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Differences of Idempotents in C^*-algebras and the Quantum Hall Effect. II. Unbounded Idempotents

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Abstract—Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, I be the unit of $\mathcal{M}$, τ be a faithful semifinite normal trace on $\mathcal{M}$. Let $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators and $L_1(\mathcal{M},\tau)$ be the Banach space of all τ-integrable operators, $P, Q \in S(\mathcal{M},\tau)$ be idempotents. If $P-Q \in L_1(\mathcal{M},\tau)$ then $\tau(P-Q) \in \mathbb{R}$. In particular, if $A = A^3 \in L_1(\mathcal{M},\tau)$, then $\tau(A) \in \mathbb{R}$. If $P-Q \in L_1(\mathcal{M},\tau)$ and $PQ \in \mathcal{M}$, then for all $n \in \mathbb{N}$ we have $(P-Q)^{2n+1} \in L_1(\mathcal{M},\tau)$ and $\tau((P-Q)^{2n+1}) = \tau(P-Q) \in \mathbb{R}$. If $A \in L_2(\mathcal{M},\tau)$ and $U \in \mathcal{M}$ is an isometry, then $||UA-A||_2^2 \le 2||(I-U)AA^*||_1$.

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1. INTRODUCTION

Let P and Q be idempotents on a Hilbert space $\mathcal{H}$. If $X = P-Q$ is a trace-class operator, then the traces of all odd powers of X coincide:

$$\mathrm{tr}(P-Q) = \mathrm{tr}((P-Q)^{2n+1}) = \dim\ker(X-I) - \dim\ker(X+I) \in \mathbb{Z}, \tag{1}$$

where I is the identity operator on $\mathcal{H}$. If X is a compact operator, then the right-hand side of (1) yields a natural "regularization" for the trace and shows that it is always an integer [1, 2]. In [3, Theorem 3] we established a C^* analogue of this statement: let φ be a trace on a unital C^*-algebra $\mathcal{A}$, let $\mathfrak{M}_\varphi$ be the definition ideal of the trace φ and consider tripotents $P, Q \in \mathcal{A}$. If $P-Q \in \mathfrak{M}_\varphi$, then $\varphi(P-Q) \in \mathbb{R}$.

Pairs of idempotents play an important role in the quantum Hall effect [4]. For idempotents P, Q, and R with the trace-class operators $P-Q$ and $Q-R$, from the equality $\mathrm{tr}(P-Q) = \mathrm{tr}(P-R) + \mathrm{tr}(R-Q)$ and (1), we obtain

$$\mathrm{tr}\left((P-Q)^3\right) = \mathrm{tr}\left((P-R)^3\right) + \mathrm{tr}\left((R-Q)^3\right). \tag{2}$$

The physical meaning of the additivity in Eq. (2) comes from the interpretation of $\mathrm{tr}((P-Q)^3)$ as *the Hall conductance*. The additivity (cubic) Eq. (2) can be considered as a variant of the Ohm's law for the additivity of conductivity [5]. In [6, Theorem 1] we established a C^* analogue of the quantum Hall effect and proved the reality of trace of differerces of wide class of symmetries from a unital C^*-algebra (see Corollaries 2 and 3 in [6]).

We generalize these results to unbounded idempotents, tripotents, and symmetries, affiliated to a von Neumann algebra (examples of such operators see in [7]). Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators, $S(\mathcal{M},\tau)^{\mathrm{id}} = \{A \in S(\mathcal{M},\tau) : A = A^2\}$, and let $L_1(\mathcal{M},\tau)$ be

