J. Math. Anal. Appl. 552 (2025) 129764

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Characterizations of the canonical trace on full matrix algebras

Mohammad Sal Moslehian [a], Airat M. Bikchentaev [b,*]

[a] *Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran*
[b] *N.I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kremlevskaya ul. 18, Kazan, Tatarstan, 420008 Russia*

ARTICLE INFO

Article history:
Received 1 April 2025
Available online 4 June 2025
Submitted by J. Bonet

Keywords:
Trace
Yang's inequality
Positive semidefinite
Characterization

ABSTRACT

We establish that a positive linear functional on the full matrix algebra $\mathbb{M}_n$ is a positive multiple of the canonical trace if and only if $\varphi(A) = \varphi(|A|)$ implies that A is positive semidefinite. Furthermore, we characterize the canonical trace on $\mathbb{M}_n$ among all positive linear functionals φ on $\mathbb{M}_n$ with $\varphi(I) = n$ via Yang's inequality $\varphi(A^{1/2}BA^{1/2})^{1/2} \leq \varphi(A+B)/2$, where $A, B \in \mathbb{M}_n$ are positive semidefinite matrices.

1. Introduction

Let $\mathbb{M}_n$ denote the $*$-algebra of $n \times n$ complex matrices equipped with the Löwner order $\geq$ on Hermitian matrices, and let $\mathbb{M}_n^+$ present the cone of positive semidefinite matrices. The identity operator is denoted by I_n or simply I. A linear functional φ on $\mathbb{M}_n$ is said to be *positive* if $\varphi(A) \geq 0$ for all $A \in \mathbb{M}_n^+$. It is called *faithful*, if $\varphi(A) = 0$ implies that $A = 0$ for any $A \in \mathbb{M}_n^+$. It is said to be a *state* if its operator norm is one. A positive linear functional φ is called *tracial* if $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathbb{M}_n$. If $A \in \mathbb{M}_n$, then its *modulus* is defined as $|A| = \sqrt{A^*A} \in \mathbb{M}_n^+$. The canonical trace is denoted by tr. For undefined notations and terminologies, readers may refer to [2,18] for the matrix theory.

In general, any trace inequality is sharp in the sense that the trace is the only positive linear functional that meets the inequality. It is well-known that inequalities such as Hölder, Cauchy–Schwarz–Buniakowski, Golden–Thompson, Peierls–Bogoliubov, Araki–Lieb–Thirring, when restricted to projections, characterize the tracial functionals among all positive functionals φ on $\mathbb{M}_n$, see [4,5,7].

* Corresponding author.
E-mail addresses: moslehian@um.ac.ir, moslehian@yahoo.com (M.S. Moslehian), Airat.Bikchentaev@kpfu.ru (A.M. Bikchentaev).

https://doi.org/10.1016/j.jmaa.2025.129764
0022-247X/

In this paper, we investigate the implication $\varphi(A) = \varphi(|A|) \Rightarrow A \geq 0$ and show that it characterizes the canonical trace on $\mathbb{M}_n$. In addition, we characterize the canonical trace on $\mathbb{M}_n$ among all positive linear functionals φ on $\mathbb{M}_n$ with $\varphi(I) = n$ via Yang's inequality [17]

$$\varphi(A^{1/2}BA^{1/2})^{1/2} \leq \varphi(A+B)/2,$$

where $A, B \in \mathbb{M}_n^+$. This inequality provides a solution to the Bellman problem on an arithmetic-geometric mean type inequality for traces of positive semidefinite matrices. Such arithmetic-geometric mean trace inequalities are well-known in the literature. For instance, it is proved in [3] that $\operatorname{tr}(|AB|^{1/2}) \leq \operatorname{tr}(A+B)/2$ for any $A, B \in \mathbb{M}_n^+$. For other characterizations of tracial functionals and trace inequalities, readers may consult [1,6–8,10,12,16] and the references therein.

