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Commutators in C^*-algebras and traces

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Abstract
Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = +\infty$. Let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then, X is a non-commutator if and only if both U and $|X|$ are non-commutators. A Hermitian operator $X \in \mathcal{B}(\mathcal{H})$ is a commutator if and only if the Cayley transform $\mathcal{K}(X)$ is a commutator. Let $\mathcal{H}$ be a Hilbert space and $\dim \mathcal{H} \leq +\infty$, $A, B, P \in \mathcal{B}(\mathcal{H})$ and $P = P^2$. If $AB = \lambda BA$ for some $\lambda \in \mathbb{C}\backslash\{1\}$ then the operator AB is a commutator. The operator AP is a commutator if and only if PA is a commutator.

Keywords Hilbert space · Linear operator · Commutator · C^*-algebra · Trace

Mathematics Subject Classification 46L05 · 46L30 · 47C15

1 Introduction

Dimension functions and traces on C^*-algebras are fundamental tools in the operator theory and its applications. Therefore, they have been actively studied in recent decades, see [12, 14, 23, 29, 32, 34]. For a C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, put

$$\mathcal{A}_0 = \left\{ X \in \mathcal{A} : \ X = \sum_{n \geq 1} [X_n, X_n^*] \ \text{ for } \ (X_n)_{n \geq 1} \subset \mathcal{A} \right\},$$

the series $\|\cdot\|$-converges. It is proved in [20, Theorem2.6] that $\mathcal{A}_0$ coincides with the zero-space of all finite traces on $\mathcal{A}^{\text{sa}}$. For a wide class of C^*-algebras that contains all von Neumann algebras, we can consider only finite sums of the indicated form, see [24]. Elements of unital C^*-algebras without tracial states can be represented as finite

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sums of commutators. Moreover, the number of commutators involved in these sums is bounded and depends only on the given C^*-algebra [31]. The characterization of traces on C^*-algebras is an urgent problem and attracts the attention of a large group of researchers. Commutation relations allowed to obtain characterizations of the traces in a broad class of weights on von Neumann algebras and C^*-algebras [6–9]. An interesting problem is representation of elements of C^*-algebras via commutators of special form [4, 13, 27].

The following results were obtained. Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = +\infty$. (1) Let a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$ be a non-commutator and $\sigma(X)$ be the spectrum of X. Then, $f(X)$ is a non-commutator for every continuous function $f : \sigma(X) \to \mathbb{R}$ with $f(x) \neq 0$ (Lemma 3.13). (2) Let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then, the following conditions are equivalent: (i) X is a non-commutator; (ii) U and $|X|$ are non-commutators (Theorem 3.15). (3) For a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent: (i) X is a commutator; (ii) the Cayley transform $\mathcal{K}(X)$ is a commutator (Theorem 3.17). (4) Let $\mathcal{H}$ be a Hilbert space and $\dim \mathcal{H} \leq +\infty$, $A, B \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})$, $P = P^2$. If $AB = \lambda BA$ for some $\lambda \in \mathbb{C}\setminus\{1\}$ then the operator AB is a commutator. The operator AP is a commutator if and only if PA is a commutator (Theorem 3.19).

2 Preliminaries

Let $\mathcal{A}$ be an algebra, $\mathcal{A}^{\mathrm{id}} = \{A \in \mathcal{A} : A^2 = A\}$ be the set of all idempotents in $\mathcal{A}$. An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$. For $X, Y \in \mathcal{A}$ define their Jordan product by the equality $X \circ Y = \frac{XY+YX}{2}$. For $A, B \in \mathcal{A}$ we write $A \sim B$ if there exist $X, Y \in \mathcal{A}$ so that $XY = A$, $YX = B$ (hence $A - B = [X, Y]$). If $\mathcal{A}$ is unital and $A, B \in \mathcal{A}$ are similar then $A \sim B$.

A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$, and $\mathcal{A}^+$ we denote its projections ($A = A^* = A^2$), Hermitian elements, and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. In a C^*-algebra $\mathcal{A}$, two projections P and Q are $*$-*equivalent* if there exists an element X in $\mathcal{A}$ (necessarily a partial isometry) such that $P = X^*X$ and $Q = XX^*$. If $R \in \mathcal{A}^{\mathrm{id}}$ then $R \sim T$ for some $T \in \mathcal{A}^{\mathrm{pr}}$; if $P, Q \in \mathcal{A}^{\mathrm{pr}}$ and $P \sim Q$ then P and Q are $*$-*equivalent* [21, Proposition IV.1.1]. As is well known, in a unital C^*-algebra $\mathcal{A}$, the Cayley transform

$$\mathcal{K}(X) = \frac{X + \mathrm{i}I}{X - \mathrm{i}I} = (X - \mathrm{i}I)^{-1}(X + \mathrm{i}I) = (X + \mathrm{i}I)(X - \mathrm{i}I)^{-1}$$

of an element $X \in \mathcal{A}^{\mathrm{sa}}$ is a unitary element of $\mathcal{A}$.

A mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is called *a trace* on a C^*-algebra $\mathcal{A}$, if $\varphi(X+Y) = \varphi(X)+\varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ (moreover, $0\cdot(+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$. For a trace φ, define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ :\ \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can always be extended by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ.

Lemma 2.1 *Let φ be a trace on a C^*-algebra $\mathcal{A}$. Then, $\varphi(AB) = \varphi(BA)$ for all $A \in \mathfrak{M}_\varphi$ and $B \in \mathcal{A}$.*

Proof See, for example, [22, §6, item (iii) of Proposition 6.1.2]. □

A positive linear functional φ on $\mathcal{A}$ with $\|\varphi\| = 1$ is called *a state*. A trace φ is called *faithful*, if $\varphi(X) = 0$ $(X \in \mathcal{A}^+) \Rightarrow X = 0$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. An operator $X \in \mathcal{B}(\mathcal{H})$ possesses a left (resp., right) essential inverse X_l^{-1} (resp., X_r^{-1}) if $X_l^{-1}X = I+K$ (resp., $XX_r^{-1} = I+K$) for some compact operator $K \in \mathcal{B}(\mathcal{H})$. We have $\mathfrak{M}_{\mathrm{tr}} = \mathfrak{S}_1(\mathcal{H})$, the set of all trace class operators on $\mathcal{H}$. By Gelfand–Naimark Theorem every C^*-algebra is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [16, II.6.4.10]. For $\dim \mathcal{H} = n < +\infty$, the algebra $\mathcal{B}(\mathcal{H})$ can be identified with the full matrix algebra $\mathbb{M}_n(\mathbb{C})$.

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. The algebra $\mathcal{B}(\mathcal{H})$ is known to contain a proper uniformly closed ideal $\mathcal{J}$ that contains all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$, see [17, Section 6]. In case $\mathcal{H}$ is separable, $\mathcal{J}$ is the ideal of compact operators. Combining Theorems 3 and 4 in [17], we get the following assertion.

Theorem 2.2 (Brown–Pearcy Theorem) *An operator $X \in \mathcal{B}(\mathcal{H})$ is a non-commutator if and only if $X = xI + J$ for some $x \in \mathbb{C}\backslash\{0\}$ and $J \in \mathcal{J}$.*

If $T \in \mathcal{B}(\mathcal{H})$ and $T = U|T|$ is its polar decomposition, the Aluthge transform of T is the operator $\Delta(T)$ defined as $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ [1]. More generally, for any real number $\lambda \in [0, 1]$, the λ-Aluthge transformation is defined as $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda} \in \mathcal{B}(\mathcal{H})$ [18]. We have $T \sim \Delta_\lambda(T)$ for any $\lambda \in [0, 1]$ (hint: put $X = |T|^\lambda$ and $Y = U|T|^{1-\lambda}$).

3 Idempotents and commutators in C^*-algebras

Lemma 3.1 *Let $\mathcal{A}$ be a unital algebra, let $A, B \in \mathcal{A}$ be such that $ABA = \lambda A$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.*

(i) *If $A \in \mathcal{A}^{\mathrm{id}}$ then the idempotents A, $\lambda^{-1}AB$ and $\lambda^{-1}BA$ are pairwise similar.*
(ii) *If $B \in \mathcal{A}^{\mathrm{id}}$ then the idempotents $\lambda^{-1}AB$, $\lambda^{-1}BA$ and $\lambda^{-1}BAB$ are pairwise similar.*
(iii) *If $A, B \in \mathcal{A}^{\mathrm{id}}$ then $A \sim \lambda^{-1}BAB$. If, moreover, $BAB = \lambda B$ then $A \sim B$.*

Proof (i) The elements $P = \lambda^{-1}AB$ and $Q = \lambda^{-1}BA$ lie in $\mathcal{A}^{\mathrm{id}}$. We have $PA = A$ and $AP = P$ (resp., $QA = Q$ and $AQ = A$) and apply [13, Lemma 2]. Therefore, the idempotents A and P (resp., A and Q) are similar. Since similarity is an equivalence relation, the idempotents P and Q are also similar. If $\mathcal{A}$ acts on a vector space $\mathcal{E}$, then by [19, Lemma 2], we have $\mathrm{Im}\, P = \mathrm{Im}\, A$ and $\mathrm{Ker}\, Q = \mathrm{Ker}\, A$.