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the Banach space of all τ-integrable operators. This paper continues the investigations of properties of τ-measurable operators, started in [7] and is an English translation of the Russian-language paper [8]. We obtain the following results: If $P, Q \in S(\mathcal{M},\tau)^{\mathrm{id}}$ and $P-Q \in L_1(\mathcal{M},\tau)$, then $\tau(P-Q) \in \mathbb{R}$ (Theorem 1). If $A = A^3 \in L_1(\mathcal{M},\tau)$, then $\tau(A) \in \mathbb{R}$ (Corollary 1). Let $A, B \in S(\mathcal{M},\tau)$ be tripotents. If $A - B \in L_1(\mathcal{M},\tau)$ and $A + B \in \mathcal{M}$, then $\tau(A-B) \in \mathbb{R}$ (Corollary 2). Let $U, V \in S(\mathcal{M},\tau)$ be symmetries ($U^2 = I$). If $U - V \in L_1(\mathcal{M},\tau)$, then $\tau(U-V) \in \mathbb{R}$ (Corollary 4). Let $P, Q \in S(\mathcal{M},\tau)^{\mathrm{id}}$ with $P - Q \in L_1(\mathcal{M},\tau)$ and $PQ \in \mathcal{M}$. Then, for all $n \in \mathbb{N}$ we have $(P-Q)^{2n+1} \in L_1(\mathcal{M},\tau)$ and $\tau((P-Q)^{2n+1}) = \tau(P-Q) \in \mathbb{R}$ (Theorem 2). If $P, Q, R \in S(\mathcal{M},\tau)^{\mathrm{id}}$ with $P-Q, Q-R \in L_1(\mathcal{M},\tau)$ and operators PQ, QR, PR lie in $\mathcal{M}$, then $\tau((P-R)^{2n+1}) = \tau((P-Q)^{2n+1}) + \tau((Q-R)^{2n+1})$ for all $n \in \mathbb{N}$ (Corollary 6). If $A = A^2 \in L_2(\mathcal{M},\tau)$ and $\mathrm{Re}(A) \geq sA^*A - (s-1)AA^*$ for some $s \in \mathbb{R}$, then A is a projection (Corollary 9). If $A \in L_2(\mathcal{M},\tau)$ and $U \in \mathcal{M}$ is a isometry, then $||UA - A||_2^2 \leq 2||(I-U)AA^*||_1$ (Theorem 5).

2. NOTATION AND DEFINITIONS

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, I be the unit of $\mathcal{M}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections ($P = P^2 = P^*$) in $\mathcal{M}$ and $P^\perp = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$, let $\mathcal{M}^+$ be the cone of all positive operators in $\mathcal{M}$. An operator $U \in \mathcal{M}$ is called an *isometry*, if $U^*U = I$; *unitary*, if $U^*U = UU^* = I$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called (see [9, Chap. V, § 2])

- *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal*, if $X_i \uparrow X (X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$;
- *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) :\ Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M},\tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [10, Chap. IX].

Let $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M},\tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M},\tau)^{\mathrm{h}}$ generated by its proper cone $S(\mathcal{M},\tau)^+$. If $X \in S(\mathcal{M},\tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M},\tau)^+$.

An operator $A \in S(\mathcal{M},\tau)$ is called an *idempotent*, if $A^2 = A$; a *tripotent*, if $A^3 = A$; a *symmetry*, if $A^2 = I$. Denote by $[A, B] = AB - BA$ the commutator of operators $A, B \in S(\mathcal{M},\tau)$.

The generalized singular value function $\mu(\cdot; X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(t; X) = \inf\{||XP|| :\ P \in \mathcal{M}^{\mathrm{pr}} \text{ and } \tau(P^\perp) \leq t\}, \quad t > 0.$$

It is a non-increasing right-continuous function, and if $A \in S(\mathcal{M},\tau)^{\mathrm{id}}$, then $\mu(t; A) \in \{0\} \cup [1, +\infty)$ for all $t > 0$ [11, Theorem 3.3].

Let m be the linear Lebesgue measure on $\mathbb{R}$. Noncommutative Lebesgue L_p-space ($0 < p < \infty$), assosiated with $(\mathcal{M},\tau)$, may be defined as

$$L_p(\mathcal{M},\tau) = \{X \in S(\mathcal{M},\tau) :\ \mu(\cdot; X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (norm for $1 \leq p < \infty$) $||X||_p = ||\mu(\cdot; X)||_p$, $X \in L_p(\mathcal{M},\tau)$. The extension of τ to the unique linear functional on the whole space $L_1(\mathcal{M},\tau)$ we denote by the same letter τ. A linear subspace $\mathcal{E} \subset S(\mathcal{M},\tau)$ is called an *ideal space* on $(\mathcal{M},\tau)$, if

1. $X \in \mathcal{E} \Rightarrow X^* \in \mathcal{E}$;

2. $X \in \mathcal{E}, Y \in S(\mathcal{M},\tau)$ and $|Y| \le |X| \Rightarrow Y \in \mathcal{E}$.

