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Continuity of Operator Functions in the Topology of Local Convergence in Measure

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Dedicated to Academician A. S. Holevo on the occasion of his 80th birthday

Abstract—Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, and let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $t_{\tau \mathrm{l}}$ be the topology of τ-local convergence in measure on the *-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators. We prove the $t_{\tau \mathrm{l}}$-continuity of the involution on the set of all normal operators in $S(\mathcal{M},\tau)$, investigate the $t_{\tau \mathrm{l}}$-continuity of operator functions on $S(\mathcal{M},\tau)$, and show that the map $A\mapsto |A|$ is $t_{\tau \mathrm{l}}$-continuous on the set of all partial isometries in $\mathcal{M}$.

Keywords—Hilbert space, linear operator, von Neumann algebra, normal trace, measurable operator, local convergence in measure, continuity of operator functions.

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1. INTRODUCTION

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, and let τ be a faithful normal semifinite trace on $\mathcal{M}$. This paper continues the research initiated in [1, 3–8, 10, 11, 15, 16] into the properties of the topologies $t_{\tau \mathrm{l}}$ and $t_{\mathrm{w}\tau \mathrm{l}}$ of τ-local and weakly τ-local convergence in measure, respectively, on the *-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators. We prove the $t_{\tau \mathrm{l}}$-continuity of the involution on the subset of all normal operators in $S(\mathcal{M},\tau)$ (Theorem 4.8), study the $t_{\tau \mathrm{l}}$-continuity of operator functions on $S(\mathcal{M},\tau)$ (Theorem 4.18) using some ideas and methods from [9, 13], and show that the map $A\mapsto |A|$ is $t_{\tau \mathrm{l}}$-continuous on the subset of all partial isometries in the algebra $\mathcal{M}$ (Corollary 4.3).

Note that the continuity of operator functions in the topology t_τ of convergence in measure on $S(\mathcal{M},\tau)$ was studied by the second author in [19], and on algebras of locally measurable operators, by M. A. Muratov and V. I. Chilin in [14]. Some of our results are new even for the *-algebra $\mathcal{M}=\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ equipped with the canonical trace $\tau=\mathrm{tr}$.

2. NOTATION AND DEFINITIONS

Let $\mathcal{M}$ be a von Neumann algebra of operators in a Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ the lattice of orthoprojections ($P=P^2=P^*$) in $\mathcal{M}$, I the identity in $\mathcal{M}$, $P^{\perp}=I-P$ for $P\in\mathcal{M}^{\mathrm{pr}}$, and $\mathcal{M}^+$ the cone of positive elements in $\mathcal{M}$.

A map $\varphi\colon \mathcal{M}^+\to[0,+\infty]$ is called a *trace* if $\varphi(X+Y)=\varphi(X)+\varphi(Y)$ and $\varphi(\lambda X)=\lambda\varphi(X)$ for all $X,Y\in\mathcal{M}^+$ and $\lambda\ge 0$ (with $0\cdot(+\infty)\equiv 0$) and in addition $\varphi(Z^*Z)=\varphi(ZZ^*)$ for all $Z\in\mathcal{M}$.

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A trace φ is said to be

- *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for all $X \in \mathcal{M}^+$;
- *normal* if whenever $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$), one has $\varphi(X) = \sup \varphi(X_i)$

(see [17, Ch. V, § 2]).

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated with the von Neumann algebra* $\mathcal{M}$ if it commutes with every unitary operator in the commutant $\mathcal{M}'$ of $\mathcal{M}$. In what follows, τ is a faithful normal semifinite trace on $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pr}}_\tau = \{P \in \mathcal{M}^{\mathrm{pr}} : \tau(P) < \infty\}$.

A closed densely defined operator X affiliated with $\mathcal{M}$, with domain $\mathcal{D}(X)$ in $\mathcal{H}$, is said to be *τ-measurable* if for every $\varepsilon > 0$ there exists a $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra with respect to taking the adjoint operator, multiplication by scalars, and the operations of strong addition and multiplication obtained by closing the ordinary operations [18, Ch. IX]. Let $S(\mathcal{M}, \tau)^{\mathrm{nor}}$ be the set of all normal ($A^*A = AA^*$) operators in $S(\mathcal{M}, \tau)$. For a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$ we denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ its positive and Hermitian parts, respectively. The partial order in $S(\mathcal{M}, \tau)^{\mathrm{h}}$ generated by the proper cone $S(\mathcal{M}, \tau)^+$ will be denoted by $\leq$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$. For operators $A \in S(\mathcal{M}, \tau)$ we will also use the notation

$$\operatorname{Re} A = \frac{1}{2}(A + A^*) \qquad \text{and} \qquad \operatorname{Im} A = \frac{1}{2\mathrm{i}}(A - A^*).$$