2. Characterization of the canonical trace via $\varphi(A) = \varphi(|A|) \Rightarrow A \geq 0$

A matrix $A \in \mathbb{M}_n$ is called *dissipative* if its imaginary part $(A - A^*)/2\mathrm{i}$ is positive semidefinite.

Proposition 2.1. *Let φ be a nonzero positive linear functional on $\mathbb{M}_n$.*

(i) *If φ has the property that "the equality $\varphi(A) = \varphi(|A|)$ for any nonzero dissipative matrix $A \in \mathbb{M}_n$ implies $A \geq 0$", then φ is faithful.*
(ii) *If φ is faithful and $\varphi(A) = \varphi(|A|)$ for a nonzero dissipative matrix $A \in \mathbb{M}_n$, then $A \geq 0$.*

Proof. (i) (*First proof.*) In order to achieve a contradiction, we assume that φ is not faithful. Take P as the support of φ, so there exists a projection P such that $\varphi(Q) = 0$ for all projections $Q \in \mathbb{M}_n$ with $Q \leq I - P$, and $\varphi(Q) > 0$ for all nonzero projections $Q \in \mathbb{M}_n$ with $Q \leq P$.

We claim that $P \neq I$. To prove this claim, we assume that $P = I$. According to the definition of the support projection, we have $\varphi(Q) > 0$ for every nonzero projection Q. By combining this with a well-known result from Choi and Wu [9, Proposition 1.4], which states that every nonzero positive semidefinite matrix is a convex combination of commuting projections, we conclude that $\varphi(T) = 0$ for some positive semidefinite matrix T only if $T = 0$. So, φ is faithful, which contradicts our initial assumption.

Now, for the dissipative matrix $A = -(I - P)$, we have $\varphi(A - |A|) = \varphi(2A) = 0$, but A is not positive semidefinite. Thus, φ is faithful.

(*Second proof.*) To achieve a contradiction, we assume that φ is not faithful. Then there exists $X \geq 0$, $X \neq 0$ such that $\varphi(X) = 0$. Hence, $\varphi(-X) = -\varphi(X) = 0 = \varphi(|-X|)$ for the dissipative matrix $-X$. However, $-X$ is not positive semidefinite.

(ii) **Step 1:** Let $A = A^*$ and $A = A_+ - A_-$ be the Jordan decomposition of A. We have

$$\varphi(A_+ - A_-) = \varphi(A) = \varphi(|A|) = \varphi(A_+ + A_-).$$

Hence, $\varphi(-A_-) = \varphi(A_-)$, and, therefore, $\varphi(A_-) = 0$. As a result, $A_- = 0$, and so $A \in \mathcal{A}^+$.

Step 2: Assume that $A \in \mathbb{M}_n$ is dissipative. Let

$$A = A_1 - A_2 + \mathrm{i}A_3,$$

where $A_1, A_2, A_3 \in \mathcal{A}^+$. Since $\varphi(A) = \varphi(|A|) \geq 0$, and, in particular, it is a real number, we have $\varphi(A_3) = 0$. Since φ is faithful, we conclude that $A_3 = 0$. Thus, $A = A_1 - A_2$ and by Step 1 $A_2 = 0$. Therefore, A is positive semidefinite. □

Remark 2.2. Proposition 2.1 (ii) may not be true if φ is not faithful. For example, consider the vector state $\tau_e(X) = \langle Xe, e\rangle$ on $\mathbb{M}_2(\mathbb{C})$, where $e = (0,1) \in \mathbb{C}^2$. Take the dissipative matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Then τ_e is not faithful, $\tau_e(A) = 0 = \tau_e(|A|)$, but A is not positive semidefinite.