Such are, for example, the algebra $\mathcal{M}$, the collection of all elementary operators $\mathcal{F}(\mathcal{M},\tau)$ and $L_p(\mathcal{M},\tau)$ for $0 < p < \infty$. For every ideal space $\mathcal{E}$ on $(\mathcal{M},\tau)$ we have $\mathcal{M}\mathcal{E}\mathcal{M} \subset \mathcal{E}$ [12, Lemma 5]. An ideal space $\mathcal{E}$ on $(\mathcal{M},\tau)$, equipped with an F-norm $||\cdot||_{\mathcal{E}}$, is called an *F-normed ideal space* on $(\mathcal{M},\tau)$, if

1. $||X||_{\mathcal{E}} = ||X^*||_{\mathcal{E}}$ for all $X \in \mathcal{E}$;

2. $X, Y \in \mathcal{E}$ and $|Y| \le |X| \Rightarrow ||Y||_{\mathcal{E}} \le ||X||_{\mathcal{E}}$ (see [13, 14]).

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the *-algebra of all bounded linear operators on $\mathcal{H}$, and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M},\tau)$ coincides with $\mathcal{B}(\mathcal{H})$, the space $L_p(\mathcal{M},\tau)$ coincides with the Shatten—von Neumann *-ideal $\mathfrak{S}_p(\mathcal{H})$ of compact operators in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t;X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X; χ_A is the indicator function of the set $A \subset \mathbb{R}$.

If $\mathcal{M}$ is Abelian (i.e., commutative), then $\mathcal{M} \simeq L^{\infty}(\Omega,\Sigma,\nu)$ and $\tau(f) = \int_{\Omega} f \,\mathrm{d}\,\nu$, where (Ω,Σ,ν) is a localized measure space, the *-algebra $S(\mathcal{M},\tau)$ coincides with the algebra of all complex measurable functions f on (Ω,Σ,ν), bounded everywhere but for a set of finite measure. The function $\mu(t;f)$ coincides with the nonincreasing rearrangement of the function $|f|$; see properties of such rearrangements in [15].

3. DIFFERENCES OF UNBOUNDED IDEMPOTENTS AND A TRACE

Lemma 1 ([10], Chap. IX, Theorem 2.13). *If $A \in \mathcal{M}$ and $B \in L_1(\mathcal{M},\tau)$*, then $AB, BA \in L_1(\mathcal{M},\tau)$.

Lemma 2 [16]. *If $A, B \in S(\mathcal{M},\tau)$ and $AB, BA \in L_1(\mathcal{M},\tau)$, then $\tau(AB) = \tau(BA)$.*

Lemma 3 ([17], Theorem 2.23). *For every $P = P^2 \in S(\mathcal{M},\tau)$ there exists the unique representation $P = \tilde{P} + Z$, where $\tilde{P} \in \mathcal{M}^{pr}$ and a nilpotent Z belongs to $S(\mathcal{M},\tau)$ with $Z^2 = 0$, moreover, $Z\tilde{P} = 0$, $\tilde{P}Z = Z$.*

Theorem 1. *If $P, Q \in S(\mathcal{M},\tau)^{id}$ and $P - Q \in L_1(\mathcal{M},\tau)$, then $\tau(P - Q) \in \mathbb{R}$.*

Proof. Let $P = \tilde{P} + Z$, $Q = \tilde{Q} + T$ be representations of Lemma 3 for $P, Q \in S(\mathcal{M},\tau)^{\mathrm{id}}$. By Lemma 1, we have

$$\tilde{P} - \tilde{Q}\tilde{P} = (P - Q)\tilde{P} - \tilde{Q}(P - Q)\tilde{P} \in L_1(\mathcal{M},\tau).$$

It can be analogously be verified that $\tilde{Q} - \tilde{P}\tilde{Q} \in L_1(\mathcal{M},\tau)$. Therefore,

$$\tilde{P} - \tilde{Q} = \tilde{P} - \tilde{Q}\tilde{P} - (\tilde{Q} - \tilde{P}\tilde{Q})^* \in L_1(\mathcal{M},\tau)$$

and $Z - T = P - Q - (\tilde{P} - \tilde{Q}) \in L_1(\mathcal{M},\tau)$. According Lemma 1 operators

$$T\tilde{P} = (T - Z)\tilde{P}, \quad Z\tilde{Q} = (Z - T)\tilde{Q}, \quad Z - \tilde{P}T = \tilde{P}(Z - T), \quad \tilde{Q}Z - T = \tilde{Q}(Z - T)$$

lie in $L_1(\mathcal{M},\tau)$, hence, $\tilde{Q}Z - \tilde{P}T = Z - \tilde{P}T + (\tilde{Q}Z - T) - (Z - T) \in L_1(\mathcal{M},\tau)$. Therefore,

$$\tilde{P}T - T = \tilde{Q}Z - T - (\tilde{Q}Z - \tilde{P}T) \in L_1(\mathcal{M},\tau).$$

