survey [5].

Clearly, the cited above formulation of the Riemann boundary value problem takes a sense for any Jordan curve including the non-rectifiable ones. At the same time, curvilinear integrals are defined for rectifiable contours only. This obstacle was overcome in 1982 by the author. The present paper is a review of the contemporary results in this field.

In the next two sections we consider two main approaches to the Riemann boundary value problem for non-rectifiable curves developed in the three last decades. Then we formulate certain open problems.

2 Quasi-solutions.

The first solution of the problems (1), (2) and (3) on non-rectifiable curves and arcs bases on the following idea (see [28, 30, 31]). We build a quasi-solution, i.e., a differentiable (but not analytic) in $\mathbb{C} \setminus \Gamma$ function φ satisfying one of these boundary relations. Then we act on it by appropriate integral-differential operator, which turn it into holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying corresponding boundary relation. This method is known as the method of regularization of quasi-solutions. We illustrate it first on example of the jump problem.

2.1 Jump problem.

Let a jump $g(t)$ have a continuation $g^e(z)$ into the whole complex plane, i. e., the function g^e is continuous in $\mathbb{C}$ and $g^e|_\Gamma = g$. Then the products $\phi_1(z) := g^e(z)\chi^+(z)$ and $\phi_2(z) := g^e(z)(\chi^+(z) - 1)\psi(z)$ are quasi-solutions of problem (2), if χ^+ is characteristic function of domain D^+ and $\psi \in C_0^\infty(\mathbb{C})$ equals to 1 on Γ and has compact support. Both quasi-solutions have compact support, too. We put

$$\Psi_{1,2}(z) := \frac{1}{2\pi i} \iint\limits_{D^+} \frac{\partial \phi_{1,2}}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z} \tag{6}$$

The properties of integral operator

$$Tf := \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{f(\zeta)d\zeta d\overline{\zeta}}{\zeta - z} \tag{7}$$

can be found in well known monographs (see, for instance, [54, 25]). If a fuction $f \in L^p$ with compact support is integrable and $p > 2$, then Tf satisfies the Hölder condition with exponent $1 - \frac{2}{p}$ in the whole complex plane, and $\frac{\partial Tf}{\partial \overline{z}} = f(z), \quad z \in \mathbb{C}$.
Thus, the differences

$$\Phi_{1,2}(z) := \phi_{1,2}(z) - \Psi_{1,2}(z) = \phi_{1,2}(z) - T\frac{\partial \phi_{1,2}}{\partial \overline{\zeta}}$$

are solutions of the jump problem (2). We will see soon that these solutions coincide under easy restrictions.
The operator $R := I - T\overline{\partial}$ is regularizator of quasi-solutions for the jump problem; here I is identical operator.

Theorem 1 *The the jump problem (2) is solvable if the jump has an extension into the whole complex plane such that the first partial derivatives of the extension are integrable a degree $p > 2$.*

In order to extend appropriate extension we apply the Whitney extension operator $\mathcal{E}_0$ (see [53], ch.1, [25], ch.2) for the set Γ. If $g \in H_\nu(\Gamma)$, then its Whitney continuation $g^w := \mathcal{E}_0 g$ belong to the Hölder space $H_\nu(\mathbb{C})$ with the same exponent ν, $g^w|_\Gamma = g$, and in $\mathbb{C}\setminus\Gamma$ the function $g^w(z)$ has partial derivatives. In addition, these derivatives satisfy inequality

$$\left|\frac{\partial^{k+m}\mathcal{E}_0 g(z)}{\partial x^k \partial y^m}\right| \le \frac{h_\nu(g,\Gamma)}{\mathrm{dist}^{k+m-\nu}(z,\Gamma)}, \quad z = x + iy. \tag{8}$$

It remains to find exponent of integrability of $\frac{\partial g^w}{\partial \overline{z}}$. For $\nu = 1$ the first derivatives are bounded. For $\nu < 1$ this exponent depends on metric characteristics of the curve Γ, which are known as *fractal dimensions*. Seemingly, the oldest of them is the Minkowski dimension known also as box-counting dimension, Kolmogorov dimension and upper metric dimension (see [17, 46]). That dimension of a compact set F in a metrc space X is defined by equality

$$\dim_{\mathrm{M}} S := \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon; S)}{-\log\varepsilon},$$

where $N(\varepsilon; S)$ is the least number of balls of radius ε covering S. If X is Euclidean space, then the balls can be changed here by non-overlapping cubes of size ε. In particular, if $X = \mathbb{C}$, $\mathcal{Q}_n$ is decomposition of $\mathbb{C}$ into non-overlapping dyadic squares with sides 2^{-n} and $m_n(S)$ stands for number of all squares from $\mathcal{Q}_n$ having non-empty intersection with S, then

$$\dim_{\mathrm{M}} S := \limsup_{\varepsilon\to 0} \frac{\log_2 m_n(S)}{n}. \tag{9}$$

For any $\varepsilon > 0$ there is valid inequality $m_n \le C2^{nd}$, where $d = \dim_{\mathrm{M}} S + \varepsilon$, $C = C(\varepsilon) > 0$.

For any metric space X the upper metric dimension of its compact subset S is greater or equal to Hausdorff dimension $\dim_{\mathrm{H}} S$ (see, for instance, [13]).
In the case $X = \mathbb{C}$ this dimension has the following additional properties:
- $\dim_{\mathrm{M}} S \leq 2$ for any $S \subset \mathbb{C}$;
- the dimension of any rectifiable curve equals to 1;
- the upper metric dimension of any continuum on the complex plane is larger or equal 1;
- if $\dim_{\mathrm{M}} S < 2$ then the set S has null square.

In the papers [30, 31] a reader can find certain examples of curves and arcs with prescribed upper metric dimension.
Now let us consider so called Whitney partition of complement of a compact set $S \subset \mathbb{C}$ (see, for instance, [53]). It is a family of non-overlapping dyadic squares Q such that

$$C^{-1} \operatorname{diam} Q \leq \operatorname{dist}(Q, S) \leq C \operatorname{diam} Q, \tag{10}$$

where $C > 0$ depends on n only. Let $w_n(\Gamma)$ be number of squares with side 2^{-n} in the Whitney partition of $\mathbb{C} \setminus S$. The sequence of these numbers is a metric characteristic of S, too. It is known as Whitney cover numbers [27]. As shown in [30], $w_n(\Gamma) \leq C m_n(\Gamma)$, where C is absolute constant.
The inequalities (8) and (10) imply that the integral of function $\left|\frac{\partial \mathcal{E}_0 g}{\partial \overline{\zeta}}\right|^p$ over D^+ does not exceed $C h_\nu(g, \Gamma) \sum\limits_{n=1}^{+\infty} m_n(\Gamma) 2^{-(2-(1-\nu)p)n}$. The series converge for

$$p < \frac{2 - \dim_{\mathrm{M}} \Gamma}{1 - \nu}. \tag{11}$$

Thus, for $\nu < 1$ the exponent of integrability of the first derivatives of Whitney continuation exceed two under condition

$$\nu > \frac{1}{2} \dim_{\mathrm{M}} \Gamma. \tag{12}$$

As a result, we obtain

Theorem 2 (see [28, 30]). *Let Γ be simple closed curve of null area and $g \in H_\nu(\Gamma)$. If the exponent ν either is unit or satisfies inequality (12), then the functions*

$$\Phi_{1,2}(z) := \phi_{1,2}(z) - \frac{1}{2\pi i} \iint_{D^+} \frac{\partial \phi_{1,2}}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z}, \tag{13}$$

where $\phi_{1,2}$ are quasi-solutions

$$\phi_1(z) := g^w(z)\chi^+(z), \quad \phi_2(z) := g^w(z)(\chi^+(z) - 1)\psi(z), \tag{14}$$

are solutions of the jump problem (2). Its boundary values $\Phi^\pm(t)$ satisfy the Hölder condition on Γ with any exponent lesser than 1 for $\nu = 1$, and with any exponent μ satisfying inequality

$$\mu < \frac{2\nu - \dim_{\mathrm{M}} \Gamma}{2 - \dim_{\mathrm{M}} \Gamma} \tag{15}$$

for $\nu < 1$. The functions χ^+ and ψ are defined above.

