



COMMUTATORS AND HYPONORMAL OPERATORS ON A HILBERT SPACE

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ABSTRACT. Let \mathcal{H} be an infinite-dimensional Hilbert space over the field \mathbb{C} , $\mathcal{B}(\mathcal{H})$ be the $*$ -algebra of all linear bounded operators on \mathcal{H} , let $|X| = \sqrt{X^*X}$ for $X \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is a commutator, if $A = [S, T] = ST - TS$ for some $S, T \in \mathcal{B}(\mathcal{H})$. Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $X^p Y X^{1-p}$ is a non-commutator for every $0 < p < 1$. Let $A \in \mathcal{B}(\mathcal{H})$ be p -hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$, then $|A|^q$ is a non-commutator for every $q > 0$. Let \mathcal{H} be separable and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal), then A is normal. We also present results in the case of a finite-dimensional Hilbert space.

1. Introduction

Let \mathcal{H} be a Hilbert space over the field \mathbb{C} , $\mathcal{B}(\mathcal{H})$ be the $*$ -algebra of all linear bounded operators on \mathcal{H} . For a C^* -subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ put

$$\mathcal{A}_0 = \{X \in \mathcal{A} : X = \sum_{n \geq 1} [X_n, X_n^*] \text{ for } (X_n)_{n \geq 1} \subset \mathcal{A}, \text{ the series } \|\cdot\| \text{-converges}\}.$$

It is proved in [26, Theorem 2.6] that \mathcal{A}_0 coincides with the zero-space of all finite traces on \mathcal{A}^{sa} . For a wide class of C^* -algebras that contains all von Neumann algebras, we can consider only finite sums of the indicated form, see [28]. Elements of unital C^* -algebras without tracial states, can be represented as finite sums of commutators. Moreover, the number of commutators involved in these

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sums is bounded and depends only on the given C^* -algebra [35]. The characterization of traces on C^* -algebras is an urgent problem and attracts the attention of a large group of researchers. Commutation relations allowed us to obtain characterizations of the traces in a broad class of weights on von Neumann algebras and C^* -algebras [8, 10, 11]. An interesting problem is the representation of elements of C^* -algebras via commutators of special form [3–5], [17–20, 24, 33, 34].

Our paper continues article [21], that possesses the following results: Let \mathcal{H} be a Hilbert space, $\dim \mathcal{H} = +\infty$.

- (1) Let a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$ be a non-commutator and $\sigma(X)$ be the spectrum of X . Then $f(X)$ is a non-commutator for every continuous function $f : \sigma(X) \rightarrow \mathbb{R}$ with $f(x) \neq 0$.
- (2) Let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:
 - (a) X is a non-commutator;
 - (b) U and $|X|$ are non-commutators.
- (3) For a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent:
 - (a) X is a commutator;
 - (b) the Cayley transform $\mathcal{K}(X)$ is a commutator.
- (4) Let \mathcal{H} be a Hilbert space, and $\dim \mathcal{H} \leq +\infty$, $A, B \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})$, $P = P^2$. If $AB = \lambda BA$ for some $\lambda \in \mathbb{C} \setminus \{1\}$, then the operator AB is a commutator. The operator AP is a commutator, if and only if PA is a commutator.

Our results here concern the facts stated above. Let $\dim \mathcal{H} = +\infty$. The algebra $\mathcal{B}(\mathcal{H})$ is known to possess a proper uniformly closed ideal \mathcal{J} , that contains all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$.

Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $A = X^p Y X^{1-p}$ is a non-commutator for every $0 < p < 1$ (Theorem 3.1).

Differences of idempotents in C^* -algebras are naturally related to the quantum Hall effect [1, 2, 13, 14, 29]. Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotents, $P^\perp = I - P$. Then $P - Q$ is a non-commutator, if and only if exactly one of the following conditions holds: (i) $Q, P^\perp \in \mathcal{J}$; (ii) $P, Q^\perp \in \mathcal{J}$ (Theorem 3.6). Let $A = A_+ - A_-$ be the Jordan decomposition of a Hermitian operator $A \in \mathcal{B}(\mathcal{H})$. Then A is a non-commutator, if and only if exactly one of A_+ or A_- is a non-commutator (Theorem 3.9). Let $A \in \mathcal{B}(\mathcal{H})$ be p -hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$, then $|A|^q$ is a non-commutator for every $q > 0$ (Theorem 3.10). Let \mathcal{H} be separable and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal), then A is normal (Theorem 3.11).