Proposition 2.3. *Let φ be a nonzero positive faithful linear functional on $\mathbb{M}_n$ and let $A \in \mathbb{M}_n$ be normal. If $\varphi(A) = \varphi(|A|)$, then $A \geq 0$.*

Proof. Assume that $A \neq 0$. It follows from [18, Theorem 9.1] that $A = \sum_{j=1}^n \lambda_j P_j$ for some complex numbers λ_j and pairwise orthogonal projections P_j with $\sum_{j=1}^n P_j = I$. Hence,

$$\left|\sum_{j=1}^n \lambda_j \varphi(P_j)\right| = |\varphi(A)| = \varphi(|A|) = \sum_{j=1}^n |\lambda_j| \varphi(P_j).$$

From the equality condition in the triangle inequality and $\varphi(P_j) \geq 0$ for $1 \leq j \leq n$ (see [18, Problem 6 of Section 5.7]), we deduce that $\lambda_j = e^{i\theta}|\lambda_j|$ for j, where $\varphi(P_j) > 0$. From $0 \leq \varphi(A) = e^{i\theta} \sum_{j=1}^n |\lambda_j| \varphi(P_j)$, we conclude that $\theta = 0$. Consequently, $A = |A| \geq 0$. □

Remark 2.4. We cannot remove the condition of normality in Proposition 2.3. For example, take nonzero positive faithful linear functional $\varphi : \mathbb{M}_2 \to \mathbb{C}$ defined by $\varphi([a_{ij}]) = a_{11} + 2a_{22}$ and $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Then $|A| = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ and $\varphi(A) = 6 = \varphi(|A|)$, but A is not normal.

However, the implication

$$\varphi(A) = \varphi(|A|) \Rightarrow A \geq 0$$

holds for the canonical trace, see [13, Lemma 2.2] for trace-class operators. We now present an alternative proof of [18, Theorem 9.6].

Theorem 2.5. *If $\mathrm{tr}(|A|) = \mathrm{tr}(A)$ for a nonzero matrix $A \in \mathbb{M}_n$, then A is positive semidefinite.*

Proof. Let $A = UDV$ be the singular value decomposition of A with $D = \mathrm{diag}(\sigma_1, \dots, \sigma_r, 0, \dots, 0)$. Put $W := VU$. Then, W is a unitary matrix, and, consequently, its column and row vectors are normalized (i.e., have unit norm). We have

$$\sum_{j=1}^r \sigma_j = \mathrm{tr}(|A|) = \mathrm{tr}(A) = \mathrm{tr}(DW) = |\,\mathrm{tr}(DW)| \leq \sum_{j=1}^r \sigma_j(D)\sigma_j(W) = \sum_{j=1}^r \sigma_j, \tag{2.1}$$

since $\sigma_j(D) = \sigma_j$ and $\sigma_j(W) = \lambda_j((W^*W)^{1/2}) = \lambda_j(I) = 1$ for $1 \leq j \leq r$. Inequality (2.1) follows from the well-known von Neumann trace inequality; for further details, see [11] (or [7, Theorem 2.1.25]).

On the other hand, if $W = [w_{ij}]$, then $\mathrm{tr}(DW) = \sum_{i=1}^r \sigma_i w_{ii}$. Therefore, (2.1) yields

$$\sum_{i=1}^r \sigma_i = \sum_{i=1}^r \sigma_i w_{ii} = \left|\sum_{i=1}^r \sigma_i w_{ii}\right| \leq \sum_{i=1}^r \sigma_i |w_{ii}| \leq \sum_{i=1}^r \sigma_i \|W_i\| = \sum_{i=1}^r \sigma_i, \tag{2.2}$$

where W_i denotes the i-th column of W. Since $\sigma_1, \dots, \sigma_r$ are nonzero, $|w_{ii}| = 1$ for all $1 \leq i \leq r$. Consequently, $w_{ij} = 0$ for all $1 \leq i \neq j \leq r$.