This result is sharp [28, 30]), i.e., for any values ν and d such that $0 < \nu \le d/2 < 1$ there exist a simple closed curve and defined on this curve function $g(t)$ such that $\dim_{\mathrm{M}} \Gamma = d$, $g \in H_\nu(\Gamma)$ and the jump problem (2) has not solutions.
There arises an intrinsic conjecture that the functions Φ_1 and Φ_2 are the same. This conjecture is a special case of the question on uniqueness of solution of the jump problem. In general the answer is negative, what is essential difference between properties of jump problems for rectifiable and non-rectifiable curves.
In the case of rectifiable curve Γ any continuous in domain $\Delta \supset \Gamma$ and holomorphic in $\Delta \setminus \Gamma$ function is holomorphic in Δ by virtue of the Painleve theorem. Hence, solution of problem (2) is unique up to additive constant. But in the case where Γ has Hausdorff dimension exceeding 1 (and, consequently, it is not rectifiable), then there exists a non-trivial function which is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and continuous in $\overline{\mathbb{C}}$ (see, for instance, [14]). Hence, the difference of two solutions of the jump problem on that curve may be non-constant.
In the same paper [14] E.P. Dolzhenko obtained the following result. If a holomorphic in $\Delta \setminus \Gamma$ function satisfies in domain $\Delta \supset \Gamma$ the Hölder condition with exponent exceeding $\dim_{\mathrm{H}} \Gamma - 1$, then it is holomorphic in Δ. Thus, we can define uniqueness classes for the jump problem.
Let us denote $\mathcal{H}_\mu(\Gamma)$ the class of all holomorphic in $\mathbb{C} \setminus \Gamma$ functions $\Phi(z)$ with boundary values Φ^+ and Φ^- belonging to class $H_\mu(\Gamma)$ and vanishing at the infinity point. If $\Phi(z) \in \mathcal{H}_\mu(\Gamma)$, then $\Phi|_{D^+}$ and $\Phi|_{D^-}$ satisfy the Hölder condition with the same exponent μ in domain D^+ and in any finite part of D^- correspondingly (see [19]). Consequently, if $\mu > \dim_{\mathrm{H}} \Gamma - 1$ then solution of the jump problem in the class $\mathcal{H}_\mu(\Gamma)$ is unique. By virtue of Theorem 2 the problem is resolvable in that class if

$$\dim_{\mathrm{H}} \Gamma - 1 < \mu < \frac{2\nu - \dim_{\mathrm{M}} \Gamma}{2 - \dim_{\mathrm{M}} \Gamma}, \tag{16}$$

or $\nu = 1$, $\dim_{\mathrm{H}} \Gamma < 2$, $\dim_{\mathrm{H}} \Gamma - 1 < \mu < 1$.
Now let us return to the question on equality $\Phi_1 = \Phi_2$. Both these functions vanish at infinity point and their boundary values satisfy the Hölder condition with any exponent lesser than 1 for $\nu = 1$, and with any exponent μ satisfying inequality (16) for $\nu < 1$. Consequently, $\Phi_1 = \Phi_2$ if either $\nu = 1$ and $\dim_{\mathrm{H}} \Gamma < 2$, or $\nu < 1$ and $\dim_{\mathrm{H}} \Gamma - 1 < \dfrac{2\nu - \dim_{\mathrm{M}} \Gamma}{2 - \dim_{\mathrm{M}} \Gamma}$.

2.2 Homogeneous Riemann problem.

We assume that coefficient $G(t)$ does not vanish (as known, this is elipticity condition for the Riemann problem, see, for instance, [52])and belongs to $H_\nu(\Gamma)$. Let $\varkappa := \frac{1}{2\pi}[\arg G]_\Gamma$, where $[\arg G]_\Gamma$ is increment of $\arg G(t)$ at a single circuit of Γ in the positive direction. Clearly, $\varkappa$ is integer. As usually, we fix a point $z_0 \in D^+$ and represent G as $G(t) = (t - z_0)^\varkappa \exp f(t)$ with $f \in H_\nu(\Gamma)$. Then we cconstruct one of quasi-solutions $\phi_{1,2}$ with f instead of g, and regularize it by the operator R. For definiteness we write here result of this procedure for ϕ_1. We obtain function

$$\Upsilon(z) := (I - T\overline{\partial}) = \chi^+(z) f^w(z) - \frac{1}{2\pi i} \iint\limits_{D^+} \frac{\partial f^w}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z}$$

being a solution of the jump problem with jump $f(t)$ under assumptions of Theorem 2. Let $X(z)$ be equal to $\exp\Upsilon(z)$ for $z\in D^+$ and $X(z)=(z-z_0)^{-\varkappa}\exp\Upsilon(z)$ for $z\in D^-$ (in [18, 49] analogous construction is called canonical function). Clearly, $X^+(t)=G(t)X^-(t), t\in\Gamma$. Hence, if $\Phi(z)$ is a solution of the problem (3) in class $\mathcal{H}_\mu(\Gamma)$ with μ satisfying (16), then ratio Φ/X is holomorphic in $\mathbb{C}$ under conditions of previous subsection. As a result, we obtain

Theorem 3 *Let assumptions of Theorem 2 be fulfilled, and, addtionally, either $\nu=1$, $0<\mu<1$ and $\dim_{\mathrm{H}}\Gamma<2$, or $\nu<1$ and inequality (16) is valid. Then any solution of problem (3) in class $\mathcal{H}_\mu(\Gamma)$ is representable as $\Phi(z)=X(z)P(z)$, where $P(z)$ is an algebraic polynomial of degree no more than $\varkappa$ for $\varkappa\geq 0$, and it is identical zero for negative $\varkappa$.*