We also present results in the setting of $\dim \mathcal{H} < +\infty$. For instance, for any unitary matrix $U \in \mathbb{M}_n(\mathbb{C})$ there exists $\varphi \in [-\pi, \pi]$, such that the inverse Cayley transform of $e^{i\varphi}U$ possesses zero trace (Proposition 3.15).

2. Preliminaries

Let \mathcal{A} be an algebra, and let $\mathcal{A}^{\text{id}} = \{A \in \mathcal{A} : A^2 = A\}$ be the set of all idempotents in \mathcal{A} . An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$. A C^* -algebra is a complex Banach $*$ -algebra \mathcal{A} , such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^* -algebra \mathcal{A} by \mathcal{A}^{pr} , \mathcal{A}^{sa} , and \mathcal{A}^+ we denote its projections ($A = A^* = A^2$), Hermitian elements, and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. As is well known, in a unital C^* -algebra \mathcal{A} the Cayley transform

$$\mathcal{K}(X) = \frac{X + iI}{X - iI} = (X - iI)^{-1}(X + iI) = (X + iI)(X - iI)^{-1}$$

of an element $X \in \mathcal{A}^{\text{sa}}$ is a unitary element of \mathcal{A} . The inverse Cayley transform of a unitary element U of \mathcal{A} is $\mathcal{K}^{-1}(U) = 2i(I - U)^{-1} - iI$, if $(I - U)^{-1} \in \mathcal{A}$. If $P \in \mathcal{A}^{\text{id}}$, then $P^\perp := I - P \in \mathcal{A}^{\text{id}}$.

Let \mathcal{H} be a Hilbert space over the field \mathbb{C} , $\mathcal{B}(\mathcal{H})$ be the $*$ -algebra of all linear bounded operators on \mathcal{H} . An operator $X \in \mathcal{B}(\mathcal{H})$, is called *p-hyponormal* for some $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$; *p-cohyponormal*, if A^* is *p-hyponormal*. By Gelfand–Naimark Theorem every C^* -algebra is isometrically isomorphic to a concrete C^* -algebra of operators on a Hilbert space \mathcal{H} [22, II.6.4.10]. For $\dim \mathcal{H} = n < \infty$, the algebra $\mathcal{B}(\mathcal{H})$ can be identified with the full matrix algebra $\mathbb{M}_n(\mathbb{C})$.

Lemma 2.1. *For $X \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = n < \infty$, the following conditions are equivalent:*

- (i) X is a commutator;
- (ii) $\text{tr}(X) = 0$;
- (iii) X is unitarily equivalent to a matrix with zero diagonal;
- (iv) $\text{tr}(|I + zX|) \geq n$ for all $z \in \mathbb{C}$.

Proof. For (i) \Leftrightarrow (ii) see [32, Ch. 24, Problem 230]; for (ii) \Leftrightarrow (iii) see [30, Chap. II, Problem 209]; for (ii) \Leftrightarrow (iv) see [12, Theorem 4.8]. □

Let \mathcal{H} be an infinite-dimensional Hilbert space. The algebra $\mathcal{B}(\mathcal{H})$ is known to contain a proper uniformly closed ideal \mathcal{J} that carries all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$, see [23, Section 6]. In case \mathcal{H} is separable, \mathcal{J} is the ideal of compact operators. Combining Theorems 3 and 4 in [23] we get the following assertion (see also [21, Theorem 2.2]).

Theorem 2.2 (Brown–Pearcy Theorem). *An operator $X \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = +\infty$, is a non-commutator, if and only if $X = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$.*

3. Idempotents and commutators in $\mathcal{B}(\mathcal{H})$

If $\dim \mathcal{H} < +\infty$, $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$, then the operator XY is a commutator, if and only if $X^p Y X^{1-p}$ is a commutator for some (hence, for all) $0 < p < 1$, see equivalence (i) \Leftrightarrow (ii) of Lemma 2.1.