The *-algebra $S(\mathcal{M}, \tau)$ is equipped with the topology t_τ of convergence in measure [18, Ch. IX, § 2], for which a fundamental system of neighborhoods of zero is formed by the sets

$$\mathcal{U}(\varepsilon, \delta) = \bigl\{X \in S(\mathcal{M}, \tau) :\ \exists Q \in \mathcal{M}^{\mathrm{pr}}\ (\|XQ\| \leq \varepsilon \ \text{and}\ \tau(Q^\perp) \leq \delta)\bigr\}, \qquad \varepsilon > 0, \quad \delta > 0.$$

The algebra $\langle S(\mathcal{M}, \tau), t_\tau\rangle$ is known to be a complete metrizable topological *-algebra, and the algebra $\mathcal{M}$ is dense in $\langle S(\mathcal{M}, \tau), t_\tau\rangle$. To denote the convergence of a net $\{X_j\}_{j\in J} \subset S(\mathcal{M}, \tau)$ to an operator $X \in S(\mathcal{M}, \tau)$ in the topology t_τ, we write $X_j \xrightarrow{\tau} X$; in this case $\{X_j\}_{j\in J}$ is said to converge to X in measure τ.

Let $\mu(X; t)$ denote the *singular value function* of an operator $X \in S(\mathcal{M}, \tau)$, i.e., the nonincreasing right continuous function $\mu(X; \cdot) : (0, \infty) \to [0, \infty)$ defined as

$$\mu(X; t) = \inf\bigl\{\|XP\| :\ P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^\perp) \leq t\bigr\}, \qquad t > 0.$$

The set $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) :\ \lim_{t\to\infty} \mu(X; t) = 0\}$ of τ-compact operators is an ideal in the *-algebra $S(\mathcal{M}, \tau)$. The topology t_τ is also generated by the F-norm $\rho_\tau(X) = \inf_{t>0} \max\{t, \mu(X; t)\}$ for $X \in S(\mathcal{M}, \tau)$.

Lemma 2.1 [12]**.** *Let $X, Y, X_j \in S(\mathcal{M}, \tau)$, $j \in J$. Then the following assertions hold*:

(i) $\mu(X; t) = \mu(|X|; t) = \mu(X^*; t)$ *for all* $t > 0$;

(ii) $\mu(X^*X; t) = \mu(XX^*; t)$ *for all* $t > 0$;

(iii) *if* $|X| \leq |Y|$, *then* $\mu(X; t) \leq \mu(Y; t)$ *for all* $t > 0$;

(iv) *if* $X \in \mathcal{M}$, *then* $\lim_{t\to+0} \mu(X; t) = \sup_{t>0} \mu(X; t) = \|X\|$;

(v) $\mu(XY; t + s) \leq \mu(X; t)\mu(Y; s)$ *for all* $t, s > 0$;

(vi) $\mu(X + Y; t + s) \leq \mu(X; t) + \mu(Y; s)$ *for all* $t, s > 0$;

(vii) $\mu(|X|^\alpha; t) = \mu(X; t)^\alpha$ *for all* $\alpha > 0$ *and* $t > 0$;

(viii) $X_j \xrightarrow{\tau} X$ *if and only if* $\mu(X_j - X; t) \to 0$ *for every* $t > 0$;

(ix) *if* $A, Z \in \mathcal{M}$, *then* $\mu(AYZ; t) \leq \|A\|\mu(Y; t)\|Z\|$ *for all* $t > 0$.

3. TOPOLOGIES OF LOCAL CONVERGENCE IN MEASURE ON $S(\mathcal{M},\tau)$

The topology t_τ of convergence in measure can be localized as follows. For $\varepsilon,\delta>0$ and $P\in\mathcal{M}_\tau^{\rm pr}$ we define the sets

$$\mathcal{V}(\varepsilon,\delta,P)=\bigl\{X\in S(\mathcal{M},\tau)\colon\ \exists Q\in\mathcal{M}^{\rm pr}\ \bigl(Q\le P,\ \|XQ\|\le\varepsilon,\ \tau(P-Q)\le\delta\bigr)\bigr\},$$
$$\mathcal{W}(\varepsilon,\delta,P)=\bigl\{X\in S(\mathcal{M},\tau)\colon\ \exists Q\in\mathcal{M}^{\rm pr}\ \bigl(Q\le P,\ \|QXQ\|\le\varepsilon,\ \tau(P-Q)\le\delta\bigr)\bigr\}.$$