By Lemmas 1 and 2, we have $0 = \tau([Z - T, \tilde{Q}]) = \tau(Z\tilde{Q} - \tilde{Q}Z + T)$. Since the operators

$$(\tilde{P} - \tilde{Q})T = \tilde{P}T - T, \quad T(\tilde{P} - \tilde{Q}) = T\tilde{P}$$

lie in $L_1(\mathcal{M},\tau)$, by Lemma 2, with $A = \tilde{P} - \tilde{Q}$, $B = T$ we obtain

$$\tau(\tilde{P}T - T) = \tau(T\tilde{P}). \tag{3}$$

Since $0 = \tau([Z - T, \tilde{P}]) = \tau(-T\tilde{P} - Z + \tilde{P}T)$, from (3) we have

$$0 = \tau(-T + \tilde{P}T - T\tilde{P}) = \tau(Z - T + (-Z + \tilde{P}T - T\tilde{P}))$$
$$= \tau(Z - T) + \tau(-Z + \tilde{P}T - T\tilde{P}) = \tau(Z - T).$$

Thus, $\tau(P - Q) = \tau(\tilde{P} - \tilde{Q}) + \tau(Z - T) = \tau(\tilde{P} - \tilde{Q}) \in \mathbb{R}$, since the operator $\tilde{P} - \tilde{Q}$ is selfadjoint. □

Corollary 1. *If $A = A^3 \in L_1(\mathcal{M}, \tau)$, then $\tau(A) \in \mathbb{R}$.*

Proof. Every tripotent ($A = A^3$) from an arbitrary algebra is the difference of two idempotents from this algebra [18, Proposition 1]. □

Note that Corollary 1 simultaneously reinforces both Corollary 2.31 from [17] (here we get rid of superfluous condition $A - A^2 \in \mathcal{M}$) and Corollary 3.13 from [7] (here we get rid of superfluous condition $A^2 \in L_1(\mathcal{M}, \tau)$).

Corollary 2. *Assume that $A, B \in S(\mathcal{M}, \tau)$ are tripotents. If $A - B \in L_1(\mathcal{M}, \tau)$ and $A + B \in \mathcal{M}$, then $\tau(A - B) \in \mathbb{R}$.*

Proof. Let $A = P_1 - Q_1$, $B = P_2 - Q_2$ be the representations from [18, Proposition 1], i.e. $P_k, Q_k \in S(\mathcal{M}, \tau)^{\mathrm{id}}$ and $P_kQ_k = Q_kP_k = 0$ for $k = 1, 2$. It seems clear that the operators $A^2 = P_1 + Q_1$ and $B^2 = P_2 + Q_2$ lie in $S(\mathcal{M}, \tau)^{\mathrm{id}}$. Since the operator $A - B = P_1 - Q_1 - P_2 + Q_2$ lies in $L_1(\mathcal{M}, \tau)$, by Lemma 1, the operator

$$A^2 - B^2 = \frac{1}{2}((A + B)(A - B) + (A - B)(A + B)) = P_1 + Q_1 - P_2 - Q_2$$

also lies in $L_1(\mathcal{M}, \tau)$. Then, the operators

$$P_1 - P_2 = \frac{1}{2}(A - B + A^2 - B^2), \quad Q_2 - Q_1 = \frac{1}{2}(A - B - (A^2 - B^2))$$

belong to $L_1(\mathcal{M}, \tau)$ and $\tau(P_1 - P_2), \tau(Q_2 - Q_1) \in \mathbb{R}$ according to Theorem 1. Thus,

$$\tau(A - B) = \tau(P_1 - Q_1 - P_2 + Q_2) = \tau(P_1 - P_2) + \tau(Q_2 - Q_1) \in \mathbb{R}$$

and the assertion is proved. □

Corollary 3. *Let $P \in S(\mathcal{M}, \tau)^{id}$ and $P = \tilde{P} + Z$ be representation of Lemma 3. We have the equivalence*

$$P \in L_1(\mathcal{M}, \tau) \Longleftrightarrow \tilde{P}, Z \in L_1(\mathcal{M}, \tau),$$

and in this case $\tau(P) = \tau(\tilde{P}) = \tau(\sqrt{|P|}|P^*|\sqrt{|P|}) = \tau(P^*) \in \mathbb{R}^+$.