Theorem 3.1. *Let $\dim \mathcal{H} = +\infty$, $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $A = X^p Y X^{1-p}$ is a non-commutator for every $0 < p < 1$.*

Proof. By Theorem 2.2 we have $XY = \lambda I + J$, for some $\lambda \in \mathbb{C} \setminus \{0\}$, and $J \in \mathcal{J}$. We show that $A = \lambda I + J_0$ for some operator $J_0 \in \mathcal{J}$ (then A is a non-commutator by Theorem 2.2). Obviously,

$$\left(X + \frac{1}{n}I\right)Y = \lambda I + \frac{1}{n}Y + J, \quad n \in \mathbb{N}.$$

Multiply these equalities by the operator $\left(X + \frac{1}{n}I\right)^{p-1}$ from the left, and by the operator $\left(X + \frac{1}{n}I\right)^{1-p}$ from the right, and obtain

$$\left(X + \frac{1}{n}I\right)^p Y \left(X + \frac{1}{n}I\right)^{1-p} = \lambda I + \frac{1}{n} \left(X + \frac{1}{n}I\right)^{p-1} Y \left(X + \frac{1}{n}I\right)^{1-p} + J_n, \tag{1}$$

where $J_n = \left(X + \frac{1}{n}I\right)^{p-1} J \left(X + \frac{1}{n}I\right)^{1-p} \in \mathcal{J}$, $n \in \mathbb{N}$. Since

$$X + \frac{1}{n}I \rightarrow X \text{ as } n \rightarrow \infty$$

in the operator norm, we have $\left(X + \frac{1}{n}I\right)^q \rightarrow X^q$ as $n \rightarrow \infty$ by the $\|\cdot\|$ -continuity of the functional calculus. Therefore,

$$\left(X + \frac{1}{n}I\right)^p Y \left(X + \frac{1}{n}I\right)^{1-p} \rightarrow A \text{ as } n \rightarrow \infty$$

in the operator norm by joint $\|\cdot\|$ -continuity of the product operation in $\mathcal{B}(\mathcal{H})$. Let us show that

$$\frac{1}{n} \left(X + \frac{1}{n}I\right)^{p-1} Y \left(X + \frac{1}{n}I\right)^{1-p} \rightarrow 0 \text{ as } n \rightarrow \infty$$

in the operator norm. Consider an Abelian unital C^* -subalgebra \mathcal{A} in $\mathcal{B}(\mathcal{H})$, generated by the operators X and I . Then $\mathcal{A} \simeq C(\Omega)$ for some compact topological space Ω (Gelfand representation) and

$$\frac{1}{\left(X + \frac{1}{n}I\right)^{1-p}} = \frac{n^{1-p}}{\left(nX + I\right)^{1-p}} \leq n^{1-p}I \text{ for all } n \in \mathbb{N},$$

hence $\left\|\left(X + \frac{1}{n}I\right)^{p-1}\right\| \leq n^{1-p}$, $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \left\|\frac{1}{n} \left(X + \frac{1}{n}I\right)^p Y \left(X + \frac{1}{n}I\right)^{1-p}\right\| &\leq \frac{1}{n} \left\|\left(X + \frac{1}{n}I\right)^p\right\| \|Y\| \left\|\left(X + \frac{1}{n}I\right)^{1-p}\right\| \\ &\leq \frac{n^{1-p}}{n} \|Y\| \left\|\left(X + \frac{1}{n}I\right)^{1-p}\right\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Now it follows from (1) that the sequence $(J_n)_{n=1}^\infty \subset \mathcal{J}$ is $\|\cdot\|$ -convergent as $n \rightarrow \infty$ to some operator $J_0 \in \mathcal{B}(\mathcal{H})$. Since the ideal \mathcal{J} is $\|\cdot\|$ -closed, we get $J_0 \in \mathcal{J}$. Then by Theorem 2.2 the operator A is a non-commutator. □

Since the exponential function $\lambda \in \mathbb{C} \mapsto \exp \lambda \in \mathbb{C}$ is an entire function, we can define $\exp X$ for all elements X of a unital Banach algebra \mathcal{A} , see [36, Chap. I, Proposition 2.7]. It also admits the representation as the absolutely convergent power series

$$\exp X = \sum_{n=0}^\infty \frac{1}{n!} X^n,$$

where $X^0 = I$ as usual.

Theorem 3.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $X = AB + (I - B)A$.

- (i) If $\dim \mathcal{H} < +\infty$ then A is a commutator, if and only if X is a commutator;
- (ii) If $\dim \mathcal{H} = +\infty$ and A is a non-commutator, then X and e^A are non-commutators;
- (iii) If $\dim \mathcal{H} < +\infty$ and $A = A^*$, then e^A is a non-commutator. There exists a normal non-commutator T such that e^T is a commutator.

Proof. The statement is obtained (i). We have $\text{tr}(X) = \text{tr}(AB - BA) + \text{tr}(A)$ and apply equivalence (i) \Leftrightarrow (ii) of Lemma 2.1.

(ii). By Theorem 2.2, we have $A = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Then

$$X = (\lambda I + J)B + (I - B)(\lambda I + J) = \lambda I + J_1$$

for $J_1 = JB - BJ + J \in \mathcal{J}$ and we apply Theorem 2.2.