The space $S(\mathcal{M},\tau)$ becomes a topological vector space with respect to the topology $t_{\tau\rm l}$ of *τ-local convergence in measure*, with a basis of neighborhoods of zero given by the family $\Theta=\{\mathcal{V}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\rm pr}}$, as well as with respect to the topology $t_{\rm w\tau l}$ of *weak τ-local convergence in measure*, with a basis of neighborhoods of zero given by the family $\Theta=\{\mathcal{W}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\rm pr}}$. We will write $X_i\xrightarrow{\tau\rm l}X$ and $X_i\xrightarrow{\rm w\tau l}X$ to denote the $t_{\tau\rm l}$- and $t_{\rm w\tau l}$-convergence, respectively. Using the standard technique of reducing von Neumann algebras, one can show (see also [11, 16]) that $X_i\xrightarrow{\tau\rm l}X$ if and only if $X_iP\xrightarrow{\tau}XP$ for all $P\in\mathcal{M}_\tau^{\rm pr}$ (cf. [8, p. 114]), and that $X_i\xrightarrow{\rm w\tau l}X$ if and only if $PX_iP\xrightarrow{\tau}PXP$ for all $P\in\mathcal{M}_\tau^{\rm pr}$ (cf. [8, p. 114] and [10, p. 746]). It is clear that $t_{\rm w\tau l}\le t_{\tau\rm l}\le t_\tau$ and the $t_{\rm w\tau l}$-convergence coincides with the convergence in measure with respect to $\langle S(P\mathcal{M}P)=PS(\mathcal{M},\tau)P,t_{\tau_P}\rangle$ for all $P\in\mathcal{M}_\tau^{\rm pr}$, where $\tau_P(X)=\tau(PXP)$. The topologies $t_{\tau\rm l}$ and $t_{\rm w\tau l}$ can also be defined in terms of nonincreasing rearrangements. The family $\widetilde{\Theta}=\{\widetilde{\mathcal{V}}(\varepsilon,\delta,P)\}_{\varepsilon,\delta>0,\ P\in\mathcal{M}_\tau^{\rm pr}}$ with $\widetilde{\mathcal{V}}(\varepsilon,\delta,P)=\{X\in S(\mathcal{M},\tau)\colon\mu(XP;\delta)<\varepsilon\}$ also defines a basis of neighborhoods of zero for $t_{\tau\rm l}$. If $\tau(I)<\infty$, then $t_\tau=t_{\tau\rm l}=t_{\rm w\tau l}$; note that t_τ is the minimal metrizable topology consistent with the ring structure in $S(\mathcal{M},\tau)$ (see [2]).

If $\mathcal{M}$ is the *-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ and $\tau=\mathrm{tr}$ is the canonical trace, then $S(\mathcal{M},\tau)$ and $S_0(\mathcal{M},\tau)$ coincide with $\mathcal{B}(\mathcal{H})$ and with the ideal $\mathfrak{S}_\infty(\mathcal{H})$ of compact operators on $\mathcal{H}$, respectively. The topology t_τ coincides with the norm topology generated by the norm $\|\cdot\|$, and $t_{\tau\rm l}$ and $t_{\rm w\tau l}$ coincide with the topologies of strong and weak operator convergence, respectively. We have $\mu(X;t)=\sum_{n=1}^\infty s_n(X)\chi_{[n-1,n)}(t)$, $t>0$, where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of a completely continuous operator X and χ_A is the indicator of a set $A\subset\mathbb{R}$.

If $\mathcal{M}$ is abelian (i.e., commutative), then $\mathcal{M}\simeq L^\infty(\Omega,\Sigma,\nu)$ and $\tau(f)=\int_\Omega f\,d\nu$, where (Ω,Σ,ν) is a localizable measure space, and the algebra $S(\mathcal{M},\tau)$ coincides with the algebra of all measurable complex functions f on (Ω,Σ,ν) that are bounded everywhere except for sets of finite measure. In this case, the topology t_τ is the ordinary topology of convergence in measure, and $t_{\tau\rm l}=t_{\rm w\tau l}$ coincide with the well-known topology of convergence in measure on sets of finite measure.

4. ON THE CONTINUITY OF OPERATOR FUNCTIONS

Lemma 4.1 [1, Theorem 1, part 1]**.** *Let a net $\{A_\alpha\}\subset S(\mathcal{M},\tau)$ converge in the topology $t_{\tau\rm l}$ to an operator $A\in S(\mathcal{M},\tau)$. Then $A_\alpha B\xrightarrow{\tau\rm l}AB$ for any $B\in S(\mathcal{M},\tau)$.*

Using the definitions of $t_{\tau\rm l}$- and $t_{\rm w\tau l}$-convergence and the t_τ-continuity of the involution and product in the algebra $S(\mathcal{M},\tau)$, we easily obtain the following.

Proposition 4.2. *If $A,A_\alpha\in S(\mathcal{M},\tau)$ and $A_\alpha\xrightarrow{\tau\rm l}A$, then $|A_\alpha|^2\xrightarrow{\rm w\tau l}|A|^2$.*

Note that by Lemma 3.1(i) in [3] the map $A\mapsto|A|$ $(A\in S(\mathcal{M},\tau))$ is $t_{\tau\rm l}$-continuous at the point $A=0$.