From the equality case in the triangle inequality and $|\sum_{i=1}^{r}\sigma_i w_{ii}| = \sum_{i=1}^{r}\sigma_i|w_{ii}|$, we deduce that $w_{ii} = \alpha|w_{ii}|$ for some complex number α with $|\alpha| = 1$. From (2.2), we conclude that $0 \le \sum_{i=1}^{r}\sigma_i w_{ii} = \alpha\sum_{i=1}^{r}\sigma_i|w_{ii}|$. Hence, $\alpha \ge 0$, and as a result, $w_{ii} \ge 0$ for $1 \le i \le r$. This leads us to $w_{ii} = 1$ for $1 \le i \le r$.

Let us apply block matrices of appropriate size for D and W (see [14]) to achieve

$$\begin{aligned} AU &= (UDV)U = U(DW) \\ &= U\begin{bmatrix} D_r & | & 0_{r\times(n-1)} \\ -- & & -- \\ 0_{(n-r)\times r} & | & 0_{(n-r)\times(n-r)} \end{bmatrix}\begin{bmatrix} I_r & | & 0_{r\times(n-1)} \\ -- & & -- \\ 0_{(n-r)\times r} & | & W_0 \end{bmatrix} = UD, \end{aligned}$$

where $D_r = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$ is an $r \times r$ diagonal matrix and $W_0 \in \mathbb{M}_{n-r}$ is a submatrix of W. Therefore, $A = UDU^*$ is positive semidefinite. □

Now, we establish the main result of this section.

Theorem 2.6. *Let φ be a nonzero positive linear functional on $\mathbb{M}_n$ such that*

$$\varphi(A) = \varphi(|A|) \Rightarrow A \ge 0 \quad \text{for every nonsingular } A \in \mathbb{M}_n. \tag{2.3}$$

Then, $\varphi = k\,\mathrm{tr}$ for some $k > 0$.

Proof. We assume that φ is a state on $\mathbb{M}_n$, which is a Hilbert space under the Frobenius inner product $\langle A, B\rangle = \mathrm{tr}(AB^*)$. By the Riesz representation theorem, $\varphi(\cdot)$ takes the form $\mathrm{tr}(S\cdot)$ for some positive semidefinite matrix S. Next, application of the Schur decomposition of S [18, Theorem 3.3], the tracial property of the canonical trace, and replacement of φ with $\varphi \circ \mathrm{Ad}_U$ for some unitary matrix U in which $Ad_U : \mathbb{M}_n \to \mathbb{M}_n$ is defined by $Ad_U(X) = UXU^*$, allows us to state that $S = \mathrm{diag}(s_1, s_2, \dots, s_n)$. Let us show that $s_1 = s_2 = \cdots = s_n = 1/n$. To prove this, it suffices to check that $s_1 = s_2$. Therefore, we can assume without loss of generality that $n = 2$. Hence, we only need to prove that if $S = \mathrm{diag}(\frac{1}{2}+s, \frac{1}{2}-s)$ for some $0 \le s \le \frac{1}{2}$, then $s = 0$.

To reach a contradiction, assume that $s \ne 0$. For real numbers $0 < t \le 1$ and $0 < x < 1/2$, set

$$U = \begin{bmatrix} t & \sqrt{1-t^2} \\ \sqrt{1-t^2} & -t \end{bmatrix}, \quad C = \begin{bmatrix} 1+x & 1 \\ 1 & 1+x \end{bmatrix},$$

and

$$A = UC = \begin{bmatrix} t + tx + \sqrt{1-t^2} & t + (1+x)\sqrt{1-t^2} \\ (1+x)\sqrt{1-t^2} - t & \sqrt{1-t^2} - t - tx \end{bmatrix}.$$

Then, U is a unitary matrix, C is positive definite, A is nonsingular as the product of nonsingular matrices, and A is not positive definite when $t \ne 0$, since it is not Hermitian. Moreover,

$$\begin{aligned} \varphi(A) &= \mathrm{tr}(SA) = \left(\frac{1}{2}+s\right)(t + tx + \sqrt{1-t^2}) + \left(\frac{1}{2}-s\right)(\sqrt{1-t^2} - t - tx) \\ &= 2st + 2stx + \sqrt{1-t^2} \end{aligned}$$

and

$$\varphi(|A|) = \varphi(C) = \mathrm{tr}(SC) = \left(\frac{1}{2}+s\right)(1+x) + \left(\frac{1}{2}-s\right)(1+x) = 1 + x.$$