2.3 Inhomogeneous Riemann problem.

Let G be the same and $g\in H_\nu(\Gamma)$. The problem (1) has at least two obvious quasi-solutions with compact supports: $\phi_1(z):=\chi^+(z)g^w(z)$ and $\phi_2(z):=(\chi^+(z)-1)\psi(z)(g/G)^w(z)$, where $\psi(z)$ is the same smooth function as above. Operator $I-XTX^{-1}\overline{\partial}$ regularizes these quasi-solutions, and, consequently, function

$$\Phi_0:=(I-XTX^{-1}\overline{\partial})\phi$$

is holomorphic in $\mathbb{C}\setminus\Gamma$ and satisfies the boundary condition (1). Here ϕ is one of functions $\phi_{1,2}$. If $\varkappa\geq 0$, then Φ_0 is regular at infinity point. But for $\varkappa<0$ it can have a pole at this point. As

$$\iint\limits_{\mathbb{C}}\frac{\partial\phi}{\partial\overline{\zeta}}\frac{d\zeta d\overline{\zeta}}{X(\zeta)(\zeta-z)}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\iint\limits_{\mathbb{C}}\frac{\partial\phi}{\partial\overline{\zeta}}\frac{\zeta^n d\zeta d\overline{\zeta}}{X(\zeta)}$$

in a neighborhood of the point at infinity, then Thus, Φ_0 is regular there $\varkappa<0$ under conditions

$$\iint\limits_{\mathbb{C}}\frac{\partial\phi}{\partial\overline{\zeta}}\frac{\zeta^n d\zeta d\overline{\zeta}}{X(\zeta)}=0,\quad n=0,1,2,\dots,-\varkappa-2. \tag{17}$$

Thus, we have proved

Theorem 4 (see [28, 30]). *Let Γ be simple closed curve, $G,g\in H_\nu(\Gamma)$, $G(t)$ does not vanish on Γ, and we solve the problem (1) in the class $\mathcal{H}_\mu(\Gamma)$. In addition, let one of the following two groups of assumptions be valid:*
(a) $\nu=1$ and $1>\mu>\dim_{\mathrm{H}}\Gamma-1$;
(b) $0<\nu<1$ and the restrictions (12), (15) are fulfilled.
Then the following propositions fulfil:
(i) for $\varkappa\geq 0$ general solution of the problem is $\Phi=\Phi_0+XP$, where P is arbitrary algebraic polynomial of degree no more than $\varkappa$;
(ii) for $\varkappa=-1$ the problem has unique solution Φ_0;
(iii) for $\varkappa<-1$ the problem is solvable if and only if $-\varkappa-1$ conditions (17) fulfil; in the later case Φ_0 is its unique solution.

This result was proved first in [30]. Then it was published in numerous surveys; see, for instance, [42].
The stability of solutions the Riemann problem on non-rectifiable curves was studied by Liu Hua [48].

2.4 Other metric characteristics.

Now there is known a number of other metric characteristics of sophisticated sets, for instance, Assouad and Aikawa dimensions and co-dimensions, Minkowski and Hausdorff contents and so on (see, for instance, [27]). Here we discuss certain characteristics of that kind, which allow to refine the obtained above results.
Summability. This concept is introduced in the paper [22]. A compact set S is called $d-$summable if

$$\int_0 N(\varepsilon; S)\varepsilon^{d-1}d\varepsilon < \infty.$$

As above, $N(\varepsilon; S)$ stands for the least number of disks of radius $\varepsilon > 0$ covering S. If a curve Γ is $d-$summable then $\dim_{\mathrm{M}} \Gamma \leq d$, and if $\dim_{\mathrm{M}} \Gamma < d$ then Γ is $d-$summable (see [22]). R. Abreu Blaya, J. Bory Reyes, and J. Marie Vilaire [10] prove that this concept is applicable for proof of solvability of the jump problem. Namely, the jump problem (2) with $g \in H_\nu(\Gamma)$ is resolvable if Γ is $d-$summable and $\nu > d/2$. R. Abreu Blaya, J. Bory Reyes and T. Moreno Garcia [6] establish that the later condition is sharp in the same sense as the condition (12).
New dimensions of decomposition type. There are introduced a number characteristics of dimensional type defined in terms of decompositions of the set $\mathbb{C} \setminus \Gamma$ into family of "well" subsets. The first characteristic of that type is the Whitney cover numbers. Then J. Harrison and A. Norton [22] defined so called $d-$mass. Let $\mathbf{Q} = \{Q_1, Q_2, \dots\}$ be a decomposition of D^+ into non-overlapping squares. If $a(Q_j)$ stand for side of square Q_j, then $d-$mass of the decomposition is sum $\mathcal{M}_d(\mathbf{Q}) = \sum_{Q_j \in \mathbf{Q}} a^d(Q_j)$ (it may be infinite). The $d-$mass of domain D^+ is the most lower bound of $d-$masses of all square decompositions of D^+. By virtue of the Peter Jones lemma (see [21]), if there exists a decomposition with finite $d-$mass then the Whitney decomposition of D^+ has finite $d-$mass, too. As shown in [10], the jump problem on boundary of a domain of finite $d-$mass is solvable for $\nu > d/2$ both for analytic and for hyperanalytic functions.
Another two metric characteristics of decomposition type are introduced by the author [37, 39, 40]. Here we describe these dimensions briefly.
We cite first the definition of approximative dimension from [39, 40]. Let us denote by $\Lambda(B)$ length of rectifiable boundary of domain B is a domain with rectifiable boundary, and by $\delta(B)$ diameter of the most disk lying inside it. We consider all possible families $\mathbf{B} = \{B_1, B_2, \dots\}$ of non-overlapping domains with rectifiable boundaries such that $\bigcup B_j = D^+$, and put $\mathcal{M}^*_d(\mathbf{B}) := \sum_{B_j \in \mathbf{B}} \Lambda(B_j)\delta^{d-1}(B_j)$, $\mathcal{M}^*_d(D^+) := \inf\{\mathcal{M}^*_d(\mathbf{B})\}$, where the least upper bound is taken for all families $\mathbf{B}$. The definition of the inner approximative dimension of the curve Γ is

$$\dim_{\mathrm{a}}{}^+ \Gamma := \inf\{d : \mathcal{M}^*_d(D^+) < \infty\},$$

and outer approximative dimension $\dim_a^- \Gamma$ is defined analogously. The approximative dimension $\dim_a \Gamma$ equals to the lesser of values $\dim_a^+ \Gamma$, $\dim_a^- \Gamma$.
The definition of refined metric dimension $\dim_r \Gamma$ (see [37])is similar to cited above one, but here we replace the families $\mathbf{B}$ by rectifiable chains, i.e., by sequences of domains with rectifiable boundaries $\mathbf{B} = \{B_1, B_2, \dots, B_n, \dots\}$ such that the signs of operations $\cup$ and $\setminus$ can be placed between members of sequence $\mathbf{B}$ so that the limit result of these operations is D^+ or D^-. Clearly, $\dim_r \Gamma \leq \dim_a \Gamma$.
All proved above theorems allow replacement of $\dim_M \Gamma$ in condition (12) by one of these dimensions. As shown in [37, 39, 40], dimensions do not exceed $\dim_M \Gamma$, and there exists curves Γ such that $\dim_a \Gamma < \dim_M \Gamma$ and $\dim_r \Gamma < \dim_M \Gamma$. Thus, the replacement of the Minkowski dimension by the refined metric dimension or by the approximative dimension sharpens these theorems. But evaluation of these new characteristics is rather complicated, and we do not know the exact values of $\dim_a \Gamma$ or $\dim_r \Gamma$ for any non-trivial curve Γ.
Marcienkiewicz exponents. Recently D.B. Katz [43, 44] introduced new characteristics of non-rectifiable curves - so called Marcienkievicz exponents. These characteristics also allow us to sharpen the solvability conditions for the Riemann problem, but they admit rather simple evaluation for a number of non-rectifiable curves.
We can define the Marcienkiewicz exponents a a metric measure space $X = (X; d; \mu)$ be equipped with a metric d and a Borel doubling regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x; r) = \{y \in X : d(y; x) < r\}, x \in X, r > 0$. For any compact subset E of a fixed open domain $Y \subset X$ we put