Corollary 4.3. *If operators $A,A_\alpha\in\mathcal{M}$ are partial isometries and $A_\alpha\xrightarrow{\tau\rm l}A$, then $|A_\alpha|\xrightarrow{\tau\rm l}|A|$.*

Proof. Since $|A_\alpha|,|A|\in\mathcal{M}^{\rm pr}$, we have $|A_\alpha|=|A_\alpha|^2\xrightarrow{\rm w\tau l}|A|^2=|A|$, i.e., $|A_\alpha|\xrightarrow{\rm w\tau l}|A|$. Now by Lemma 3.7(i) in [3] we obtain $|A_\alpha|\xrightarrow{\tau\rm l}|A|$. $\square$

Corollary 4.4. *If $A_\alpha\in\mathcal{M}_r:=\{X\in\mathcal{M}\colon\|X\|\le r\}$, $r>0$, and $A_\alpha\xrightarrow{\rm w\tau l}A\in S(\mathcal{M},\tau)$, then $A\in\mathcal{M}_r$.*

Proof. Suppose that $A \notin \mathcal{M}_r$, i.e., $r < \|A\| \le +\infty$. Since $\|A^*A\| = \|A\|^2$, we have $r^2 < \|A^*A\| \le +\infty$. There exists a spectral orthoprojection Q of the operator A^*A and a number $a > r^2$ such that

$$A^*A \ge aQ. \tag{4.1}$$

Since the trace τ is semifinite, there exists a nonzero orthoprojection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ such that $P \le Q$. Then from (4.1) we obtain $PA^*AP \ge aP$. Since $\|A_\alpha^* A_\alpha\| \le r^2$, we have $PA_\alpha^* A_\alpha P \le r^2 P$ and

$$PA^*AP - PA_\alpha^* A_\alpha P \ge (a - r^2)P.$$

Therefore, by Lemma 2.1(ii),

$$\mu\big(PA^*AP - PA_\alpha^* A_\alpha P; t\big) \ge (a - r^2)\mu(P; t) = (a - r^2)\chi_{(0,\tau(P)]}(t).$$

Consequently, from Lemma 2.1(viii) we have $PA_\alpha^* A_\alpha P \overset{\tau}{\not\to} PA^*AP$. Thus, we have arrived at a contradiction. □

Theorem 4.5. *Let $A, A_n \in S(\mathcal{M}, \tau)$, $A_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} A$ as $n \to \infty$, and the sequence $\{A_n\}$ be t_τ-bounded. Then $BA_n \xrightarrow{\tau\mathrm{l}} BA$ as $n \to \infty$ for every operator $B \in S_0(\mathcal{M}, \tau)$ and $BA_nC \xrightarrow{\tau} BAC$ as $n \to \infty$ for every pair of operators $B, C \in S_0(\mathcal{M}, \tau)$.*

Proof. Let $X, X_n \in S(\mathcal{M}, \tau)$. Then

$$X_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} X \quad \text{as} \quad n \to \infty \qquad \Leftrightarrow \qquad PX_nQ \xrightarrow{\tau} PXQ \quad \text{as} \quad n \to \infty \quad \forall P, Q \in \mathcal{M}_\tau^{\mathrm{pr}}$$

(see [1, p. 20]). Since $X_n \xrightarrow{\mathrm{w}\tau\mathrm{l}} X$ as $n \to \infty$ if and only if $X_n^* \xrightarrow{\mathrm{w}\tau\mathrm{l}} X^*$ as $n \to \infty$, we have $PA_n^* \xrightarrow{\tau\mathrm{l}} PA^*$ as $n \to \infty$ for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. Now, by [1, Theorem 2] we get

$$PA_n^*B^* \xrightarrow{\tau} PA^*B^* \qquad \text{as} \quad n \to \infty \qquad \forall P \in \mathcal{M}_\tau^{\mathrm{pr}}, \quad \forall B \in S_0(\mathcal{M}, \tau) \tag{4.2}$$

(recall that $B^* \in S_0(\mathcal{M}, \tau)$). Passing to the adjoint operators in (4.2) and taking into account the t_τ-continuity of the involution in $S(\mathcal{M}, \tau)$, we have $BA_nP \xrightarrow{\tau} BAP$ as $n \to \infty$. Since the orthoprojection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ is arbitrary, we obtain $BA_n \xrightarrow{\tau\mathrm{l}} BA$ as $n \to \infty$. Applying [1, Theorem 2] once again, we conclude that $BA_nC \xrightarrow{\tau} BAC$ as $n \to \infty$ for every pair of operators $B, C \in S_0(\mathcal{M}, \tau)$. □