(ii) The elements $P = \lambda^{-1}AB$, $Q = \lambda^{-1}BA$ and $R = \lambda^{-1}BAB$ lie in $\mathcal{A}^{\mathrm{id}}$. We have $PR = P$ and $RP = R$ (resp., $QR = R$ and $RQ = Q$) and apply [13, Lemma 2]. Therefore, the idempotents P and R (resp., Q and R) are similar. Since similarity is an equivalence relation, the idempotents P and Q are also similar. If $\mathcal{A}$ acts on a vector space $\mathcal{E}$ then by [19, Lemma 2] we have $\mathrm{Im}\, Q = \mathrm{Im}\, R$ and $\mathrm{Ker}\, P = \mathrm{Ker}\, R$.

(iii) Put $X = \lambda^{-1}AB$ and $Y = BA$. □

Projections $P, Q \in \mathcal{B}(\mathcal{H})$ are called *isoclinic* with angle $\theta \in (0, \pi/2)$, if $PQP = \cos^2\theta\, P$ and $QPQ = \cos^2\theta\, Q$ [33, Definition 10.4]. Thus, in this case, the idempotents P, Q, $\cos^{-2}\theta\, PQ$, $\cos^{-2}\theta\, QP$ are pairwise similar.

Example Consider the following complex 2×2 matrices:

$$P = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}, \quad X = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}.$$

Then, $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $PXP = \lambda P$, $PQP = (1 + z^2)P$, $QPQ = (1 + z^2)Q$. For an arbitrary $A \in \mathbb{M}_n(\mathbb{C})$, there exists a pseudo-inverse $B \in \mathbb{M}_n(\mathbb{C})$ such that $ABA = A$ (see [30, Theorem 1.4.15]).

Lemma 3.2 *Let $\mathcal{A}$ be an algebra, $A, B \in \mathcal{A}^{\mathrm{id}}$ be such that $ABA = \lambda A$ and $BAB = \lambda B$ for some $\lambda \in \mathbb{C}\backslash\{0, 1\}$. Then, the idempotents A, $\lambda^{-1}AB$, B and $\lambda^{-1}BA$ are pairwise similar and $P = \frac{1}{1-\lambda}(A - B)^2 \in \mathcal{A}^{id}$. We have $[A, B]^{2k} = \lambda^k(\lambda - 1)^k P$ and $[A, B]^{2k+1} = \lambda^k(\lambda - 1)^k[A, B]$ for all $k \in \mathbb{N}$.*

If $\mathbb{J}$ is an ideal in $\mathcal{A}$ then $[A, B]^n \in \mathbb{J} \Leftrightarrow A, B \in \mathbb{J}$ for all $n \in \mathbb{N}$.

Proof By Lemma 3.1, the idempotents A, $\lambda^{-1}AB$, B and $\lambda^{-1}BA$ are pairwise similar. We have

$$[A, B]^2 = ABA \cdot B + BAB \cdot A - ABA - BAB = -\lambda(A - B)^2. \tag{3.1}$$

On the other hand, for all $A, B \in \mathcal{A}^{\mathrm{id}}$, we have

$$[A, B]^2 = (A - B)^4 - (A - B)^2.$$

Thus, by (3.1), we obtain $(A - B)^4 = (1 - \lambda)(A - B)^2$. Multiply both sides of the last equality by the number $(1 - \lambda)^{-2}$ and obtain $P = \frac{1}{1-\lambda}(A - B)^2 \in \mathcal{A}^{\mathrm{id}}$. Since $(A - B)^2$ commutes with A and B for all $A, B \in \mathcal{A}^{\mathrm{id}}$, we have $PA = AP = A$ and $PB = BP = B$. Since $[A, B]^2 = \lambda(\lambda - 1)P$, we conclude that $[A, B]^{2k} = \lambda^k(\lambda - 1)^k P$ for all $k \in \mathbb{N}$. Since $[A, B]^{2k+1} = [A, B]^{2k} \cdot [A, B] = \lambda^k(\lambda - 1)^k[A, B]$, the element $[A, B]^{2k+1}$ is a commutator for all $k \in \mathbb{N}$.