We also have $e^A = e^\lambda I + J_2$, for some $J_2 \in \mathcal{J}$. Indeed, consider the partial sums

$$\begin{aligned} I + A + A^2 + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} &= I + \lambda I + \frac{\lambda^2}{2!}I + \dots + \frac{\lambda^n}{n!}I + \\ &+ J + \lambda J + \frac{\lambda^2}{2!}J^2 + \dots + \frac{\lambda^n}{n!}J^n, \quad n \in \mathbb{N}. \end{aligned}$$

Since $I + \lambda I + \frac{\lambda^2}{2!}I + \dots + \frac{\lambda^n}{n!}I \rightarrow e^\lambda I$ as $n \rightarrow \infty$ in the operator norm, the sequence $J + \lambda J + \frac{\lambda^2}{2!}J^2 + \dots + \frac{\lambda^n}{n!}J^n$ is also $\|\cdot\|$ -convergent to some operator $J_2 \in \mathcal{B}(\mathcal{H})$ as $n \rightarrow \infty$. Since the ideal \mathcal{J} is $\|\cdot\|$ -closed, we get $J_2 \in \mathcal{J}$. Then by Theorem 2.2 the operator e^A is a non-commutator.

(iii). If $\dim \mathcal{H} = n < +\infty$ and $A = A^*$ then without loss of generality put

$$A = \text{diag}(a_1, \dots, a_n)$$

with $a_1, \dots, a_n \in \mathbb{R}$. Then $e^A = \text{diag}(e^{a_1}, \dots, e^{a_n})$, $\text{tr}(e^A) = e^{a_1} + \dots + e^{a_n} > 0$ and we apply equivalence (i) \Leftrightarrow (ii) of Lemma 2.1.

Finally, put $T = \text{diag}(0, i\pi)$ in $\mathbb{M}_2(\mathbb{C})$ and apply Lemma 2.1. □

Let $X \in \mathcal{B}(\mathcal{H})$, $P \in \mathcal{B}(\mathcal{H})^{\text{id}}$ and $S = 2P - I$. Consider the following conditions:

- (A) X is a non-commutator;
- (B) $PX + XP^\perp$ is a non-commutator;
- (C) $X + SXS$ is a non-commutator.

Theorem 3.3. Let operators $X, P, S \in \mathcal{B}(\mathcal{H})$ be as above.

- (i) If $\dim \mathcal{H} < +\infty$, then (A) \Leftrightarrow (B) \Leftrightarrow (C).
- (ii) If $\dim \mathcal{H} = +\infty$, then (A) \Rightarrow (B) \Rightarrow (C) and in the general case the implications (C) \Rightarrow (B), (C) \Rightarrow (A) and (B) \Rightarrow (A) are false.

Proof. (i). Follows from equivalence (i) \Leftrightarrow (ii) of Lemma 2.1.

(ii), (A) \Rightarrow (B). Consider a non-commutator $X = \lambda I + J$, for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$, see Theorem 2.2. Then

$$PX = \lambda P + PJ, \quad XP^\perp = \lambda P^\perp + JP^\perp.$$

We sum these equalities term-by-term, conclude that $PX + XP^\perp = \lambda I + J_1$, where $J_1 = PJ + P^\perp J \in \mathcal{J}$, and apply Theorem 2.2.

(ii), (B) \Rightarrow (C). Consider a non-commutator $PX + XP^\perp = \lambda I + J$ with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$, see Theorem 2.2. Then

$$PXP = (PX + XP^\perp)P = \lambda P + JP, \quad P^\perp XP^\perp = P^\perp(PX + XP^\perp) = \lambda P^\perp + JP^\perp.$$

By summing these equalities term-by-term we get

$$PXP + P^\perp XP^\perp = \frac{1}{2}(X + SXS) = \lambda I + J_2,$$

where $J_2 = JP + P^\perp J \in \mathcal{J}$, and apply Theorem 2.2.

Now we show that for an infinite dimensional separable Hilbert space \mathcal{H} implications (C) \Rightarrow (B), (C) \Rightarrow (A) and (B) \Rightarrow (A) are false. Fix some $X \in \mathcal{B}(\mathcal{H})^{\text{pr}}$ with $\dim X\mathcal{H} = \dim X^\perp\mathcal{H} = +\infty$. Then in the direct sum $\mathcal{H} = X\mathcal{H} \oplus X^\perp\mathcal{H}$ we have $X = \text{diag}(1, 0)$ and for

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

with $S = 2P - I$ we obtain $SXS = \text{diag}(0, 1) = X^\perp$. Hence $X + SXS = I$ and condition (C) holds by Theorem 2.2. It is clear that condition (A) does not hold by Theorem 2.2. Since

$$PX + XP^\perp = \begin{pmatrix} 1 & -1/2 \\ 1/2 & 0 \end{pmatrix},$$

condition (B) is false by Theorem 2.2.