$$I_p(E, t, r, \mu) := \int\limits_{B(t,r)\setminus E} \frac{d\mu}{\text{dist}^p(z, E)},$$

where $t \in E$, and define the local Marcinkiewicz exponent of set E with respect to measure μ as the least upper bound of set $\{p : \lim\limits_{r\to 0} I_p(E, t, r, \mu) < \infty\}$. We denote it $\mathfrak{m}(E, t, \mu)$.
If Γ is a closed curve on the complex plane, then its inner and outer local Marcinkiewicz exponents are equal to $\mathfrak{m}^+(\Gamma, t) := \mathfrak{m}(\Gamma, t, \mu^+)$ and $\mathfrak{m}^-(\Gamma, t) := \mathfrak{m}(\Gamma, t, \mu^-)$, where $\mu^\pm$ are restrictions of the plane Lebesgue measure on domains $D^\pm$, and $\mathfrak{m}^*(\Gamma; t) := \max\{\mathfrak{m}^+(\Gamma; t), \mathfrak{m}^-(\Gamma; t)\}$.
As shown in [43, 44, 45], all proved above theorems keep their validity if we replace there the condition (12) by

$$\nu > 1 - \frac{1}{2}\,\mathfrak{m}^*(\Gamma; t), \quad t \in \Gamma. \tag{18}$$

In the same papers D.B. Katz proves that $\mathfrak{m}^*(\Gamma; t) \geq 2 - \dim_M(\Gamma)$ for any curve Γ, and this inequality is strict for certain curves. Thus, this replacement also sharpens our existence theorems.

3 Generalizations of curvilinear integral.

The Riemann boundary value problem on non-rectifiable curves was solved in the preceding section was solved without application of curvilinear integrals. At the same time, it is of interest to define the operation of integration over non-rectifiable paths and to apply it for solving of the Riemann problem. This operation is useful also for theory of elasticity (see, for instance, [12]). In this section we consider certain intrinsic approaches to building of that generalized integral, and its application.

3.1 Stieltjes approach.

One of customary definitions of curvilinear integral $\int_\Gamma f(z)dz$ is based on theory of Stieltjes integral. Indeed, let $z = z(t); [0,1] \mapsto \Gamma$ be homeomorphism of the segment $[0,1]$ onto curve Γ. Then $\int_\Gamma f(z)dz = \int_0^1 f(z(t))dz(t)$, where the right side is Stieltjes integral. If the mapping $z(t)$ has bounded variation, then Γ is rectifiable. But the boundedness of variation of $z(t)$ is not necessary for existence of the Stieltjes integral. It exists, for instance, for functions with bounded $\Phi-$variation (see [47]). The images of segments under mappings with bounded $\Phi-$variation are $\Phi-$rectifiable curves. Generally speaking, the class of $\Phi-$rectifiable curves contains curves, which are non-rectifiable in usual sense. Let us cite results of papers [35, 38, 36] concerning this approach to integration over non-rectifiable paths.

We fix a real increasing function $\Phi(x)$ such that $\Phi(0) = 0$. For any enumerated in positive order finite sequence of points $\tau = \{t_0, t_1, \dots, t_n\} \subset \Gamma$ we put

$$\sigma_\Phi(\tau) := \sum_{j=1}^{\infty} \Phi(|t_j - t_{j-1}|).$$

Is the set of all that sums is bounded, then we say that the curve Γ is $\Phi-$rectifiable. In particular, for $\Phi(x) = x^q, q > 1$, we call it $q-$rectifiable . For instance, the Von Koch snowflake is $\log_3 4-$rectifiable. In general, for $q > 1$ the class of $q-$rectifiable curves is wider than customary class of rectifiable curves. Ler us consider the Cauchy integral o

$$\mathcal{CS}_\Gamma f(z) = \frac{1}{2\pi i} \int\limits_\Gamma f(\xi) d\log(\xi - z), \quad \xi \in \Gamma, z \notin \Gamma,$$

over $\Phi-$rectifiable curve. The following theorems are proved in papers [35, 38, 36].

Theorem 5 *Let Γ be a closed q-rectifiable curve, $f \in H_\nu(\Gamma)$, $\nu > q - 1$ and $\nu > \frac{\dim_M \Gamma}{2}$. Then the Cauchy-Stieltjies integral $\mathcal{CS}_\Gamma f(z)$ exists and has continuous limit values on Γ satisfying relation*

$$\mathcal{CS}^+_\Gamma f(t) - \mathcal{CS}^-_\Gamma f(t) = f(t).$$

Theorem 6 *If a curve Γ is Φ-rectifiable for a convex function $\Phi(x)$ and $f \in H_1(\Gamma)$, then the Cauchy-Stieltjes integral exists and has continuous limit values satisfying the same boundary value condition under restriction*

$$\sum_{n=1}^{\infty} \phi^2(\frac{1}{n}) < \infty,$$

where ϕ is the inverse function for Φ.

These results allow us to solve the Riemann boundary value problem on $\Phi-$rectifiable curves.

3.2 Stokes integral.

If a function $u(z)$ is continuous in closure of a finite domain D^+ bounded by rectifiable curve Γ and has in D^+ integrable derivatives of the first order, then by virtue of the Stokes formula

$$\int\limits_{\Gamma} u(\zeta)d\zeta = -\iint\limits_{D^+} \frac{\partial u}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}. \tag{19}$$

For non-rectifiable path Γ then we can use the right side of equality (19) as definition of the left side. By this definition a function f, which is defined on closed non-rectifiable curve Γ, is integrable over this curve if it allows an extension $u(z)$ into domain D^+ with integrable derivatives. Under this condition integral $\int_\Gamma f(\zeta)d\zeta$ equals to $-\iint\limits_{D^+} \frac{\partial f^e}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}$, were f^e is the mentioned extension of f. Seemingly, this definition was offered first in [29]; then it was repeated and studied in the papers [34, 21, 23, 24] and others.

The Stokes integral is independent on the choice of the extension f^e in the following sense. Let $f \in H_\nu(\Gamma)$, $\nu > \dim_{\mathrm{M}} \Gamma - 1$, and f has extensions u_1 and u_2 into D^+ such that $u_{1,2}(z) \in H_\nu(\overline{D^+})$, then

$$\iint\limits_{D^+} \frac{\partial u_1}{\partial \overline{\zeta}} d\zeta d\overline{\zeta} = \iint\limits_{D^+} \frac{\partial u_2}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}.$$

The Stokes can be defined by means of rectifiable approximations of the curve Γ. For any sequence $\{\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots\}$ of simple closed polygonal paths bounding polygonal domains D_n^+, $n = 1, 2, \dots\}$ such that $D_1^+ \subset D_2^+ \subset \cdots \subset D_n^+ \subset \cdots \subset D^+$ and $\bigcup\limits_{n=1}^{+\infty} D_n^+ = D^+$ and any function $f \in H_\nu(\Gamma)$, $\nu > \dim_{\mathrm{M}} \Gamma - 1$, we have

$$\lim_{n\to+\infty} \int\limits_{\Gamma_n} f^e(\zeta)d\zeta = -\lim_{n\to+\infty} \iint\limits_{D_n^+} \frac{\partial f^e}{\partial \overline{\zeta}} d\zeta d\overline{\zeta} = -\iint\limits_{D^+} \frac{\partial f^e}{\partial \overline{\zeta}} d\zeta d\overline{\zeta},$$

i.e., the Stokes integral is the limit of customary curvilinear integrals over approximating Γ polygonal paths.