Let $\mathbb{J}$ be an ideal in $\mathcal{A}$, $\lambda \neq 0$, $n \in \mathbb{N}$ and $[A, B]^n \in \mathbb{J}$.

Step 1. If n is even then by the equality $[A, B]^{2k} = \lambda^k(\lambda - 1)^k P$, we have $(A - B)^2 \in \mathbb{J}$ and $(1 - \lambda)A = A - ABA = A(A - B)^2 A \in \mathbb{J}$. Thus, $A \in \mathbb{J}$.

Step 2. If n is odd then for the even number $n + 1$, we have $[A, B]^{n+1} = [A, B]^n \cdot [A, B] \in \mathbb{J}$ and apply Step 1. □

Example Consider the following complex 2×2 matrices:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}.$$

Then, $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $PQP = P$, $QPQ = Q$.

Theorem 3.3 *Let φ be a faithful trace on a C^*-algebra $\mathcal{A}$; let $A, B \in \mathcal{A}^{\mathrm{id}} \setminus \{0\}$ be such that $ABA = \lambda A$ and $BAB = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then, $[A, B]^n \neq 0$ for all $n \in \mathbb{N}$.*

Proof It suffices to show that $[A, B]^n \neq 0$ for all even $n \in \mathbb{N}$.

Case 1: $[A, B]^{2k} \notin \mathfrak{M}_\varphi$. Then, $[A, B]^{2k} \neq 0$.

Case 2: $[A, B]^{2k} \in \mathfrak{M}_\varphi$. Then, by Lemma 3.2 we obtain $A, B \in \mathfrak{M}_\varphi$. We have $\varphi(A), \varphi(B) \in \mathbb{R}^+$ by [10, Theorem 4.6] and recall that the trace φ is faithful. Now, apply Lemma 3.2 and by Lemma 2.1 obtain

$$\begin{aligned}
\varphi([A, B]^{2k}) &= \lambda^k (\lambda - 1)^k \varphi(P) \\
&= -\lambda^k (\lambda - 1)^{k-1} (\varphi(A) - \varphi(AB) + \varphi(B) - \varphi(BA)) \\
&= -\lambda^k (\lambda - 1)^{k-1} (\varphi(A) - \varphi(ABA) + \varphi(B) - \varphi(BAB)) \\
&= \lambda^k (\lambda - 1)^k (\varphi(A) + \varphi(B)) \neq 0.
\end{aligned}$$

Thus, $[A, B]^n \neq 0$ for all $n \in \mathbb{N}$. □

Corollary 3.4 *Let φ be a faithful tracial state on a C^*-algebra $\mathcal{A}$, let $A, B \in \mathcal{A}^{\mathrm{id}} \setminus \{0\}$ be such that $ABA = \lambda A$ and $BAB = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then, the element $[A, B]^{2n}$ is a non-commutator for all $n \in \mathbb{N}$.*

Proof We have $\varphi([A, B]^{2n}) \neq 0$ for all $n \in \mathbb{N}$ (see the proof of Theorem 3.3). □

Theorem 2.2 allows us to state

Lemma 3.5 *Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = \infty$. If operators $X, Y \in \mathcal{B}(\mathcal{H})$ are non-commutators then XY and $X \circ Y$ are non-commutators. In particular, X^n is a non-commutator for every $n \in \mathbb{N}$.*

Theorem 3.6 *Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = +\infty$, and let an operator $X \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{B}(\mathcal{H})$ and $A_1 B_1, \ldots, A_n B_n$ are non-commutators, then the operator $A_n \cdots A_1 X B_1 \cdots B_n$ is a non-commutator.*

Proof We apply Theorem 2.2. Let $X = \lambda I + J$ and $A_k B_k = \lambda_k I + J_k$ for some $\lambda, \lambda_k \in \mathbb{C} \setminus \{0\}$ and operators $J, J_k \in \mathcal{J}$ for $k = 1, \ldots, n$. The proof is by induction. For $n = 1$, we have

$$\begin{aligned}
A_1 X B_1 &= A_1(\lambda I + J) B_1 = \lambda A_1 B_1 + A_1 J B_1 = \lambda(\lambda_1 I + J_1) + A_1 J B_1 \\
&= \lambda\lambda_1 I + \lambda J_1 + A_1 J B_1.
\end{aligned}$$

Note that $\lambda\lambda_1 \neq 0$ and the operator $\lambda J_1 + A_1 J B_1$ lies in $\mathcal{J}$. The case of $n \geq 2$ follows by induction. □