Corollary 4.6. *Let $A, A_n \in \mathcal{B}(\mathcal{H})$, $A_n \to A$ as $n \to \infty$ in the weak operator topology, and the sequence $\{A_n\}$ be $\|\cdot\|$-bounded. Then $BA_n \to BA$ as $n \to \infty$ in the strong operator topology for every operator $B \in \mathfrak{S}_\infty(\mathcal{H})$, and $\|B(A_n - A)C\| \to 0$ as $n \to \infty$ for every pair of operators $B, C \in \mathfrak{S}_\infty(\mathcal{H})$.*

Example 4.7. The condition that the sequence $\{A_n\}$ is t_τ-bounded is essential in Theorem 4.5. In the abelian von Neumann algebra $\mathcal{M} \simeq L^\infty(\mathbb{R}^+, \mathrm{d}\nu)$ with linear Lebesgue measure ν, consider the faithful normal semifinite trace $\tau(f) = \int_{\mathbb{R}^+} f \,\mathrm{d}\nu$ and set

$$f_n = n\chi_{[n,2n]}, \qquad n \in \mathbb{N}.$$

Then $f_n \xrightarrow{\tau\mathrm{l}} 0$ as $n \to \infty$ and for τ-compact g and h, $g = h$, given by the function $\min\{1, x^{-1/2}\}$, $x \in \mathbb{R}^+$, we have $\mu(gf_nh; t) \ge \chi_{(0,n]}(t)/2 \not\to 0$ as $n \to \infty$ for every $t > 0$. Therefore, $gf_nh \overset{\tau}{\not\to} 0$ as $n \to \infty$ by Lemma 2.1(viii).

Theorem 4.8. *If $A, A_\alpha \in S(\mathcal{M}, \tau)^{\mathrm{nor}}$ and $A_\alpha \xrightarrow{\tau\mathrm{l}} A$, then $A_\alpha^* \xrightarrow{\tau\mathrm{l}} A^*$.*

Proof. *Step* 1. We have $PA_\alpha^* \xrightarrow{\tau} PA^*$ for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ by the t_τ-continuity of the involution in $S(\mathcal{M}, \tau)$. Therefore, by the t_τ-continuity of the product in $S(\mathcal{M}, \tau)$, we obtain

$$PA_\alpha \cdot A_\alpha^* P = PA_\alpha^* \cdot A_\alpha P \xrightarrow{\tau} PA^* \cdot AP = PA \cdot A^*P$$

for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. Thus, $\mu(PA_\alpha A_\alpha^* P - PAA^*P; t) \to 0$ for every $t > 0$ by Lemma 2.1(viii).

Step 2. For every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ and $t > 0$ we estimate

$$\begin{aligned}
\mu(A_\alpha^* P - A^* P; t)^2 &= \mu(|A_\alpha^* P - A^* P|; t)^2 = \mu(|A_\alpha^* P - A^* P|^2; t) \\
&= \mu\big((PA_\alpha - PA)(A_\alpha^* P - A^* P); t\big) \\
&= \mu\big(PA_\alpha A_\alpha^* P + PAA^* P - PA_\alpha A^* P - PAA_\alpha^* P; t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P + PA^* AP - 2\,\mathrm{Re}(PA_\alpha A^* P); t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P + PA^* AP - 2\,\mathrm{Re}(PA^* AP) - 2\,\mathrm{Re}(PA_\alpha A^* P - PA^* AP); t\big) \\
&= \mu\big(PA_\alpha^* A_\alpha P - PA^* AP - 2\,\mathrm{Re}(PA_\alpha A^* P - PA^* AP); t\big) \\
&\le \mu\Big(PA_\alpha^* A_\alpha P - PA^* AP; \frac{t}{2}\Big) + 2\mu\Big(\mathrm{Re}(PA_\alpha A^* P - PA^* AP); \frac{t}{2}\Big) \qquad (4.3)
\end{aligned}$$

by Lemma 2.1(vi), (vii). According to step 1 we have

$$\mu\Big(PA_\alpha^* A_\alpha P - PA^* AP; \frac{t}{2}\Big) \to 0$$

for every $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ and $t > 0$. Let us estimate the second term in the last inequality in (4.3):

$$\begin{aligned}
2\mu\Big(\mathrm{Re}\big(PA_\alpha A^* P - PA^* AP\big); \frac{t}{2}\Big) &= 2\mu\Big(P\big(\mathrm{Re}(A_\alpha A^* - A^* A)P\big); \frac{t}{2}\Big) \\
&= \mu\Big(P(A_\alpha - A)A^* P + PA(A_\alpha^* - A^*)P; \frac{t}{2}\Big) \\
&\le \mu\Big(P(A_\alpha - A)A^* P; \frac{t}{4}\Big) + \mu\Big(PA(A_\alpha^* - A^*)P; \frac{t}{4}\Big) \\
&\le \|P\|\,\mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) + \|P\|\,\mu\Big(PA(A_\alpha^* - A^*); \frac{t}{4}\Big) \\
&= \mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) + \mu\Big((PA(A_\alpha^* - A^*))^*; \frac{t}{4}\Big) \\
&= 2\mu\Big((A_\alpha - A)A^* P; \frac{t}{4}\Big) \to 0
\end{aligned}$$

by Lemma 2.1(vi), (ix), (viii) and the t_τ-continuity of multiplication by the operator A^* on the left (see Lemma 4.1). Thus, $\mu(A_\alpha^* P - A^* P; t) \to 0$ for arbitrary $t > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. This completes the proof of the theorem. □

For an operator $A \in S(\mathcal{M}, \tau)$, let $R_\lambda(A)$ denote its resolvent.