J. Harrison [24] developed essentially the method of integration over non-rectifiable path by means of its approximation by polygonal constructions (so called chainlets).

The Stokes approach allows us to represent the solutions of the Riemann boundary value problem on non-rectifiable curves in terms of the Cauchy type generalized integrals. In particular, the right side of (13) is representable as $\frac{1}{2\pi i}\int_\Gamma \frac{g(t)dt}{t-z}$, where the integration is understood in the sense of this item.

3.3 Integrations as distributions.

Recently the author proposed the following scheme (see, for instance, [2, 41]).

Any function $F(\zeta)$ on the complex plane determines distribution

$$F : C_0^\infty(\mathbb{C}) \ni \omega \mapsto \iint\limits_{\mathbb{C}} F(\zeta)\omega(\zeta)d\zeta d\overline{\zeta},$$

if the integral exists. If F is holomorphic in $\mathbb{C}\setminus\Gamma$, then support of its distributional derivative $\overline{\partial}F$ belongs to the curve Γ, i.e., it is compact. We assume also that F has limit values $F^{\pm}$ from both sides at any point $t\in\Gamma$. For rectifiable Γ we have

$$\langle\overline{\partial}F,\omega\rangle=\int\limits_{\Gamma}(F^{+}(\zeta)-F^{-}(\zeta))\omega(\zeta)d\zeta,\quad \omega\in C^{\infty}(\mathbb{C}).$$

If Γ is not rectifiable, then this distribution is a generalized integration over Γ with weight $F^{+}(\zeta)-F^{-}(\zeta)$. If function F has unit jump on Γ, then we obtain integration without weight. An example of that function for closed Jordan curve Γ is the characteristic function $\chi^{+}(z)$ of domain D^{+}; it equals to 1 in D^{+} and to 0 in D^{-}. If Γ is non-closed arc, then an example of holomorphic function with unit jump on this arc is single-valued holomorphic branch of logarithmic function

$$k_{\Gamma}(z):=\frac{1}{2\pi i}\ln\frac{z-a_2}{z-a_1},\tag{20}$$

defined in $\overline{\mathbb{C}}\setminus\Gamma$ by condition $k_{\Gamma}(\infty)=0$; here a_1 is beginning of Γ, and a_2 is its end point. Let $\mathfrak{X}$ be a Banach space with norm $\|\cdot\|_{\mathfrak{X}}$ of functions defined on Γ. We assume that it contains all restrictions on Γ of functions $f\in C^{\infty}(\mathbb{C})$ and closure $\mathfrak{Y}$ of the set of all these restrictions in $\mathfrak{X}$ satisfies condition $\mathfrak{Y}C^{\infty}(\mathbb{C})=\mathfrak{Y}$.
We assume also that

$$|\langle\overline{\partial}F,\omega\rangle|\le C\|\omega\|_{\mathfrak{X}}\tag{21}$$

for any $\omega\in C^{\infty}(\mathbb{C})$, where C is a positive constant. Then the distribution $\overline{\partial}F$ is continuable up to functional on $\mathfrak{Y}$. In turn, for any $f\in\mathfrak{Y}$ this functional determines distribution

$$\langle f\overline{\partial}F,\omega\rangle:=\overline{\partial}F(f\omega).\tag{22}$$

We keep notation $\overline{\partial}F$ for the mentioned above functional and understand all constructed distributions as integrations over Γ with weights $f(\zeta)(F^{+}(\zeta)-F^{-}(\zeta))$. Thus, here we approximate the integrands unlike the preceding item, where we were dealing with approximations of the curve.
The realization of this scheme for the Hölder space $H_{\nu}(B)$ with norm $\|f\|_{\nu}:=h_{\nu}(f,B)+\sup\{|f(z)|:z\in B\}$ (see (5)) as $\mathfrak{X}$ leads to the following result.

Theorem 7 (cf. [41]). *Let Ω be a finite domain such that $\Gamma\subset\Omega$, and $1\ge\nu>\dim_{\mathrm{M}}\Gamma-1$. If a holomorphic in $\mathbb{C}\setminus\Gamma$ function F is integrable in Ω with any degree $p\ge1$, then inequality (21) is valid for $\mathfrak{X}=H_{\nu}(\overline{\Omega})$.*

The distribution $\overline{\partial}F$ is extendable up to continuous functional on $H_{\nu}(\overline{\Omega})$ for $\nu>\dim_{\mathrm{M}}\Gamma-1$. Hence, we are able to define distribution (22) if $f\in H_{\nu}(\overline{\Omega})$. Thus, for any function $f\in H_{\nu}(\Gamma)$ there is defined distribution $f\overline{\partial}F$, and it is unique in the following sense: if $f,g\in H_{\nu}(\overline{\Omega})$ and $f|_{\Gamma}=g|_{\Gamma}$, then $f\overline{\partial}F=g\overline{\partial}F$ (see [2]).
We consider Cauchy transform of a distribution ϕ with compact support S on the complex plane, i.e., the holomorphic in $\mathbb{C}\setminus S$ function

$$\mathcal{C}\mathrm{au}\,\phi:=\frac{1}{2\pi i}\left\langle\phi,\frac{1}{\zeta-z}\right\rangle,$$

where ϕ is applied to the Cauchy kernel $\frac{1}{2\pi i(\zeta - z)}$ as to a function of variable ζ. It is generalization of the Cauchy type integral, which is the Cauchy transform of distribution

$$C^\infty(\mathbb{C}) \ni \omega \mapsto \int\limits_\Gamma f(t)\omega(t)dt.$$

Theorem 8 (cf. [41]). *Let $F(z)$ be holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$, continuous in $\overline{D^+}$ and $\overline{D^-}$, and $F(\infty) = 0$. If $g \in H_\nu(\Gamma)$, $\nu > \frac{1}{2}\dim_{\mathrm{M}} \Gamma$, then function $\Phi(z) := \mathcal{C}\mathrm{au}\, g\overline{\partial}F(z)$ is holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$, continuous in $\overline{D^+}$ and $\overline{D^-}$, $\Phi(\infty) = 0$, and*

$$\Phi^+(t) - \Phi^-(t) = g(t)(F^+(t) - F^-(t)), \quad t \in \Gamma.$$

Clearly, this theorem shows that solutions of the Riemann boundary value problem on non-rectifiable curves are representable in terms of the Cauchy transforms. In particular, we obtain

Corollary 1 (cf. [42]). *Let $J(\Gamma)$ be set of all continuous functions g such that the jump problem on the curve Γ has a solution. If $\nu > \dfrac{\dim_{\mathrm{M}} \Gamma}{2}$, then $fg \in J(\Gamma)$ for any $g \in J(\Gamma)$ and any $f \in H_\nu(\Gamma)$, i.e., $J(\Gamma)H_\nu(\Gamma) = J(\Gamma)$.*

Both later theorems and the corollary allow sharpening in terms of the Marcinkiewicz exponents.
As the Cauchy transform $\mathcal{C}\mathrm{au}\,\phi$ is convolution of ϕ with fundamental solution of the $\overline{\partial}-$equation, then there exists certain duality between the jump problem for analytic functions and the problem of generalization of curvilinear integral.