Consider now $P \in \mathcal{B}(\mathcal{H})^{\text{pr}}$ with $\dim P\mathcal{H} = \dim P^\perp\mathcal{H} = +\infty$. Then in the direct sum $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ we have $P = \text{diag}(1, 0)$, $P^\perp = \text{diag}(0, 1)$. For

$$X = \begin{pmatrix} \lambda & b \\ c & \lambda \end{pmatrix}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$, $b \in \mathcal{J}$ and $c \notin \mathcal{J}$, the operator

$$PX + XP^\perp = \begin{pmatrix} \lambda & 2b \\ 0 & \lambda \end{pmatrix}$$

is a non-commutator by Theorem 2.2, i.e., condition (B) holds. Since X is a commutator by Theorem 2.2, condition (A) does not hold. For $S = 2P - I$ we obtain $X + SXS = 2\lambda I$ and condition (C) holds by Theorem 2.2. \square

Let \mathcal{A} be an algebra, let $A, B \in \mathcal{A}$ be such that $AB = -BA$, i.e., A and B anticommute. Then AB and BA are commutators: $AB = [\frac{A}{2}, B]$, $BA = [B, \frac{A}{2}]$.

Example 3.4. *Let \mathcal{A} be a unital algebra. Then*

- (i) *if $P, Q \in \mathcal{A}^{\text{id}}$ then $A = P - Q$ and $B = I - P - Q$ anticommute;*
- (ii) *if $P \in \mathcal{A}^{\text{id}}$, $X \in \mathcal{A}$ then $A = 2P - I$ and $B = [X, P]$ anticommute;*
- (iii) *if $X, Y, T \in \mathcal{A}$ and T is left invertible then $T[X, Y]T_l^{-1} = [TXT_l^{-1}, TYT_l^{-1}]$.*

Note that even matrices with zero trace may not only anticommute but enjoy more peculiar properties, cf. item (i) of [21, Theorem 3.19].

Example 3.5. For $A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda^2 & \dots & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & \lambda^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu & 0 & \dots & 0 \\ 0 & 0 & 0 & \mu^2 & \dots & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & \mu^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$

$\lambda, \mu \in \mathbb{C}$, we have $\lambda AB = \mu BA$.

Theorem 3.6. Let $\dim \mathcal{H} = +\infty$ and $P, Q \in \mathcal{B}(\mathcal{H})^{id}$. Then $A = P - Q$ is a non-commutator if and only if exactly one of the following conditions holds:

- (i) $Q, P^\perp \in \mathcal{J}$;
- (ii) $P, Q^\perp \in \mathcal{J}$.

Proof. Assume that $A = P - Q$ is a non-commutator. Then by Theorem 2.2 we have

$$P - Q = \lambda I + J \tag{2}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Hence

$$P = Q + \lambda I + J = (Q + \lambda I + J)^2$$

and

$$\lambda I + J = \lambda^2 I + 2\lambda Q + J_1, \tag{3}$$

where $J_1 = J^2 + 2\lambda J + QJ + JQ \in \mathcal{J}$. Consider two cases.

a) If $Q \in \mathcal{J}$ then $\lambda = \lambda^2$. Since $\lambda \neq 0$, we have $\lambda = 1$ and by (2) infer that $P^\perp = -J - Q \in \mathcal{J}$. Thus condition (i) holds.

b) If $Q \notin \mathcal{J}$ then by (2) we know that $Q^\perp = -J - Q \in \mathcal{J}$ and $\lambda^2 + \lambda = 0$. Since $\lambda \neq 0$, we have $\lambda = -1$ and by (2) achieve the equality $P = -Q^\perp + J \in \mathcal{J}$. Thus condition (ii) holds.

Let us show the reverse implication of Theorem. If condition (i) holds then $-I + P - Q := J \in \mathcal{J}$ and $P - Q = I + J$. If condition (ii) holds then $P - Q + I := J \in \mathcal{J}$ and $P - Q = -I + J$. In both of these cases $A = P - Q$ is a non-commutator by Theorem 2.2. □

Let \mathcal{A} be an algebra and $A = A^3 \in \mathcal{A}$. Then $A = P - Q$ for some $P, Q \in \mathcal{A}^{id}$ with $PQ = QP = 0$, see [7, Proposition 1].