Corollary 3.7 *If $A, B \in \mathcal{B}(\mathcal{H})$ and the operator AB is a non-commutator, then the operator $A^n B^n$ is a non-commutator for every $n \in \mathbb{N}$.*

Lemma 3.8 (on division) *Let $\mathcal{H}$ be a Hilbert space,* $\dim \mathcal{H} = +\infty$ *and $X, Y \in \mathcal{B}(\mathcal{H})$.*

(i) *If XY and X (resp., Y) are non-commutators, then Y (resp., X) is a non-commutator.*
(ii) *If $X \circ Y$ and X (resp., Y) are non-commutators then Y (resp., X) is a non-commutator.*

Proof (i) Let operators XY and X be non-commutators. Then, by Theorem 2.2, we have

$$XY = \lambda I + J, \quad X = \mu I + J_1$$

for some $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ and certain operators $J, J_1 \in \mathcal{J}$. Therefore,

$$\lambda I + J = XY = (\mu I + J_1)Y = \mu Y + J_1 Y$$

and $Y = \frac{\lambda}{\mu} I + J_2$ with the operator $J_2 = \frac{1}{\mu}(K - J_1 Y) \in \mathcal{J}$. Thus, Y is a non-commutator by Theorem 2.2.

In particular, if $X \in \mathcal{B}(\mathcal{H})$ is left (resp., right) invertible, then X is a non-commutator if and only if X_l^{-1} (resp., X_r^{-1}) is a non-commutator.

(ii) Let operators $X \circ Y$ and X be non-commutators. Then, by Theorem 2.2, we have

$$X \circ Y = \lambda I + J, \quad X = \mu I + J_1$$

for some $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ and certain operators $J, J_1 \in \mathcal{J}$. Therefore,

$$\lambda I + J = X \circ Y = \frac{(\mu I + J_1)Y + Y(\mu I + J_1)}{2} = \mu Y + \frac{J_1 Y + Y J_1}{2}$$

and $Y = \frac{\lambda}{\mu} I + J_2$ with the operator $J_2 = \frac{1}{\mu}(J - J_1 \circ Y) \in \mathcal{J}$. Thus, Y is a non-commutator by Theorem 2.2. □

Corollary 3.9 [15, Corollary 14] *If $\mathcal{H}$ is separable and an operator $X \in \mathcal{B}(\mathcal{H})$ admits a left (resp., right) essential inverse X_l^{-1} (resp., X_r^{-1}) then X_l^{-1} (resp., X_r^{-1}) is a non-commutator if and only if X is a non-commutator.*

Corollary 3.10 *Let $\lambda \in \mathbb{C}$ be a regular point of $X \in \mathcal{B}(\mathcal{H})$ and $R_\lambda = (X - \lambda I)^{-1}$ be the resolvent of X. If X is a non-commutator, then R_λ is a non-commutator.*

Proof By Theorem 2.2. we have $X = xI + J$ for some $x \in \mathbb{C}\backslash\{0\}$ and an operator $J \in \mathcal{J}$. Since every operator from $\mathcal{J}$ is non-invertible, we infer that $x \neq \lambda$ and apply Corollary 3.9. □

Corollary 3.11 *Let $\mathcal{H}$ be a Hilbert space,* $\dim \mathcal{H} = \infty$. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $ABA = \lambda A + J$ for some $\lambda \in \mathbb{C}\setminus\{0\}$ and an operator $J \in \mathcal{J}$. If A is a non-commutator then B is also a non-commutator.*

Proof Note that an operator $\lambda A + J$ is a non-commutator, see Theorem 2.2. Apply Lemma 3.8 with $X = AB$, $Y = A$ and conclude that AB is non-commutator. Again apply Lemma 3.8 with $X = A$, $Y = B$ and infer that B is non-commutator. □

Lemma 3.12 *Let $\mathcal{J}$ be a proper uniformly closed ideal in a unital C^*-algebra $\mathcal{A}$. Let a Hermitian element $X \in \mathcal{A}$ be of the form $X = xI + J_1$, where $x \in \mathbb{R}$ and $J_1 \in \mathcal{J}$. The equality $f(X) = f(x)I + J$ holds for any continuous real-valued function f on the spectrum $\sigma(X)$, here $J \in \mathcal{J}$.*