Proof. If $P \in L_1(\mathcal{M}, \tau)$, then $P\tilde{P} = \tilde{P} \in L_1(\mathcal{M}, \tau)$, by Lemma 1, and the operator $Z = P - \tilde{P}$ lies in $L_1(\mathcal{M}, \tau)$. From Theorem 1 for $Q = 0$, we obtain $\tau(P) = \tau(\tilde{P})$; hence, $\tau(Z) = \tau(P - \tilde{P}) = 0$. We have $P = |P^*|\,|P|$ [7, Theorem 3.3] and $\tau(P) = \tau(\sqrt{|P|}|P^*|\sqrt{|P|})$ [7, Corollary 3.4]. In particular, $\tau(P^*) = \overline{\tau(P)} = \tau(\tilde{P}) = \tau(P) \in \mathbb{R}^+$. □

Corollary 4. *Let $U, V \in S(\mathcal{M}, \tau)$ be symmetries. If $U - V \in L_1(\mathcal{M}, \tau)$, then $\tau(U - V) \in \mathbb{R}$.*

Proof. The formula $U = 2P - I$ ($P \in S(\mathcal{M}, \tau)^{\mathrm{id}}$) establishes a bijection between $S(\mathcal{M}, \tau)^{\mathrm{id}}$ and the set of all symmetries from $S(\mathcal{M}, \tau)$. □

Corollary 5. *Let $\tau(I) < +\infty$ and $P, Q \in S(\mathcal{M}, \tau)^{id}$. If $P + Q \in L_1(\mathcal{M}, \tau)$, then $\tau(P + Q) = \tau(\tilde{P}) + \tau\left(\left(\widetilde{Q^\perp}\right)^\perp\right) = \tau(\tilde{P}) + \tau(\tilde{Q}) \in \mathbb{R}^+$.*

Proof. Since $P + Q - I = P - Q^\perp \in L_1(\mathcal{M}, \tau)$, by Theorem 1, we have

$$\tau(P + Q) = \tau(P + Q - I) + \tau(I) = \tau(P - Q^\perp) + \tau(I)$$
$$= \tau\left(\tilde{P} - \widetilde{Q^\perp}\right) + \tau(I) = \tau(\tilde{P}) + \tau\left(I - \widetilde{Q^\perp}\right) = \tau(\tilde{P}) + \tau\left(\left(\widetilde{Q^\perp}\right)^\perp\right) \in \mathbb{R}^+.$$

On the other hand, $\tilde{P}+\tilde{Q} \in L_1(\mathcal{M},\tau)$, so, $Z+T = P+Q-(\tilde{P}+\tilde{Q}) \in L_1(\mathcal{M},\tau)$. Then, the operators

$$T\tilde{P} = (Z+T)\tilde{P}, \quad Z\tilde{Q} = (Z+T)\tilde{Q}, \quad Z+\tilde{P}T = \tilde{P}(Z+T), \quad T+\tilde{Q}Z = \tilde{Q}(Z+T)$$

lie in $L_1(\mathcal{M},\tau)$. Therefore,

$$\tilde{Q}Z+\tilde{P}T = (Z+\tilde{P}T)+(\tilde{Q}Z+T)-(Z+T) \in L_1(\mathcal{M},\tau)$$

and $\tilde{P}T-T = (\tilde{Q}Z+\tilde{P}T)-(\tilde{Q}Z+T) \in L_1(\mathcal{M},\tau)$. Since $(\tilde{P}-\tilde{Q})T = \tilde{P}T-T \in L_1(\mathcal{M},\tau)$ and $T(\tilde{P}-\tilde{Q}) = T\tilde{P} \in L_1(\mathcal{M},\tau)$, equation (3) holds true via Lemma 2 with $A=\tilde{P}-\tilde{Q}$, $B=T$. Hence,