Corollary 3.7. Let $\dim \mathcal{H} = +\infty$ and $A = A^3 \in \mathcal{B}(\mathcal{H})$, let $A = P - Q$ be a representation as above. Then A is a non-commutator if and only if exactly one of conditions (i) or (ii) of Theorem 3.6 holds.

Corollary 3.8. Let $\dim \mathcal{H} = +\infty$ and $P, Q \in \mathcal{B}(\mathcal{H})^{id}$. Then $A = P + Q$ is a non-commutator, if and only if exactly one of the following conditions holds:

- (i) $P - Q^\perp \in \mathcal{J}$;
- (ii) $P^\perp, Q^\perp \in \mathcal{J}$.

Proof. Assume that $A = P + Q$ is a non-commutator. Then by Theorem 2.2 we have

$$P + Q = \lambda I + J,$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. If $\lambda = 1$ then $P - Q^\perp := J \in \mathcal{J}$ and condition (i) holds. If $\lambda \notin \{0, 1\}$ then the equality

$$P - Q^\perp = (\lambda - 1)I + J$$

allows us to apply Theorem 3.6 to the idempotent pair $\{P, Q^\perp\}$. Therefore, $P^\perp, Q^\perp \in \mathcal{J}$ by item (i) of Theorem 3.6 and condition (ii) of Corollary 3.7 holds. Moreover, condition (ii) of Theorem 3.6 leads us to $P, Q \in \mathcal{J}$ and corresponds to the prohibited value $\lambda = 0$.

Let us show the reverse implication of Corollary 3.8. If condition (i) holds then $P - Q^\perp := J \in \mathcal{J}$ and $P + Q = I + J$. If condition (ii) holds then $P^\perp + Q^\perp := J \in \mathcal{J}$ and $P + Q = 2I - J$. In both of these cases, $A = P + Q$ is a non-commutator by Theorem 2.2. \square

Theorem 3.9. *Let $\dim \mathcal{H} = +\infty$ and $A = A_+ - A_-$ be the Jordan decomposition of an operator $A \in \mathcal{B}(\mathcal{H})^{\text{sa}}$. Then A is a non-commutator if and only if exactly one of A_+ or A_- is a non-commutator.*

Proof. “ \Rightarrow ”. Assume that $A \in \mathcal{B}(\mathcal{H})^{\text{sa}}$ and $A = A_+ - A_-$ with $A_+, A_- \in \mathcal{B}(\mathcal{H})^+$, $A_+A_- = 0$. Let P_+ and P_- be the support projections (=carriers) of A_+ and A_- , respectively; put $S := P_+ - P_-$. Then the polar decomposition of A is $A = S|A|$ with $|A| = A_+ + A_-$. By [21, Theorem 3.15] the operators S and $|A|$ are non-commutators. Thus by Theorem 3.6 we have one of the following conditions: either (i) $P_-, P_+^\perp \in \mathcal{J}$, or (ii) $P_+, P_-^\perp \in \mathcal{J}$. In case (i), the projection P_+ is a non-commutator by Theorem 2.2 and $A_+ = P_+A$ is a non-commutator by [21, Lemma 3.5]. In case (ii), the projection P_- is a non-commutator by Theorem 2.2 and $A_- = -P_-A$ is a non-commutator by [21, Lemma 3.5]. \square

Theorem 3.10. *Let $\dim \mathcal{H} = +\infty$ and $A \in \mathcal{B}(\mathcal{H})$ be p -hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$ then $|A|^q$ is a non-commutator for every $q > 0$.*

Proof. By [21, Remark 3.14] the operator $|A^*|^{2p} = (|A^*|^r)^{\frac{2p}{r}}$ is also a non-commutator. By Theorem 2.2 we have

$$|A|^{2p} \geq |A^*|^{2p} = \lambda I + J \tag{4}$$

for some $\lambda > 0$ and $J \in \mathcal{J}^{\text{sa}}$, $J \geq -\lambda I$.