Lemma 4.9. *If a net $\{A_\alpha\}$ in the *-algebra $S(\mathcal{M}, \tau)^{\mathrm{h}}$ converges to A in the topology $t_{\tau \mathrm{l}}$, then $R_\lambda(A_\alpha) \xrightarrow{\tau \mathrm{l}} R_\lambda(A)$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. As is well known,

$$R_\lambda(A) - R_\lambda(A_\alpha) = R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A).$$

Take a $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$. Since $A_\alpha - A \xrightarrow{\tau \mathrm{l}} 0$, we have $(A_\alpha - A)R_\lambda(A) \xrightarrow{\tau \mathrm{l}} 0$ by Lemma 4.1. Therefore, $(A_\alpha - A)R_\lambda(A)Q \xrightarrow{\tau} 0$, i.e., $\mu((A_\alpha - A)R_\lambda(A)Q; s) \to 0$ for any $s > 0$. Now, using

$$\begin{aligned}
\mu\big(R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A)Q; s\big) &\le \|R_\lambda(A_\alpha)\|\,\mu\big((A_\alpha - A)R_\lambda(A)Q; s\big) \\
&\le |\mathrm{Im}\,\lambda|^{-1}\mu\big((A_\alpha - A)R_\lambda(A)Q; s\big),
\end{aligned}$$

we obtain $\mu(R_\lambda(A_\alpha)(A_\alpha - A)R_\lambda(A)Q; s) \to 0$.

Thus, $(R_\lambda(A) - R_\lambda(A_\alpha))Q \xrightarrow{\tau} 0$ for every $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$; i.e., $R_\lambda(A_\alpha) \xrightarrow{\tau \mathrm{l}} R_\lambda(A)$. □

Lemma 4.10. *Let f and g be two continuous functions from $\mathbb{R}$ (or $\mathbb{C}$) to $\mathbb{C}$, and let g be bounded. If the operator functions f and g are $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$), then the operator function fg is also $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, respectively).*

Proof. Let $A_\alpha \xrightarrow{\tau l} A$. We can write

$$(gf)(A) - (gf)(A_\alpha) = (g(A) - g(A_\alpha))f(A) + g(A_\alpha)(f(A) - f(A_\alpha)).$$

Lemma 4.1 implies that $(g(A) - g(A_\alpha))f(A) \xrightarrow{\tau l} 0$. Using the estimate

$$\mu\big(g(A_\alpha)(f(A) - f(A_\alpha))Q; s\big) \le |g|\, \mu\big((f(A) - f(A_\alpha))Q; s\big)$$

for $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$, we find that $g(A_\alpha)(f(A) - f(A_\alpha)) \xrightarrow{\tau l} 0$. Thus, the operator function gf is $t_{\tau l}$-continuous. □

Lemma 4.11. *Let a sequence (f_n) of continuous functions acting from $\mathbb{R}$ (or $\mathbb{C}$) to $\mathbb{C}$ converge to a function f uniformly on $\mathbb{R}$ (on $\mathbb{C}$, respectively). If the operator functions f_n are $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$), then the operator function f is also $t_{\tau l}$ -continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$ (on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, respectively).*

Proof. We will give a proof for functions on $\mathbb{C}$.

Take an $\varepsilon > 0$ and choose n_0 such that $\sup_{x\in\mathbb{C}}|f(x) - f_{n_0}(x)| \le \varepsilon/3$. Let $A_\alpha \xrightarrow{\tau l} A$ and $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$. For $s > 0$, by Lemma 2.1(iv)–(vi), (ix) we have

$$\begin{aligned}
&\mu\big((f(A) - f(A_\alpha))Q; s\big) \\
&\qquad = \mu\Big(\big(f(A) - f_{n_0}(A)\big)Q + \big(f_{n_0}(A) - f_{n_0}(A_\alpha)\big)Q + \big(f_{n_0}(A_\alpha) - f(A_\alpha)\big)Q; s\Big) \\
&\qquad \le \mu\Big((f - f_{n_0})(A)Q; \frac{s}{3}\Big) + \mu\Big(\big(f_{n_0}(A) - f_{n_0}(A_\alpha)\big)Q; \frac{s}{3}\Big) + \mu\Big((f - f_{n_0})(A_\alpha)Q; \frac{s}{3}\Big) \\
&\qquad \le \|(f - f_{n_0})(A)Q\| + \mu\Big(\big(f_{n_0}(A) - f_{n_0}(A_\alpha)\big)Q; \frac{s}{3}\Big) + \|(f - f_{n_0})(A_\alpha)Q\| \\
&\qquad \le \frac{2\varepsilon}{3} + \mu\Big(\big(f_{n_0}(A) - f_{n_0}(A_\alpha)\big)Q; \frac{s}{3}\Big).
\end{aligned}$$