$$\tau(Z+\tilde{P}T) = \tau(\tilde{P}(Z+T)) = \tau((Z+T)\tilde{P}) = \tau(T\tilde{P}) = \tau(\tilde{P}T-T)$$

according to Lemma 2 with $A=\tilde{P}$, $B=Z+T$ and $\tau(Z+\tilde{P}T-(\tilde{P}T-T)) = \tau(Z+T)=0$. Thus, $\tau(P+Q) = \tau(\tilde{P})+\tau(\tilde{Q})$ and $\tau\left(\left(\widetilde{Q^\perp}\right)^\perp\right) = \tau(\tilde{Q})$. □

Example 1. Let $\tau(I) < +\infty$ and an idempotent $P \in S(\mathcal{M},\tau)^{\mathrm{id}}$ be represented as the sum $P = \tilde{P}+Z$ by Lemma 3. Since $\tilde{P} \in L_1(\mathcal{M},\tau)$, we have $P \in L_1(\mathcal{M},\tau) \Leftrightarrow Z \in L_1(\mathcal{M},\tau)$. Examples of such unbounded idempotents are [7, Example 3.2] and [17, Example 2.4]. Let $Z \notin L_1(\mathcal{M},\tau)$ and $Q = P^\perp$. Then, $P+Q = I \in L_1(\mathcal{M},\tau)$, but $\{P,Q\}\bigcap L_1(\mathcal{M},\tau) = \emptyset$ (cf. with item (ii) of Lemma 3 from [19]).

Theorem 2. *Let $P,Q \in S(\mathcal{M},\tau)^{id}$ with $P-Q \in L_1(\mathcal{M},\tau)$ and $PQ \in \mathcal{M}$. Then, for all $n \in \mathbb{N}$ we have $(P-Q)^{2n+1} \in L_1(\mathcal{M},\tau)$ and $\tau((P-Q)^{2n+1}) = \tau(P-Q) \in \mathbb{R}$.*

Proof. We may easily verify by induction that

$$(P-Q)^{2n+1} = P-Q+\lambda_1(PQP-QPQ)+\cdots+\lambda_n(\underbrace{PQP\cdots QP}_{2n+1}-\underbrace{QPQ\cdots PQ}_{2n+1})$$

with some $\lambda_k \in \mathbb{Z}$, $k=1,2,\ldots,n$, see step 1 of the proof of Theorem 1 from [6]. By Lemma 1, the operators $PQP-QPQ = PQ(P-Q)+(P-Q)PQ$ and $PQ-QPQ = (P-Q)PQ$ lie in $L_1(\mathcal{M},\tau)$. Since $\tau([P-Q,PQ]) = 0$, see Lemma 2, we have

$$\tau(PQP-QPQ) = \tau(PQP-QPQ+[P-Q,PQ]) = \tau(PQ-OPQ). \tag{4}$$

For operators $A=PQ$, $B=P-QP$ we have $AB=0 \in L_1(\mathcal{M},\tau)$ and $BA = PQ-OPQ \in L_1(\mathcal{M},\tau)$. Therefore, $0=\tau(0)=\tau(AB)=\tau(BA)$ via Lemma 2. Thus, from (4) we obtain $\tau(PQP-QPQ)=0$. Now, we apply the mathematical induction. Consider a number $n \geq 2$ and an operator

$$X := \underbrace{PQP\cdots QP}_{2n-1}-\underbrace{QPQ\cdots PQ}_{2n-1} \in L_1(\mathcal{M},\tau)$$

with $\tau(X)=0$. Then, the operators

$$\underbrace{PQP\cdots QP}_{2n+1}-\underbrace{PQP\cdots PQ}_{2n} = PQ\cdot X, \quad Y := \underbrace{PQP\cdots QP}_{2n+1}-\underbrace{QPQ\cdots PQ}_{2n+1} = PQ\cdot X+X\cdot PQ$$

lie in $L_1(\mathcal{M},\tau)$ according to Lemma 1. For the operators

$$A_1 := PQ, \quad B_1 := \underbrace{PQP\cdots QP}_{2n-1}-\underbrace{QPQ\cdots QP}_{2n}$$

we have $A_1B_1 = 0 \in L_1(\mathcal{M},\tau)$ and

$$B_1A_1 = \underbrace{PQP\cdots PQ}_{2n}-\underbrace{QPQ\cdots PQ}_{2n+1} = X\cdot PQ \in L_1(\mathcal{M},\tau).$$