Proof Since the ideal $\mathcal{J}$ is proper, $I \notin \mathcal{J}$, the elements of $\mathcal{J}$ are irreversible and $x \in \sigma(X)$. Since $X^n = x^n I + J_n$ with $J_n \in \mathcal{J}$, for a polynomial $p(t) = a_0 + a_1 t + \cdots + a_k t^k$ we have $p(X) = a_0 I + a_1 X + \cdots + a_k X^k = p(x)I + J'$, where $J' \in \mathcal{J}$. By the Weierstrass Theorem, there exists a sequence $\{p_m\}_{m=1}^{\infty}$ of polynomials, which converges uniformly on $\sigma(X)$ to the function f as $m \to \infty$. For each $m \in \mathbb{N}$, $p_m(X) = p_m(x)I + J^{(m)}$, where $J^{(m)} \in \mathcal{J}$. Since $p_m(X) \to f(X)$ and $p_m(x)I \to f(x)I$ as $m \to \infty$, the sequence $\{J^{(m)}\}_{m=1}^{\infty}$ also converges. The limit of $\{J^{(m)}\}_{m=1}^{\infty}$ lies in $\mathcal{J}$, because $\mathcal{J}$ is uniformly closed. It follows that $f(X) = f(x)I + J$ with $J \in \mathcal{J}$. □

By Lemma 3.12 and Theorem 2.2, we get

Lemma 3.13 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Let an operator $X \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ be a non-commutator, $X = xI + J$ for some $x \in \mathbb{R}\setminus\{0\}$ and $J \in \mathcal{J}$ (see* Theorem 2.2*). Then, $f(X)$ is a non-commutator for every continuous function $f : \sigma(X) \to \mathbb{R}$ with $f(x) \neq 0$.*

Remark 3.14 In particular, an operator $X \in \mathcal{B}(\mathcal{H})^+$ is a non-commutator if and only if an operator X^q is a non-commutator for some (consequently, for all) $q > 0$ (recall that $\dim \mathcal{H} = \infty$). This fact also follows by [5, Remark 4] (hint: consider the odd continuation of the function $f(t) = t^q$ from $[0, +\infty)$ to $\mathbb{R}$). If $\mathcal{H}$ is a separable space and an operator $X \in \mathcal{B}(\mathcal{H})^+$ is a non-commutator, then the projection X^0 on the closure of the range of X is a non-commutator. Indeed, if $0 \le X \le I$, then $\{X^{\frac{1}{n}}\}_{n=1}^{\infty}$ is a monotone increasing sequence of operators whose strong-operator limit is the projection X^0 on the closure of the range of X [28, Lemma 5.1.5]. If $X \in \mathcal{B}(\mathcal{H})^+$ is a non-commutator, then $\dim X^{0\perp}\mathcal{H} < \infty$ and $X^0 = I - X^{0\perp}$ is a non-commutator by Theorem 2.2.

Theorem 3.15 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then, the following conditions are equivalent:*

(i) *X is a non-commutator;*
(ii) *U and $|X|$ are non-commutators.*

Proof (i)⇒(ii). By Theorem 2.2, we have $X = xI + J$ for some $x \in \mathbb{C}\setminus\{0\}$ and an operator $J \in \mathcal{J}$. Since an operator J^* is lies in $\mathcal{J}$, $X^* = \overline{x}I + J^*$ is a non-commutator. Now, by Lemma 3.5, the operator X^*X is a non-commutator. Therefore, $|X| = \sqrt{X^*X}$ is a non-commutator by Lemma 3.13 with $f(t) = \sqrt{t}$, $t \geq 0$. Since $X = U|X|$, an operator U is a non-commutator by Lemma 3.8.

(ii)⇒(i). Since $X = U|X|$, the assertion follows by Lemma 3.5. □

Corollary 3.16 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{B}(\mathcal{H})$.*

(i) *If T is a non-commutator, then for any real number $\lambda \in [0, 1]$, the λ-Aluthge transformation $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ is a non-commutator.*
(ii) *If $|T|$ and $\Delta_\lambda(T)$ for some number $\lambda \in [0, 1]$ are non-commutators, then T is a non-commutator.*

Proof (i) By Theorem 3.15, the operators U and $|T|$ are non-commutators. Then, we apply Theorem 3.6 with $A_1 = |T|^\lambda$, $B_1 = |T|^{1-\lambda}$ and $X = U$.