4 Certain versions of the Riemann problem.

We have considered the Riemann boundary value problem on closed non-rectifiable Jordan curve. But the theory of this problem on piecewise-smooth curves contains various versions of its formulation. Here we transfer on non-rectifiable curves two that versions: the Riemann problem for open arcs and semi-continuous Riemann problem.

4.1 Riemann problem for non-rectifiable arcs.

We consider a Jordan arc Γ beginning at point a_1 and ending at point a_2. Generally speaking, the boundary conditions (1), (2) and (3) become meaningless at the points $a_{1,2}$, and we have to rewrite them in the form

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{23}$$

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{24}$$

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{25}$$

Clearly, in order to obtain customary sets of solutions we have to restrict behavior of desired function at the points a_1 and a_2. According classic monographs [18, 26, 49, 52] we assume that the desired function is either bounded there, or satisfies the bound

$$|\Phi(z)| \le C|z-a_j|^{-\gamma}, \quad j=1,2, \quad \gamma=\gamma(\Phi)\in(0,1) \tag{26}$$

(in the mentioned monographs this restriction is called integrability).
In order to construct a quasi-solution for the jump problem (24) we apply function (20). It has unit jump on the arc Γ, and the product $\phi(z)=\omega(z)k_\Gamma(z)g^w(z)$ is a quasi-solution. Here ω is a smooth function with compact support such that $\omega|_\Gamma=1$. But we meet the following obstakle: singularities of the function ϕ at the points a_1 and a_2 may be arbitrarily strong. Therefore, we need additional restrictions on the arc Γ. One of appropriate restrictions is square integrability of $k_\Gamma(z)$ points $a_{1,2}$. If this restriction and the condition (12) are fulfilled, then the function $\Phi(z)=(I-T\overline{\partial})\phi(z)$ satisfies condition (26) by virtue of well known properties of the operator T (see [54]).
The concept of uniqueness classes must be modified in the following way. If Γ is a non-closed arc , then class $\mathcal{H}_\mu(\Gamma)$ consists of all holomorphic in $\mathbb{C}\setminus\Gamma$ functions $\Phi(z)$ such that for any $\epsilon>0$ their boundary values $\Phi^\pm(t)$ satisfy the Hölder condition with exponent μ on the set $\Gamma\setminus\bigcup\limits_{j=1}^{2}\{z:|z-a_j|<\epsilon\}$.

Theorem 9 (cf. [28, 31]). *Let Γ be a simple arc of null area, the kernel k_Γ is square integrable near the ends of Γ, and $g\in H_\nu(\Gamma)$. If $\nu=1$ or $\nu>\frac{1}{2}\dim_{\mathrm{M}}\Gamma$, then function*

$$\Phi(z):=\omega(z)k_\Gamma(z)\mathcal{E}_0g(z)-\frac{1}{2\pi i}\iint_{\mathbb{C}}\frac{\partial\omega\mathcal{E}_0g}{\partial\overline{\zeta}}\frac{k_\Gamma(\zeta)d\zeta d\overline{\zeta}}{\zeta-z} \tag{27}$$

is a solution of the jump problem (24) in the class (26). It is a unique (up to additive constant) solution of the problem in the class $\mathcal{H}_\mu(\Gamma)$ if either $\nu=1$ and $1>\mu>\dim_{\mathrm{H}}\Gamma-1$ or $1>\nu>\frac{1}{2}\dim_{\mathrm{M}}\Gamma$ and μ satisfies (16).

In what follows we suppose that

$$k_\Gamma(z)=O(\ln|z-a_j|^{-1}), \quad z\to a_j, \quad j=1,2. \tag{28}$$

As above, we assume that $G\in H_\nu(\Gamma)$ and $G(t)\ne 0$ for $t\in\Gamma$. Then $f(t):=\ln G(t)\in H_\nu(\Gamma)$, and function

$$\Upsilon(z):=\omega(z)k_\Gamma(z)\mathcal{E}_0f(z)-\frac{1}{2\pi i}\iint_{\mathbb{C}}\frac{\partial\omega\mathcal{E}_0f}{\partial\overline{\zeta}}\frac{k_\Gamma(\zeta)d\zeta d\overline{\zeta}}{\zeta-z}$$

satisfies estimates $\Upsilon(z)=f(a_j)k_\Gamma(z)+O(1)$ at the points $a_{1,2}$ under assumptions of the later theorem and condition (28). Let $u(t)$ and $v(t)$ be real and imaginary parts of function $f(t)$. Clearly, $u(t)=\ln|G(t)|$, and $v(t)$ is a fixed single-valued branch of $\arg G(t)$. Obviously, $\mathrm{Re}\,\Upsilon(z)=(-1)^j(2\pi)^{-1}(u(a_j)\arg(z-a_j)+v(a_j)\ln|z-a_j|)+O(1)$ at a_j, $j=1,2$, and canonical function

$$X(z):=(z-a_1)^{-\varkappa_1}(z-a_2)^{-\varkappa_2}\exp\Upsilon(z)$$

satisfies the condition (26) if

$$\varkappa_j = 1 + \left]\frac{(-1)^j v(a_j)}{2\pi} + \liminf_{z\to a_j} \frac{(-1)^j u(a_j)\arg(z-a_j)}{2\pi \ln|z-a_j|}\right[, \quad j=1,2.$$

The symbol $]x[$ means value $\{n \in \mathbb{Z} : n < x\}$, and branch of $\arg(z-a_j)$ is determined by cut along Γ. We obtain

Theorem 10 (see [28, 31]). *The general solution of problem (25) in the class $\mathcal{H}_\mu(\Gamma)$ satisfying condition (26) for $\varkappa := \varkappa_1 + \varkappa_2 \ge 0$ is $\Phi(z) = X(z)P(z)$, where P is arbitrary algebraic polynomial of degree no more than $\varkappa$. For $\varkappa < 0$ the problem has null solution only.*

Then we need a quasi-solution ϕ for the problem (23). If it is continuous on Γ, then we have $\phi^+(t) = \phi^-(t)$, and, consequently, $\phi(t) = g(t)/(1-G(t))$, $t \in \Gamma$. Hence, $\phi(z) = \omega(z)\left(\frac{g}{1-G}\right)^w(z)$ is a quasi-solution of the problem (23), but only if $G(t) \neq 1$, $t \in \Gamma$. In the paper [32] a reader will find another construction of quasi-solution, which is valid independently on this condition. In addition, in the case under consideration the function X^{-1} can have singularities of high orders at the points $a_{1,2}$, what makes the regularizing operator $I - XTX^{-1}\overline{\partial}$ unapplicable. We exclude this possibility by means of requirement of existence of finite limits

$$\lim_{z\to a_j} \frac{\arg(z-a_j)}{\ln|z-a_j|}, \quad j=1,2,$$

for the same branch of $\arg(z-a_j)$. As a result, we get a theorem on solvability of the problem (23), which is similar to Theorem 4.
The dependence of index $\varkappa$ on geometry of the boundary was described first for non-smooth rectifiable arcs; see [51].