Therefore,

$$\begin{aligned}
&\varphi(A) = \varphi(|A|) \\
&\quad \iff \sqrt{1-t^2} = 1 + x - 2st - 2stx \\
&\quad \iff (1 + 4s^2 + 8s^2x + 4s^2x^2)t^2 - (4s + 8sx + 4sx^2)t + 2x + x^2 = 0 \\
&\quad \iff t_\pm = \frac{2s + 4sx + 2sx^2 \pm \sqrt{4s^2 + 8s^2x + 4s^2x^2 - x^2 - 2x}}{1 + 4s^2 + 8s^2x + 4s^2x^2}.
\end{aligned}$$

Thus, for sufficiently small x (for example, if $x^2 + 2x < s^2$), we obtain $t_+ \in \mathbb{R}$, $t_+ > 0$, and $\varphi(A) = \varphi(|A|)$. However, A is not positive semidefinite. This contradicts (2.3).

Thus, $S = \frac{1}{2}I$, and, as a result, $\varphi = \frac{1}{2}\,\mathrm{tr}$. $\square$

3. Trace characterization via Yang's inequality

In order to achieve the main result of this section, we need some lemmas. The first one is known as Taylor's formula with Peano's remainder.

Lemma 3.1. *If $b \in \mathbb{R}$, then*

$$(1+x)^b = 1 + bx + \frac{1}{2!}b(b-1)x^2 + \ldots + \frac{1}{n!}b(b-1)\cdots(b-n+1)x^n + o(x^n) \quad as \quad x \to 0.$$

Lemma 3.2. *Let numbers $s_1, \ldots, s_n \geq 0$ be such that $s_1 + \ldots + s_n = n$ and let there exist $k, i \in \{1, \ldots, n\}$ with $s_k \neq s_i$.*

(i) *If $n = 2$, then there exists $0 < s \leq 1$ such that $\{s_1, s_2\} = \{1 - s, 1 + s\}$.*
(ii) *If $n \geq 3$, then there exist $m, j \in \{1, \ldots, n\}$, $m \neq j$ such that $s_m + s_j < 2$.*

Proof. (i). Since $0 \leq \min\{s_1, s_2\} < 1 < \max\{s_1, s_2\} \leq 2$, we have $s = |s_1 - s_2|/2$.

(ii). Let $n \geq 3$. Obviously, $a := \min_{1 \leq k \leq n} s_k < 1$. Without loss of generality, we may assume that $s_1 = a$. We show that there exists $k \in \{2, \ldots, n\}$ such that $s_1 + s_k < 2$:

Assume that for every $k \in \{2, \ldots, n\}$ the inequality $s_1 + s_k \geq 2$ holds true. Add these inequalities, term by term, and obtain

$$(n-1)s_1 + s_2 + \ldots + s_n \geq 2(n-1).$$

Since $s_1 + \ldots + s_n = n$, we have $(n-2)s_1 \geq n - 2$. Therefore, $s_1 \geq 1$, which leads to a contradiction. $\square$

Theorem 3.3. *For a positive linear functional φ on $\mathbb{M}_n$ with $\varphi(I) = n$, the following conditions are equivalent:*

(i) $\varphi = \mathrm{tr}$;
(ii) $\varphi(A^{1/2}BA^{1/2})^{1/2} \leq \varphi(A+B)/2$ *for all $A, B \in \mathbb{M}_n^+$.*

Proof. (i)⇒(ii). This result is known, see [17]. For completeness, we provide the new proof as follows:

By the Cauchy–Schwarz inequality with respect to the Frobenius inner product, we derive that

$$\mathrm{tr}(A^{1/2}BA^{1/2}) = \mathrm{tr}(AB) \leq \mathrm{tr}(A^2)^{\frac{1}{2}}\,\mathrm{tr}(B^2)^{\frac{1}{2}} \leq \mathrm{tr}(A)\,\mathrm{tr}(B) \leq \left(\frac{\mathrm{tr}(A+B)}{2}\right)^2.$$

The second inequality follows from the well-known submultiplicativity of the trace and the last inequality follows from the classical arithmetic-geometric mean inequality.