(ii) For $\lambda \in [0, 1]$, the operators $|T|^\lambda$, $|T|^{1-\lambda}$ are non-commutators, see Remark 3.14. For $X = |T|^\lambda$, $Y = U|T|^{1-\lambda}$ Lemma 3.8 implies that $U|T|^{1-\lambda}$ is non-commutator. Thus, $T = U|T|^{1-\lambda} \cdot |T|^\lambda$ is non-commutator as a product of two non-commutators by Lemma 3.5. □

For $T \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} < \infty$, we have $\mathrm{tr}(T) = \mathrm{tr}(\Delta_\lambda(T))$ for any number $\lambda \in [0, 1]$. Thus, T is a commutator if and only if $\Delta_\lambda(T)$ is a commutator for some (consequently, for all) $\lambda \in [0, 1]$ by [26, Ch. 24, Problem 230].

Example Let $X = U|X|$ be the polar decomposition of a matrix $X \in \mathbb{M}_2(\mathbb{C})$. If X is an invertible commutator, then U is a commutator. Indeed, we have $X = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$ in some basis in $\mathbb{C}^2$ by [25, Ch. II, Problem 209] and $ab \neq 0$. Let $a = e^{\mathrm{i}\alpha}|a|$, $b = e^{\mathrm{i}\beta}|b|$ for $0 \leq \alpha, \beta < 2\pi$. Then, $|X| = \sqrt{X^*X} = \mathrm{diag}(|b|, |a|)$. For the unitary matrix $U = [u_{ij}]_{i,j=1}^2$ from equality $X = U|X| = \begin{pmatrix} u_{11}|b| & u_{12}|a| \\ u_{21}|b| & u_{22}|a| \end{pmatrix}$, we have $u_{11} = u_{22} = 0$, $u_{12} = e^{\mathrm{i}\alpha}$, $u_{21} = e^{\mathrm{i}\beta}$ and U is a commutator by [25, Ch. II, Problem 209].

Theorem 3.17 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Then, for $X \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$, the following conditions are equivalent:*

(i) *X is a commutator;*
(ii) *the Cayley transform $\mathcal{K}(X)$ is a commutator.*

Proof (ii)⇒(i). Let X be a non-commutator. By Theorem 2.2, we have $X = xI + J$ for some $x \in \mathbb{R}\setminus\{0\}$ and a Hermitian operator $J \in \mathcal{J}$. The operators $X \pm \mathrm{i}I = (x \pm \mathrm{i})I + J$ are non-commutators by Theorem 2.2. Therefore, the operator $(X - \mathrm{i}I)^{-1}$ is a non-commutator by Corollary 3.9 and we apply Lemma 3.5. Thus, the Cayley transform $\mathcal{K}(X)$ is a non-commutator.

(i)⇒(ii). Let $\mathcal{K}(X)$ be a non-commutator. By Theorem 2.2 for the unitary operator $\mathcal{K}(X)$, we have $\mathcal{K}(X) = xI + J$ for some $x \in \mathbb{C}\backslash\{0\}$ with $|x| = 1$ and an operator $J \in \mathcal{J}$. We have

$$X + \mathrm{i}I = \mathcal{K}(X)(X - \mathrm{i}I) = (xI + J)(X - \mathrm{i}I) = xX - \mathrm{i}xI + JX - \mathrm{i}J. \quad (3.2)$$

Therefore, if $x = 1$, then $I \in \mathcal{J}$; if $x = -1$ then $X \in \mathcal{J}$. In both cases, we arrive to a contradiction. Thus, $x \neq \pm 1$. By (3.2), we have $(1 - x)X = -\mathrm{i}(1 + x)I + JX - \mathrm{i}J$ and apply Theorem 2.2. Thus, X is a non-commutator. □

Let $\mathcal{A}$ be an algebra, let $A, B \in \mathcal{A}$ be such that $AB = -BA$, i.e., A and B *anticommute*. Then, AB and BA are commutators: $AB = [\frac{A}{2}, B]$, $BA = [B, \frac{A}{2}]$.

Lemma 3.18 *Let $\mathcal{A}$ be a unital algebra, let $A, B \in \mathcal{A}$ be such that $AB = \lambda BA$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. Then, we have the spectral relation $\sigma(AB) = \sigma(BA) = \lambda\sigma(BA)$. Moreover, if B is invertible, then $\sigma(A) = \lambda\sigma(A)$.*