4.2 Semi-continuous Riemann problem.

Let coefficients and solutions of the Riemann boundary value problem admit discontinuities at prescribed points of the boundary. That version of the Riemann problem is called *semi-continuous* (see [18, 26, 49, 52]). Here we will solve the semi-continuous on closed non-rectifiable curve.
Let $E = \{a_1, a_2, \ldots, a_m\} \subset \Gamma$ be a finite set of points of a closed Jordan curve Γ. The corresponding semi-continuous problem consists from boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus E \tag{29}$$

and condition

$$|\Phi(z)| \le C|z-a|^{-\gamma}, \quad a \in E, \quad \gamma = \gamma(\Phi) \in (0,1) \tag{30}$$

on singularities of the desired functions at the points of set E. We consider here the case where g has there singularities of power type, and G is Hölder-continuous.
In the paper [33] this problem is solved in the following way. We assume that $g(t) = \mathfrak{w}(t)g_0(t)$, where $\mathfrak{w}(t) := \prod_{a\in E} |t-a|^{-p(a)}$, $0 < p(a) < 1$, $g_0 \in H_\nu(\Gamma)$. Then a

quasi-solution of the semi-continuous jump problem (i.e., the problem (29 with $G \equiv 1$) and the Riemann problem (29 itself is $\phi(z) := \chi^+(z)\mathfrak{w}(z)(g_0)^w(z)$. By means of this quasi-solution we prove semi-continuous analogs of all cited above theorems on solvability of the Riemann problem; a reader can find these analogs in [33]. But recently they are improved by D.B. Katz [45] in terms of Marcienkiewicz exponents.

5 Conclusion

Let us note that described above methods have proven useful not only in complex analysis, but in Clifford [4, 7] and quaternionic [1] analysis, in fractal analysis [3, 8], for study of hyper-analytic [9] and $\beta-$analytic [11] functions and others.
As a conclusion, we cite certain open questions concerning the problem under consideration from the paper [42].
1. In 1958 V.P. Havin [20] obtained the following result. We consider a bounded, closed and connected set Γ on the complex plane and an essential measure μ on this set. A positive measure is called essential on Γ if any its null set $E \subset \Gamma$ satisfies condition $\overline{\Gamma \setminus E} = \Gamma$. Then any holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function is representable as

$$\Phi(z) = const + \sum_{k=0}^{\infty} \int_{\Gamma} \frac{Y_k(t)dt}{(t-z)^{k+1}},$$

where $\int_{\Gamma} |Y_k(t)|^2 d\mu < +\infty$ for any k, and $\lim_{k\to+\infty} \left(\int_{\Gamma} |Y_k(t)|^2 d\mu \right)^{1/k} = 0$. V.P. Havin obtained also a criterion for finiteness of the sum in this representation.
If Γ is rectifiable curve, μ is length, and $\Phi(z)$ is a solution of the jump problem on Γ, then the Havin representation for Φ contains only one term. It is of interest to find that representation for a solution Φ of the jump problem (2) on non-rectifiable curve Γ and for a given essential measure μ.
2. The cited above results imply that for $g \in H_\nu(\Gamma)$, $\frac{1}{2}\dim_{\mathrm{M}}\Gamma > \nu > \dim_{\mathrm{M}}\Gamma - 1$, the Cauchy transform $\Phi(z) = \mathcal{C}\mathrm{au}\, g\overline{\partial}\chi^+(z)$ exists, but it can be discontinuous on Γ. The problem is to describe its boundary behavior.
3. The author supposes that if Γ is self-similar non-rectifiable curve, $f \in H_\nu(\Gamma)$ and $1 > \nu > 0$, then the Cauchy transform $\mathcal{C}\mathrm{au}\, g\overline{\partial}\chi^+(z)$ is continuous in $\overline{D^+}$ and in $\overline{D^-}$, and its boundary values on Γ from the left and from the right satisfy the Hölder condition with any exponent $\mu < \nu$.
This supposition means that the self-similarity of non-rectifiable curve can improve boundary properties of the Cauchy transform in almost the same degree as the smoothness improves boundary behavior of the Cauchy integral over rectifiable curve.
4. There exists a lot of open questions concerning the Riemann boundary value problem on non-rectifiable arcs. Here we formulate one of them. Let arc Γ satisfy restriction (28), and

$$\vartheta_j := \limsup_{z\to a_j} \frac{\mathrm{Re}\, k_\Gamma(z)}{\ln|z-a_j|} - \liminf_{z\to a_j} \frac{\mathrm{Re}\, k_\Gamma(z)}{\ln|z-a_j|}.$$

Assume that $\ln G(t)$ is restriction on Γ of a function, which has partial derivatives of order $[\vartheta_j]+1$ at point a_j, and P_j is its Taylor polynomial at this point, $j=1,2$. The problem is to describe full pattern of solvability of the problems (25) and (23) in terms of these polynomials.

References

[1] R. Abreu and J. Bory. Boundary value problems for quaternionic monogenic functions on nonsmooth surfaces, Adv. Appl. Clifford Algebras, 1999, V.9, No.1, P. 1-22.

[2] Ricardo Abreu-Blaya, Juan Bory-Reyes, Boris A. Kats, Integration over non-rectifiable curves and Riemann boundary value problems. Journal of Mathematical Analysis and Applications, 2011, V. 380, No. 1, P. 177-187

[3] Ricardo Abreu-Blaya, Juan Bory-Reyes, Boris A. Kats. Approximate dimension applied to criteria for monogenicity on fractal domains, Bulletin of the Brazilian Mathematical Society, New Series, 2012, V.43(4), P. 529-544.

[4] R. Abreu-Blaya, J. Bory-Reyes, B. A. Kats. On the solvability of the jump problem in clifford analysis. Journal of Mathematical Sciences, February 2013, V.189, No.1, P. 1-9

[5] R. Abreu-Blaya, J. Bory-Reyes, B. A. Kats. The Cauchy Type Integral and Singular Integral Operator over closed Jordan curves. Monatshefte fur Mathematik. Volume 176, Issue 1. 2015. pp. 1-15

[6] R. Abreu Blaya, J. Bory Reyes, T. Moreno Garcia, The sharpness of condition for solving the jump problem, Comm. Math. Anal., 2012, V.12, No.2, P. 26-33.

[7] R. Abreu Blaya; J. Bory Reyes; T. Moreno Garcia. Minkowski dimension and Cauchy transform in Clifford analysis, Compl. Anal. Oper. Theory, 2007, V.1, No.3, P, 301-315.

[8] R. Abreu Blaya, J. Bory Reyes, T. Moreno Garcia. Hermitian decomposition of continuous functions on a fractal surface, Bull. Braz. Math. Soc., 2009, V.40(1), P. 107-115.

[9] R. Abreu Blaya, J. Bory Reyes and D. Pena Pena. Riemann boundary value problem for hyperanalytic functions. Int. J. Math. Math. Sci., 2005, V.17, P. 2821-2840.

[10] R. Abreu Blaya, J. Bory Reyes, J. Marie Vilaire, Hyperanalytic Riemann boundary value problem on $d-$summable closed curves, J. Math. Anal. Appl., 2010, V.361, P. 579-586.