(ii)⇒(i). The positive linear functional φ can be represented as $\varphi(\cdot) = \mathrm{tr}(S_\varphi \cdot)$ for some positive semidefinite matrix S_φ. Choose an orthonormal basis $\xi_1, \ldots, \xi_n \in \mathbb{C}^n$ such that S_φ has the form

$$S_\varphi = \mathrm{diag}(s_1, s_2, \ldots, s_n)$$

with $s_k \geq 0$, $k = 1, \ldots, n$, and $s_1 + s_2 + \cdots + s_n = n$. Then, $\varphi(X) = \mathrm{tr}(S_\varphi X)$ for all $X \in \mathbb{M}_n$. Assume that $\varphi \neq \mathrm{tr}$, then there exists $j \in \{1, \ldots, n\}$ such that $s_j < 1$.

Step 1: Let $n = 2$. By item (i) of Lemma 3.2, we can assume that

$$S_\varphi = \mathrm{diag}(1 - s, 1 + s)$$

for some $0 < s \leq 1$. Let $\delta \in \mathbb{C}$ with $|\delta| = 1$ and $f(t) = \sqrt{t - t^2}$ for $0 \leq t \leq 1$. We define the projection $R^{(t,\delta)}$ in $\mathbb{M}_2(\mathbb{C})$ as follows:

$$R^{(t,\delta)} = \begin{bmatrix} t & \delta f(t) \\ \bar{\delta} f(t) & 1 - t \end{bmatrix}. \tag{3.1}$$

Put $A := R^{(1/2-\varepsilon,1)}$ and $B := R^{(1/2+\varepsilon,1)}$ where $0 < \varepsilon < 1/2$. Then,

$$\varphi(A) = (1/2 - \varepsilon)(1 - s) + (1/2 + \varepsilon)(1 + s) = 1 + 2\varepsilon s.$$

Similarly, we have $\varphi(B) = 1 - 2\varepsilon s$. Since

$$A^{1/2} B A^{1/2} = ABA = 4f(1/2 - \varepsilon)^2 A,$$

we can rewrite inequality (ii) as

$$2f(1/2 - \varepsilon)(1 + 2\varepsilon s)^{1/2} \leq \frac{1}{2}(1 + 2\varepsilon s + 1 - 2\varepsilon s) = 1.$$

Squaring both sides of this inequality gives $4(1/4 - \varepsilon^2)(1 + 2\varepsilon s) \leq 1$, which simplifies to

$$2s\varepsilon - 4\varepsilon^2 - 8s\varepsilon^3 \leq 0.$$

Divide both sides of the latter inequality by -2ε, and achieve $4s\varepsilon^2 + 2\varepsilon - s \geq 0$. However, this inequality is false for

$$0 < \varepsilon < \frac{\sqrt{1 + 4s^2} - 1}{4s},$$

when $s > 0$. Thus, $s = 0$, and, hence, $\varphi = \mathrm{tr}$.

Step 2: Let $n \geq 3$.