If $A = U|A|$ is the polar decomposition of A then $A^* = U^*|A^*|$ is the polar decomposition of A^* and $A = (A^*)^* = |A^*|U$, hence $|A| = U^*A = U^*|A^*|U$. Therefore, $|A|^n = U^*|A^*|^nU$ for all $n \in \mathbb{N}$. By the Weierstrass Theorem there exists a sequence $\{f_n\}_{n=1}^\infty$ of polynomials, which converges uniformly on the interval $[0; 2\|A\|]$ to the function $f(t) = t^q$ as $n \rightarrow \infty$. Hence $|A|^q = U^*|A^*|^qU$ for all $q > 0$. Therefore, by (4) we have

$$|A|^{2p} = \lambda U^*U + U^*JU = \lambda P + J_1$$

with the projection $P = U^*U$ and $J_1 = U^*JU \in \mathcal{J}^{\text{sa}}$, $J_1 \geq -\lambda P$. Thus,

$$\lambda P + J_1 \geq \lambda I + J = \lambda P + \lambda P^\perp + J$$

and $0 \leq \lambda P^\perp \leq J_1 - J_2 \in \mathcal{J}^+$. If $X, Y \in \mathcal{B}(\mathcal{H})^+$ and $X \leq Y$ then $X = VYV^*$ for some $V \in \mathcal{B}(\mathcal{H})$ with $\|V\| \leq 1$, see [27, Chap. 1, Sect. 1, Lemma 2]. So, $\lambda P^\perp \in \mathcal{J}^+$ and $|A|^{2p} = \lambda P + J_1 = \lambda I + J_2$ with $J_2 = J_1 - \lambda P^\perp \in \mathcal{J}$. Hence $|A|^{2p}$ is a non-commutator by Theorem 2.2 and $|A|^q = (|A|^{2p})^{\frac{q}{2p}}$ is a non-commutator by [21, Remark 3.14]. \square

If \mathcal{H} is separable and $\dim \mathcal{H} = +\infty$, then there exists a hyponormal operator $A \in \mathcal{B}(\mathcal{H})$ such that A^*A is a non-commutator, but AA^* is a commutator (hint: consider an isometry $A \in \mathcal{B}(\mathcal{H})$ with $\dim(\text{Ker}(AA^*)) = +\infty$).

Theorem 3.11. *Let \mathcal{H} be separable, $\dim \mathcal{H} = +\infty$ and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal) then A is normal.*

Proof. By Theorem 2.2 we have

$$A = \lambda I + J$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Since $A^*A \geq AA^*$, we obtain $J^*J \geq JJ^*$. Since \mathcal{J} is the set of compact operators (when \mathcal{H} is separable), by Ando–Berberian–Stampfli Theorem (see [32, Chap. 21, Problem 206]) we obtain $J^*J = JJ^*$. Therefore, $A^*A = AA^*$, i.e., A is normal.

If A is cohyponormal ($A^*A \leq AA^*$) then A^* is hyponormal. If \mathcal{A} is a $*$ -algebra, then $X \in \mathcal{A}$ is a commutator, if and only if X^* is a commutator (hint: if $X = [Y, Z]$ then $X^* = [Z^*, Y^*]$). \square

Theorem 3.12. *Let $P_1, \dots, P_n \in \mathcal{B}(\mathcal{H})^{id}$ and $P_1 + \dots + P_n = I$. Put $\mathcal{P}(A) = \sum_{k=1}^n P_k A P_k$ for $A \in \mathcal{B}(\mathcal{H})$.*

- (i) *If $\dim \mathcal{H} < +\infty$ then A is a commutator if and only if $\mathcal{P}(A)$ is a commutator;*
- (ii) *If $\dim \mathcal{H} = +\infty$ and A is a non-commutator then $\mathcal{P}(A)$ is a non-commutator.*

Proof. (i). If $\dim \mathcal{H} < +\infty$ then $\text{tr}(A) = \text{tr}(\mathcal{P}(A))$ for all $A \in \mathcal{B}(\mathcal{H})$ by [15, Lemma 1] and the assertion follows by equivalence (i) \Leftrightarrow (ii) of Lemma 2.1.

(ii). By Theorem 2.2 we have $A = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Then

$$\mathcal{P}(A) = \lambda I + \sum_{k=1}^n P_k J P_k = \lambda I + J_1$$

with $J_1 = \sum_{k=1}^n P_k J P_k \in \mathcal{J}$ and $\mathcal{P}(A)$ is a non-commutator by Theorem 2.2. \square

Recall that for $P_1, \dots, P_n \in \mathcal{B}(\mathcal{H})^{pr}$, the mapping \mathcal{P} coincides with the block projection operator, which was investigated in [9, 25, 31] and [16].