Therefore, $\tau(B_1A_1) = \tau(A_1B_1) = \tau(0) = 0$ by Lemma 2. Thus,

$$\tau(Y) = \tau(Y+B_1A_1) = \tau(\underbrace{PQP\cdots QP}_{2n+1}-\underbrace{PQP\cdots PQ}_{2n}).$$

Since $(PQ)^n \in \mathcal{M}$ and $P-Q \in L_1(\mathcal{M},\tau)$, the operator

$$Z := [(PQ)^n, P-Q] = \underbrace{PQP\cdots QP}_{2n+1} - 2\underbrace{PQP\cdots PQ}_{2n} + \underbrace{QPQ\cdots PQ}_{2n+1}$$

belongs to $L_1(\mathcal{M},\tau)$. Hence, $\tau(Z)=0$ via Lemma 2 with $A_2=(PQ)^n$ and $B_2=P-Q$. Since $0=\tau(Z)=\tau(Y-B_1A_1)$ and $\tau(B_1A_1)=0$, we have $\tau(Y)=0$. Now $\tau((P-O)^{2n+1})=\tau(P-O)\in\mathbb{R}$ by Theorem 1. □

Corollary 6. *If $P,Q,R\in S(\mathcal{M},\tau)^{id}$ with $P-Q, Q-R\in L_1(\mathcal{M},\tau)$ and operators $PQ, QR, PR\in\mathcal{M}$, then $\tau\left((P-R)^{2n+1}\right)=\tau\left((P-Q)^{2n+1}\right)+\tau\left((Q-R)^{2n+1}\right)$ for all $n\in\mathbb{N}$.*

Corollary 7. *Let $U,V,W\in S(\mathcal{M},\tau)$ be symmetries with $U-V, V-W\in L_1(\mathcal{M},\tau)$ and operators $UV+U+V, UW+U+W, VW+V+W\in\mathcal{M}$. Then,*

$$\tau\left((U-W)^{2n+1}\right)=\tau\left((U-V)^{2n+1}\right)+\tau\left((V-W)^{2n+1}\right)$$

for all $n\in\mathbb{N}$.

Proof. Let $U=2P-I$, $V=2Q-I$ and $W=2R-I$ with $P,Q,R\in S(\mathcal{M},\tau)^{\mathrm{id}}$. Then, $U-W=2(P-R)$ and, according to Corollary 6 for every $n\in\mathbb{N}$, we have

$$\begin{aligned}\tau((U-W)^{2n+1}) &= 2^{2n+1}\tau\left((P-R)^{2n+1}\right)=2^{2n+1}\left(\tau\left((P-Q)^{2n+1}\right)+\tau\left((Q-R)^{2n+1}\right)\right)\\ &= \tau\left((U-V)^{2n+1}\right)+\tau\left((V-W)^{2n+1}\right).\end{aligned}$$

□

Theorem 3. *Let an operator $P\in S(\mathcal{M},\tau)^{id}$. Then,*

(i) $|P|=|P|P=P^*|P|$;

(ii) *if $P^*=\tilde{P}+Z$ is the representation of Lemma 3, then $|P|\ge\tilde{P}$ and $|P|\ge|Z^*|$.*

Proof. (i) Let $P=U|P|$ be the polar decomposition of the operator P. Then, $P^*=U^*|P^*|$ is the polar decomposition of the operator P^* and $U^*U|P|=|P|$. Since $P=|P^*|\,|P|$ [7, Theorem 3.3], left multiplying both parts of the equality $U|P|=|P^*|\,|P|$ by the operator U^*, allows us to conclude that $|P|=P^*|P|$. Passing to adjoint operators, we obtain $|P|=(P^*|P|)^*=|P|P$.