1) Assume that the system $\{s_1, \ldots, s_n\}$ contains at most one 0. Rearrange the elements of the basis $\{\xi_1, \ldots, \xi_n\}$, if necessary, and consider the inequality

$$0 < s_1 + s_2 < 2; \tag{3.2}$$

see item (ii) of Lemma 3.2. For $0 < \varepsilon \leq 1/2$, put

$$A := \operatorname{diag}(R^{(1/2+\varepsilon,1)}, \underbrace{0,\dots,0}_{n-2}) \quad \text{and} \quad B := \operatorname{diag}(R^{(1/2,1)}, \underbrace{0,\dots,0}_{n-2});$$

see (3.1). Then,

$$\varphi(A) = \left(\frac{1}{2}+\varepsilon\right)s_1 + \left(\frac{1}{2}-\varepsilon\right)s_2 = \frac{s_1+s_2}{2} + \varepsilon(s_1-s_2) \quad \text{and} \quad \varphi(B) = \frac{s_1+s_2}{2}.$$

Since

$$R^{(1/2+\varepsilon,1)}R^{(1/2,1)}R^{(1/2+\varepsilon,1)} = \left(\frac{1}{2} + f\left(\frac{1}{2}+\varepsilon\right)\right)R^{(1/2+\varepsilon,1)},$$

we have $A^{1/2}BA^{1/2} = ABA = (1/2 + f(1/2+\varepsilon))A$. Therefore,

$$\varphi(A^{1/2}BA^{1/2}) = \left(\frac{1}{2} + f\left(\frac{1}{2}+\varepsilon\right)\right)\varphi(A) = \left(\frac{1}{2} + f\left(\frac{1}{2}+\varepsilon\right)\right)\left(\frac{s_1+s_2}{2} + \varepsilon(s_1-s_2)\right). \tag{3.3}$$

By the Taylor formula with Peano remainder (see Lemma 3.1), we have:

$$f\left(\frac{1}{2}+\varepsilon\right) = \left(\frac{1}{4}-\varepsilon^2\right)^{1/2} = \frac{1}{2}(1-4\varepsilon^2)^{1/2} = \frac{1}{2}(1-2\varepsilon^2+o(\varepsilon^2)) = \frac{1}{2} - \varepsilon^2 + o(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0^+.$$

Now, from (3.3), we obtain

$$\varphi(A^{1/2}BA^{1/2}) = \frac{s_1+s_2}{2} + (s_1-s_2)\varepsilon - \frac{s_1+s_2}{2}\varepsilon^2 + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0^+.$$

Squaring both sides of inequality (ii) yields

$$\frac{s_1+s_2}{2} + (s_1-s_2)\varepsilon - \frac{s_1+s_2}{2}\varepsilon^2 + o(\varepsilon^2) \le \frac{(s_1+s_2)^2}{4} + \frac{s_1^2-s_2^2}{2}\varepsilon + \frac{(s_1-s_2)^2}{4}\varepsilon^2.$$

By taking the limit as $\varepsilon \to 0^+$ in this inequality, we obtain

$$\frac{s_1+s_2}{2} \le \left(\frac{s_1+s_2}{2}\right)^2,$$

which contradicts (3.2).

2) Let us assume that the system $\{s_1,\dots,s_n\}$ contains more than one 0. By rearranging the elements of the basis $\{\xi_1,\dots,\xi_n\}$, if necessary, we may consider the matrix

$$S_\varphi = \operatorname{diag}(0, s_2, s_3, \dots, s_n),$$

where $0 < s_2 \le n$. If $s_2 < 2$, then the matrices mentioned in Step 1 do the job. Assume that $s_2 \ge 2$. Put

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} p^2 & p \\ p & 1 \end{bmatrix} \quad \text{for } \ p \ge 0.$$