Example 3.13. *For an infinite dimensional separable Hilbert space \mathcal{H} , consider $P_1 \in \mathcal{B}(\mathcal{H})^{pr}$ with $\dim P_1 \mathcal{H} = \dim P_1^\perp \mathcal{H} = +\infty$, and put $P_2 = P_1^\perp$, $\mathcal{P}(X) = P_1 X P_1 + P_2 X P_2$ for all $X \in \mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})^+$ admit the operator matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in the direct sum $\mathcal{H} = P_1 \mathcal{H} \oplus P_2 \mathcal{H}$. Then $P_1 = \text{diag}(1, 0)$, $P_2 = \text{diag}(0, 1)$ and A is a commutator, but $\mathcal{P}(A) = I$ is a non-commutator by Theorem 2.2.*

Note that the Cayley transform of a commutator in the finite dimensional case, is not necessarily a matrix with zero trace, cf. the infinite dimensional case of [21].

Example 3.14. (i) *Scalar multiples of the Pauli matrices are the unitary matrices with zero trace whose inverse Cayley transform also possesses zero trace.*

$$i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) *A set of unitary matrices with zero trace whose inverse Cayley transform also possesses zero trace: for $a, b \in \mathbb{R}$ with $a^2 + b^2 \leq 1$, $c := \sqrt{1 - a^2 - b^2}$, we have*

$$\begin{pmatrix} ia & b - ic \\ -b - ic & -ia \end{pmatrix} \mapsto \begin{pmatrix} -a & c + ib \\ c - ib & a \end{pmatrix}.$$

(iii) *Unitary matrices with zero trace whose inverse Cayley transform possesses nonzero trace:*

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & -e^{i\varphi} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sin(\varphi)}{\cos(\varphi)-1} & 0 \\ 0 & \frac{\sin(\varphi)}{\cos(\varphi)+1} \end{pmatrix}, \varphi \neq \pi k/2, k \in \mathbb{Z}.$$

$$\begin{pmatrix} 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \\ e^{i\gamma} & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} i \frac{e^{i(\alpha+\beta+\gamma)}+1}{1-e^{i(\alpha+\beta+\gamma)}} & \frac{2ie^{i\alpha}}{1-e^{i(\alpha+\beta+\gamma)}} & \frac{2ie^{i(\alpha+\beta)}}{1-e^{i(\alpha+\beta+\gamma)}} \\ \frac{2ie^{i(\gamma+\beta)}}{1-e^{i(\alpha+\beta+\gamma)}} & i \frac{e^{i(\alpha+\beta+\gamma)}+1}{1-e^{i(\alpha+\beta+\gamma)}} & \frac{2ie^{i\beta}}{1-e^{i(\alpha+\beta+\gamma)}} \\ \frac{2ie^{i\gamma}}{1-e^{i(\alpha+\beta+\gamma)}} & \frac{2ie^{i(\alpha+\gamma)}}{1-e^{i(\alpha+\beta+\gamma)}} & i \frac{e^{i(\alpha+\beta+\gamma)}+1}{1-e^{i(\alpha+\beta+\gamma)}} \end{pmatrix},$$

$\alpha + \beta + \gamma \neq \pi k, k \in \mathbb{Z}$.

Despite last of these examples we have a

Proposition 3.15. *For any unitary matrix $U \in \mathbb{M}_n(\mathbb{C})$, there exists $\varphi \in [-\pi, \pi]$ such that the inverse Cayley transform of $e^{i\varphi}U$ possesses zero trace.*

Proof. Indeed, if we diagonalize U so that $U = \text{diag}\{z_1, \dots, z_n\}$, $|z_k| = 1$ the inverse Cayley transform of U is also a diagonal real matrix $\mathcal{K}^{-1}(U) = \text{diag}\{i \frac{1+z_1}{1-z_1}, \dots, i \frac{1+z_n}{1-z_n}\}$. Consider the adjacent numbers z_k, z_{k+1} of the unit circle \mathbb{S}^1 . Now for $z_k \rightarrow 1$ from below the number $i \frac{1+z_k}{1-z_k} \rightarrow -\infty$ and for $z_{k+1} \rightarrow 1$ from above the number $i \frac{1+z_{k+1}}{1-z_{k+1}} \rightarrow +\infty$. The function $\text{tr}(\mathcal{K}^{-1}(U))$ is continuous. Hence there exists $\varphi \in [-\pi, \pi]$ so that the trace of $\mathcal{K}^{-1}(e^{i\varphi}U)$ equals zero, thus $\mathcal{K}^{-1}(e^{i\varphi}U)$ is a commutator by equivalence (i) \Leftrightarrow (ii) of Lemma 2.1. □

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