(ii) We have $0=Z\tilde{P}=(Z\tilde{P})^*=\tilde{P}Z^*$ and $|P|=\sqrt{(\tilde{P}+Z)(\tilde{P}+Z)^*}=\sqrt{\tilde{P}+ZZ^*}$. Since $\tilde{P}, ZZ^*\in S(\mathcal{M},\tau)^+$, by the operator monotonocity of the function $f(t)=\sqrt{t}$ $(t\ge 0)$ [20, Chap. 1, Proposition 4.4], we obtain

$$\sqrt{\tilde{P}+ZZ^*}\ge\sqrt{\tilde{P}}=\tilde{P}\quad\text{and}\quad\sqrt{\tilde{P}+ZZ^*}\ge\sqrt{ZZ^*}=|Z^*|.$$

□

Corollary 8. *Let $\langle\mathcal{E},||\cdot||_{\mathcal{E}}\rangle$ be an F-normed ideal space on $(\mathcal{M},\tau)$ and $P=P^2\in\mathcal{E}$, $P=\tilde{P}+Z$ be the representation of Lemma 3. Then, $\tilde{P}, Z\in\mathcal{E}$ and*

$$||\tilde{P}||_{\mathcal{E}}+||Z||_{\mathcal{E}}\ge||P||_{\mathcal{E}}=||P^*||_{\mathcal{E}}\ge\max\{||\tilde{P}||_{\mathcal{E}},\ ||Z||_{\mathcal{E}}\}.$$

Proof. Let $P^*=\tilde{P}+Z$ be the representation of Lemma 3. By item (ii) of Theorem 3, we have $\tilde{P},Z\in\mathcal{E}$. By properties of the F-norm $||\cdot||_{\mathcal{E}}$, we obtain $||P^*||_{\mathcal{E}}=||P||_{\mathcal{E}}=|||P|||_{\mathcal{E}}\ge||\tilde{P}||_{\mathcal{E}}$ and $||P^*||_{\mathcal{E}}=||P||_{\mathcal{E}}=|||P|||_{\mathcal{E}}\ge|||Z^*|||_{\mathcal{E}}=||Z^*||_{\mathcal{E}}=||Z||_{\mathcal{E}}$. The rest is clear. □

Theorem 4. *Let an operator $A\in L_2(\mathcal{M},\tau)$ and $A^2+A^{2*}\ge tA^*A-(t-2)AA^*$ for some $t\in\mathbb{R}$. Then, $A=A^*$.*

Proof. We have $\tau(A^*A-AA^*)=||A||_2^2-||A^*||_2^2=0$ and

$$\begin{aligned}0\le||A-A^*||_2^2&=\tau((A^*-A)(A-A^*))=\tau\left(A^*A-A^{*2}-A^2+AA^*\right)\\ &\le(1-t)\tau(A^*A-AA^*)=0.\end{aligned}$$

Hence, $A=A^*$ by faithfulness of the norm $||\cdot||_2$. □

Corollary 9. *If an operator $A=A^2\in L_2(\mathcal{M},\tau)$ and $Re(A)\ge sA^*A-(s-1)AA^*$ for some $s\in\mathbb{R}$, then $A\in\mathcal{M}^{\mathrm{pr}}$.*

Theorem 5. *Let an operator $A \in L_2(\mathcal{M}, \tau)$ and $U \in \mathcal{M}$ be an isometry. Then,* $||UA - A||_2^2 \leq 2||(I-U)AA^*||_1$. *In particular, if $A = A^*$, then* $||UA - A||_2^2 \leq 2||UA^2 - A^2||_1$.

Proof. We have

$$||UA - A||_2^2 = \tau((UA-A)^*(UA-A)) = \tau(A^*A - A^*U^*A - A^*UA + A^*A)$$
$$= \tau(A^*(I-U^*)A + A^*(I-U)A) = 2\tau(\mathrm{Re}(A^*(I-U)A)) = 2\tau(A^*(I-\mathrm{Re}(U))A)$$
$$\leq 2|\tau(A^*(I-\mathrm{Re}(U))A) - i\tau(A^*(\mathrm{Im}(U))A)|$$
$$= 2|\tau(A^*(I-U)A)| = 2|\tau((I-U)AA^*)| \leq 2\tau(|(I-U)AA^*|) = 2||(I-U)AA^*||_1,$$

according to Lemma 2 with the operators A^* and $(I-U)A$ and the inequality $|\tau(X)| \leq \tau(|X|)$ for all $X \in L_1(\mathcal{M}, \tau)$, see [21, p. 1463]. □

For the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$, endowed with the trace $\tau = \mathrm{tr}$, an operator $A \geq 0$ and a unitary U Theorem 5 was established in [22, Lemma 1].

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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