Since the operator function f_{n_0} is $t_{\tau l}$-continuous, the second term in the last expression does not exceed $\varepsilon/3$ for sufficiently large values of α. □

Proposition 4.12. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ with $f(x) = O(x)$ as $x \to \infty$. Then the operator function f is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$.*

Proof. Let us first consider the case where $f(x) \to 0$ as $x \to \infty$. If $p(x)$ and $q(x)$ are real polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$ and $q(x)$ has no real roots, then the rational function $r(x) = p(x)/q(x)$ can be represented as a finite linear combination of functions of the form $(x - \lambda)^{-n}$ ($\lambda \in \mathbb{C} \setminus \mathbb{R}$, $n \in \mathbb{N}$). By Lemmas 4.9 and 4.10, we conclude that the operator function r is $t_{\tau l}$-continuous. By Stone's theorem, $f(x)$ can be uniformly approximated on $\mathbb{R}$ by a sequence of rational functions $r(x)$ of the form considered above, and then Lemma 4.11 implies the $t_{\tau l}$-continuity of the operator function f.

Now we turn to the general case. Let us represent f in the form

$$f(x) = f(x)\frac{1}{1 + x^2} + f(x)\frac{x^2}{1 + x^2}.$$

Denote the first term by $f_1(x)$ and the second by $f_2(x)$. Then $f_1(x) \to 0$ as $x \to \infty$; therefore, the operator function f_1 is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$. Successive application of Lemma 4.10 to the functions $xf_1(x)$ and $x(xf_1(x))$ shows that the operator function f_2 is $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{h}}$. □

Lemma 4.13. *The maps $A \mapsto \operatorname{Re} A$ and $A \mapsto \operatorname{Im} A$ are $t_{\tau l}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$.*

Proof. This follows from Theorem 4.8. □

Lemma 4.14. *The map* $A \mapsto I + |\mathrm{Re}\, A| + |\mathrm{Im}\, A|$ *is* $t_{\tau\mathrm{l}}$*-continuous on* $S(\mathcal{M},\tau)^{\mathrm{nor}}$.

Proof. Apply Lemma 4.13 and Proposition 4.12. □

Proposition 4.15. *Let a continuous function* $f\colon \mathbb{C} \to \mathbb{C}$ *be* $O(|z|)$ *as* $z \to \infty$. *Then the corresponding operator function* f *is* $t_{\tau\mathrm{l}}$*-continuous on* $S(\mathcal{M},\tau)^{\mathrm{nor}}$.

Proof. Let $\mathcal{A}$ be the algebra of $\mathbb{R}$-valued functions on $\mathbb{C}$ generated by functions of the form $f(\mathrm{Re}\, z)$ and $g(\mathrm{Im}\, z)$ for continuous $\mathbb{R}$-valued functions $f(x)$ and $g(x)$ on $\mathbb{R}$ that tend to zero as $x \to \infty$. By Stone's theorem, the algebra $\mathcal{A}$ is uniformly dense in the algebra $\overline{\mathcal{A}}$ of all continuous functions from $\mathbb{C}$ to $\mathbb{R}$ that tend to zero at infinity. From Lemma 4.13, Proposition 4.12, and Lemma 4.10, it follows that the operator functions corresponding to the functions in $\mathcal{A}$ are $t_{\tau\mathrm{l}}$-continuous, and by Lemma 4.11 the same holds for the functions in $\overline{\mathcal{A}}$.

Now let a continuous function $f\colon \mathbb{C} \to \mathbb{R}$ be $O(|z|)$ as $z \to \infty$. We can write

$$f(z) = (1 + |\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2 \frac{f(z)}{(1 + |\mathrm{Re}\, z| + |\mathrm{Im}\, z|)^2}$$

and apply Lemmas 4.14 and 4.10 twice.

Finally, let a continuous function $f\colon \mathbb{C} \to \mathbb{C}$ be $O(|z|)$ as $z \to \infty$. Then the functions $\mathrm{Re}\, f(z)$ and $\mathrm{Im}\, f(z)$ act continuously from $\mathbb{C}$ to $\mathbb{R}$ and are $O(|z|)$ as $z \to \infty$. The corresponding operator functions $\mathrm{Re}\, f(A)$ and $\mathrm{Im}\, f(A)$ are $t_{\tau\mathrm{l}}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$, and hence the operator function $f(A) = \mathrm{Re}\, f(A) + \mathrm{i}\ \mathrm{Im}\, f(A)$ is $t_{\tau\mathrm{l}}$-continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}$. □

Corollary 4.16. *The map* $A \mapsto |A|$ *is* $t_{\tau\mathrm{l}}$*-continuous on* $S(\mathcal{M},\tau)^{\mathrm{nor}}$.