[11] R. Abreu Blaya, J. Bory Reyes, D. Pena Pena and J.-M. Vilaire. Riemann boundary value problem for $\beta-$analytic functions. Int. J. Pure Appl. Math., 2008, V.42, No.1, P. 19-37.

[12] F. M. Borodich, A.Y. Volovikov. Surface integrals for domains with fractal boundaries and some applications to elasticity. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000), No. 1993, 1–24.

[13] L. Carleson, Selected Problems on Exceptional Sets, D. Van Nostrand Company, INC. Princeton, New Jersey, 1967

[14] E.P. Dolzhenko, On ”erasing” of singularities of analytical functions, Uspekhi Mathem. Nauk, 1963, V.18, No.4, P. 135-142 (Russian)

[15] E. M. Dynkin, On the smoothness of Cauchy type integral. Doklady AN USSR, 1979, V.250, P. 794-797 (Russian)

[16] E. M. Dynkin, Smoothness of the Cauchy type integral. Zapiski nauchn. sem. Leningr. dep. mathem. inst. AN USSR, 1979, V.92, P. 115–133 (Russian)

[17] I. Feder, Fractals, Mir Publishers, Moscow, 1991.

[18] F.D. Gakhov, Boundary value problems, Nauka publishers, Moscow, 1977.

[19] F. W. Gehring, W. K. Hayman and A. Hinkkanen, Analytic functions satisfying Hölder conditions on the boundary, J. Approx. Theory 35 (1982), P. 243-249.

[20] V.P. Havin, An analog of the Laurent series. Studies on contemporary problems of theory of functions of complex variable, Collection of papers, Editor A.I. Marcushevich, GIFML, Moscow, 1961, P. 121-131 (Russian)

[21] J. Harrison and A. Norton, Geometric integration on fractal curves in the plane, Indiana Univ. Math. J., 1991, V.40, No.2, P. 567-594

[22] J. Harrison, A. Norton, The Gauss-Green theorem for fractal boundaries, Duke Mathematical Journal, 67(3) (1992), P. 575-588.

[23] J. Harrison, Stokes' theorem for nonsmooth chains. Bulletin of the American Math Society, October, 1993.

[24] J. Harrison. Lectures on chainlet geometry – new topological methods in geometric measure theory. arXiv:math-ph/0505063v1 24 May 2005; Proceedings of Ravello Summer School for Mathematical Physics, 2005.

[25] L. Hörmander, The Analysis of Linear Partial Differential Operators I. Distribution theory and Fourier Analysis, Springer Verlag, 1983.

[26] Jian-Ke Lu, *Boundary Value Problems for Analytic Functions*, (World Scientific, 1993).

[27] Antti Käenmäki, Juha Lehrbäck and Matti Vuorinen, *Dimension, Whitney Covers, and Tubular Neighborhoods*, Indiana Univ. Math. J., V. 62, no.6 (2013), pp. 1861-1889

[28] B.A. Kats, The Riemann boundary value problem on non-rectifiable Jordan curve. Doklady AN USSR. 1982. V.267, No.4. P. 789-792 (Russian)

[29] B.A. Kats, The integration over non-rectifiable curve. Moscow Civil Engineering Institute. Questions of mathematics, mechanics of solid media and application of mathematical methods in constructions. Collection of scientific papers. Moscow, MISI, 1982, P. 63-69 (Russian)

[30] B.A. Kats, The Riemann problem on closed Jordan curve. Izvestija vuzov. Matematika. 1983, No.4. P. 68-80 (Russian)

[31] B.A. Kats, The Riemann problem on Jordan arc. Izvestija vuzov. Matematika. 1983, No.12. P. 30-38 (Russian)

[32] B.A. Kats, The inhomogeneous Riemann problem on Jordan arc, Trudy seminara po kraevym zadacham. Kazan Univ., 1984, V. 21. P. 87-93 (Russian)

[33] B.A. Kats. The Riemann problem on closed Jordan curve in semi-continuous statement. Izv. vuzov. Mathematics, 1987, No.5, P. 49-57. (Russian)

[34] B.A. Kats, The jump prolem and integral over non-rectifiable curve. Izv. vuzov, Mathem., 1987, No.5, P. 49-57 (Russian)

[35] B.A. Kats, The Cauchy integral along $\Phi-$rectifiable curves. Lobatchevskii Journal of Mathematics, 2000, V.7, P. 15-29

[36] B.A. Kats, On a generalization of a theorem of N. A. Davydov. Izv. vuzov. Mathem., 2002, No.1, P. 39-44 (Russian).

[37] B. A. Kats. The Refined Metric Dimension with Applications, Computation Methods and Function Theory, 2007, No.1, P. 77-89.

[38] B.A. Kats, The Cauchy integral over non-rectifiable paths. Contemporary Mathematics, 2008, V.455, P. 183-196.

[39] B. A. Kats. On solvability of the jump problem. J. Math. Anal. Appl.,2009, V.356, No.2, P. 577-581.

[40] B.A. Kats. Approximation dimensions with applications. Proceedings of Institute of Mathematics of Ukranian National Academy of Science, 2010, V.7, No.2, P. 73-80

[41] B. A. Kats, The Cauchy Transform of Certain Distributions with Application. Complex Analysis and Operator Theory, 2012, V.6, No.6, P. 1147-1156

[42] B. A. Kats. The Riemann boundary value problem on non-rectifiable curves and related questions. Complex Variables and Elliptic Equations: An International Journal Volume 59, Issue 8, pp. 1053-1069, 2014

[43] D.B. Kats, The Marcinkiewicz exponents with application in boundary value problems. Izvestia vuzov. Mathematics. 2014, no 3, p.68-71. (Russian)

[44] D.B. Kats, Local Marcinkiewicz exponents with application, Uchen. Zapiski Kazan Univ. Ser. phys.-math. nauki. 2014, V.156, book 4, p.31-38. (Russian)

[45] D.B. Kats, Weighted Marcinkiewicz exponents with application, to appear

[46] A.N. Kolmogorov, V.M. Tikhomirov, $\varepsilon-$entropy and capasity of set in functional spaces. Uspekhi Math. Nauk, 1959, V.14, P. 3-86 (Russian)

[47] R. Lesniewicz and W. Orlicz, On generalized variations (II). Studia Mathematica, 1973, V.45, No.1, P 71-109.

[48] Liu H. A note about Riemann boundary value problems on non-rectifiable curves. Complex Var. Elliptic Equ. 52, No.10-11, (2007) P. 877-882.

[49] N.I. Muskhelishvili, Singular integral equations, Nauka publishers, Moscow, 1962.

[50] T. Salimov, A direct estimate for a singular Cauchy integral over a closed curve. Azerbaidzhan. Gos. Univ. Uchen. Zap., 1979, No.5, P. 59-75 (Russian)

[51] R.K. Seifullaev, The Riemann boundary value problem on non-smooth arc, Mathem. Sb., 1980, V.112(154), No.2(6), P. 147-161 (Russian)

[52] A.P. Soldatov, *One-dimensional singular operators and boundary value problems of theory of functions* (Moscow, Vyssh. shc., 1991).

[53] E.M. Stein, Singular integrals and differential properties of functions, Princeton University Press, Princeton, 1970.

[54] I.N. Vekua, Generalized analytical functions, Nauka publishers, Moscow, 1988.