Then, $A_1 = 2R^{(1/2,1)}$ and $A_1^{1/2} = 2^{1/2}R^{(1/2,1)} = 2^{-1/2}A_1$. Moreover,

$$A_1^{1/2}B_1A_1^{1/2} = \frac{A_1B_1A_1}{2} = \frac{(p+1)^2}{2}A_1.$$

Since $\varphi(A_1) = \varphi(B_1) = s_2$, for matrices

$$A = \operatorname{diag}(A_1, \underbrace{0, \dots, 0}_{n-2}) \quad \text{and} \quad B = \operatorname{diag}(B_1, \underbrace{0, \dots, 0}_{n-2})$$

we can rewrite inequality (ii) as

$$\frac{p+1}{2^{1/2}} s_2^{1/2} \leq \frac{1}{2}(s_2 + s_2),$$

that is, $(p+1)s_2^{1/2} \leq 2^{1/2} s_2$. Dividing both sides of the latter inequality by $s_2^{1/2} > 0$, we obtain $p + 1 \leq (2s_2)^{1/2}$. This inequality does not hold for $p > (2s_2)^{1/2} - 1$. $\square$

Remark 3.4. For another proof of (i) ⇒ (ii), see [15].

Declaration of competing interest

On behalf of the authors, the corresponding author states that there is no conflict of interest.

Acknowledgment

The authors would like to sincerely thank Professor Fuzhen Zhang for his insightful discussions. They are grateful to the referee for carefully reading the manuscript and providing constructive comments. The work was performed under the development program of Volga Region Mathematical Center (agreement no. 075-02-2024-1438). The authors also appreciate the referee for providing a short proof of Proposition 2.1(i) and for the valuable comments.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] S.A. Abed, An inequality for projections and convex functions, Lobachevskii J. Math. 39 (9) (2018) 1287–1292.
[2] R. Bhatia, Matrix Analysis, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997.
[3] R. Bhatia, F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl. 11 (2) (1990) 272–277.
[4] A.M. Bikchentaev, Commutation of projections and trace characterization on von Neumann algebras. II, Math. Notes 89 (3–4) (2011) 461–471.
[5] A.M. Bikchentaev, The Peierls-Bogoliubov inequality in C^*-algebras and characterization of tracial functionals, Lobachevskii J. Math. 32 (3) (2011) 175–179.
[6] A.M. Bikchentaev, Inequalities for determinants and characterization of the trace, Sib. Math. J. 61 (2) (2020) 248–254.
[7] A.M. Bikchentaev, F. Kittaneh, M.S. Moslehian, Y. Seo, Trace Inequalities for Matrices and Hilbert Space Operators, Forum for Interdisciplinary Mathematics, Springer, 2024.
[8] K. Cho, T. Sano, Young's inequality and trace, Linear Algebra Appl. 431 (8) (2009) 1218–1222.
[9] M.D. Choi, P.Y. Wu, Convex combinations of projections, Linear Algebra Appl. 136 (1990) 25–42.
[10] M. Dalloul, Characterization of tracial functionals by inequalities for the spread of a matrix, Lobachevskii J. Math. 45 (6) (2024) 2478–2484.
[11] R.D. Grigorieff, A note on von Neumann's trace inequality, Math. Nachr. 151 (1991) 327–328.
[12] D.T. Hoa, H. Osaka, H.M. Toan, On generalized Powers-Størmer's inequality, Linear Algebra Appl. 438 (1) (2013) 242–249.
[13] Y. Li, U.-E. Li, Some characterizations of the trace norm triangle equality, Linear Algebra Appl. 484 (2015) 396–408.
[14] M.S. Moslehian, H. Osaka, Advanced Techniques with Block Matrices of Operators, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2024, x+218 pp. ISBN 978-3-031-64545-7; 978-3-031-64546-4.
[15] H. Neudecker, A matrix trace inequality, J. Math. Anal. Appl. 166 (1) (1992) 302–303.

[16] A.I. Stolyarov, O.E. Tikhonov, A.N. Sherstnev, Characterization of normal traces on von Neumann algebras by inequalities for the modulus, Math. Notes 72 (3–4) (2002) 411–416.
[17] Y.S. Yang, A matrix trace inequality, J. Math. Anal. Appl. 133 (2) (1988) 573–574.
[18] F. Zhang, Matrix Theory. Basic Results and Techniques, second edition, Universitext, Springer, New York, 2011.