The following result is a corollary to [3, Lemma 3.4].

Lemma 4.17. *Let* $\{A_\alpha\}$ *and* $\{B_\alpha\}$ *be two nets in* $\mathcal{M}^+$ *such that* $\{A_\alpha\}$ *is uniformly bounded,* $A_\alpha \xrightarrow{\tau\mathrm{l}} 0$, *and* $B_\alpha \le A_\alpha$ *for all* α. *Then* $B_\alpha \xrightarrow{\tau\mathrm{l}} 0$.

Next, for $\Omega \subset \mathbb{C}$ we set $S(\mathcal{M},\tau)^{\mathrm{nor}}_{\Omega} = \{A \in S(\mathcal{M},\tau)^{\mathrm{nor}}\colon \mathrm{Sp}(A) \subset \Omega\}$, where $\mathrm{Sp}(A)$ is the spectrum of the operator A.

Theorem 4.18. *Let* $\Omega \subset \mathbb{C}$ *and* $A \in S(\mathcal{M},\tau)^{\mathrm{nor}}_{\Omega}$. *Let* $f\colon \Omega \to \mathbb{C}$ *be a function such that its restriction to any bounded subset of* Ω *that is closed in* $\mathbb{C}$ *is Borel measurable,* $\sup\{|f(z)|/(1+|z|)\colon z \in \Omega\} < \infty$, *and* f *is continuous at every point in* $\mathrm{Sp}(A)$. *If a net* $\{A_\alpha\}$ *of operators in* $S(\mathcal{M},\tau)^{\mathrm{nor}}_{\Omega}$ *converges to* A *in the topology* $t_{\tau\mathrm{l}}$, *then* $f(A_\alpha) \xrightarrow{\tau\mathrm{l}} f(A)$.

Proof. It suffices to prove the theorem for a real-valued function f, and we will do this in two stages.

1. Suppose that f is bounded on Ω, say $|f| \le 1$, and is continuous at all points in $\mathrm{Sp}(A)$. By the Tietze–Urysohn theorem, there exists a continuous function $g\colon \mathbb{C} \to [-1,1]$ that coincides with f on $\mathrm{Sp}(A)$. By Proposition 4.15,

$$g(A_\alpha) \xrightarrow{\tau\mathrm{l}} g(A) = f(A).$$

According to [20, Lemma 2], there exists a bounded continuous function h on Ω such that $h = 0$ on $\mathrm{Sp}(A)$ and $|f - g| \le h$ on Ω.

1a. Now assume additionally that $\Omega = \mathbb{C}$. Then $h(A_\alpha) \xrightarrow{\tau} h(A) = 0$ by Proposition 4.15. Since $0 \le (f - g + h)(A_\alpha) \le 2h(A_\alpha)$, we have $(f - g + h)(A_\alpha) \xrightarrow{\tau\mathrm{l}} 0$ by Lemma 4.17. Hence,

$$f(A_\alpha) = (f - g + h)(A_\alpha) + g(A_\alpha) - h(A_\alpha) \xrightarrow{\tau\mathrm{l}} f(A).$$

1b. If $\Omega \ne \mathbb{C}$, then according to [20, Lemma 1] we construct a bounded function $k\colon \mathbb{C} \to \mathbb{R}$ that extends h, is continuous at all points in Ω, and is upper semicontinuous on $\mathbb{C}$ (so it is Borel measurable). Using case 1a, we obtain

$$h(A_\alpha) = k(A_\alpha) \xrightarrow{\tau l} k(A) = h(A) = 0,$$

and it remains to repeat verbatim the last two sentences of the previous paragraph.

2. In the general case, we set $g(z) = f(z)/(1+|z|)$. Then $g(A_\alpha) \xrightarrow{\tau l} g(A)$ by the above and $I + |A_\alpha| \xrightarrow{\tau l} I + |A|$ by Proposition 4.15. Hence by [1, Theorem 3] we obtain

$$f(A_\alpha) = g(A_\alpha)(I + |A_\alpha|) \xrightarrow{\tau l} g(A)(I + |A|) = f(A). \qquad \square$$

Corollary 4.19. *Let $\Omega \subset \mathbb{C}$ and $f\colon \Omega \to \mathbb{C}$ be a continuous function on Ω such that $\sup\{|f(z)|/(1+|z|)\colon z \in \Omega\} < \infty$. Then the corresponding operator function is continuous on $S(\mathcal{M},\tau)^{\mathrm{nor}}_{\Omega}$ in the topology $t_{\tau l}$.*

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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