Proof We have $\lambda\sigma(BA) = \sigma(\lambda BA) = \sigma(AB)$. Since

$$\sigma(XY) \cup \{0\} = \sigma(YX) \cup \{0\} \quad \text{for all} \quad X, Y \in \mathcal{A}, \quad (3.3)$$

see [26, Ch. 9, Problem 76], we obtain $\lambda\sigma(BA) \cup \{0\} = \sigma(BA) \cup \{0\}$. Then, we consider two cases: 1) $0 \in \sigma(BA)$, and 2) $0 \notin \sigma(BA)$. In both cases, we have $\lambda\sigma(BA) = \sigma(BA)$. Thus,

$$\sigma(AB) = \lambda\sigma(BA) = \sigma(BA) = \lambda\sigma(AB).$$

For an invertible B, we have $A = AB \cdot B^{-1} = \lambda BA \cdot B^{-1}$ and $\sigma(A) = \lambda\sigma(BAB^{-1}) = \lambda\sigma(A)$ since similarity preserves spectra [26, Ch. 9, Problem 75]. □

In particular, if $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ and $\det(AB) \neq 0$, then $\lambda^n = 1$ by the theorem on the determinant of a matrix product.

Example In $\mathbb{M}_2(\mathbb{C})$ for matrices $A = \mathrm{diag}(1, -1)$ and $B = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, we have $AB = -BA$. Consider the primitive cubic roots of 1: $\omega_1 = 1$, $\omega_2 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$, $\omega_3 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$. In $\mathbb{M}_3(\mathbb{C})$ for the matrices

$$A = \begin{pmatrix} 0 & 0 & \omega_1 \\ \omega_2 & 0 & 0 \\ 0 & \omega_3 & 0 \end{pmatrix}$$

and $B = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$, we have $AB = \omega_3 BA$.

Theorem 3.19 *Let $\mathcal{H}$ be a Hilbert space and* $\dim \mathcal{H} \leq +\infty$, *$A, B \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$.*

(i) *If $AB = \lambda BA$ for some $\lambda \in \mathbb{C}\backslash\{1\}$, then the operator AB is a commutator.*

(ii) *If* $\dim \mathcal{H} < +\infty$*, then* AB *is a commutator if and only if* BA *is a commutator.*
(iii) *The operator* AP *is a commutator if and only if* PA *is a commutator.*

Proof (i) For $\lambda = 0$, the assertion is trivial. Assume that $\lambda \neq 0$ and consider two cases.

Case 1: let $\dim \mathcal{H} < +\infty$. Then,

$$\mathrm{tr}(BA) = \mathrm{tr}(AB) = \mathrm{tr}(\lambda BA) = \lambda \mathrm{tr}(BA)$$

and $\mathrm{tr}(BA) = \mathrm{tr}(AB) = 0$. Thus, AB and BA are commutators by [26, Ch. 24, Problem 230].

Case 2: let $\dim \mathcal{H} = +\infty$. Assume that the operator AB is a non-commutator. Then,

$$AB = \mu I + J \tag{3.4}$$

for some $\mu \in \mathbb{C}\backslash\{0\}$ and an operator $J \in \mathcal{J}$ by Theorem 2.2. Multiply both sides of equality (3.4) by the operator A from the right and obtain

$$ABA = \mu A + JA. \tag{3.5}$$

Since $\lambda BA = \mu I + J$, multiply both sides of the last equality by the operator A from the left and obtain

$$\lambda ABA = \mu A + AJ. \tag{3.6}$$

By (3.5), we have $\lambda ABA = \mu\lambda A + \lambda JA$; subtract this relation term by term from equality (3.6) and conclude that $\mu(\lambda - 1)A = AJ - \lambda JA$. Therefore, $A \in \mathcal{J}$. Thus $AB \in \mathcal{J}$ and we have a contradiction with representation (3.4).

(ii) If $\dim \mathcal{H} < +\infty$, then $\mathrm{tr}(BA) = \mathrm{tr}(AB)$ and the assertion follows by [26, Ch. 24, Problem 230].

(iii) Let $\dim \mathcal{H} = +\infty$. Assume that the operator AP is a non-commutator. Then, by Theorem 2.2, we have $AP = xI + J$ for some $x \in \mathbb{C}\backslash\{0\}$ and an operator $J \in \mathcal{J}$. Then, for the idempotent $P^{\perp} = I - P$, we conclude that

$$0 = AP \cdot P^{\perp} = xP^{\perp} + JP^{\perp}.$$

Hence, $P^{\perp} \in \mathcal{J}$ and $P = I - P^{\perp}$ is a non-commutator by Theorem 2.2. Since AP and P are non-commutators, the operator A is a non-commutator via Lemma 3.8. Since A and P are non-commutators, the operator PA is a non-commutator via Lemma 3.5.

For the proof of the inverse implication, note that if PA is a non-commutator, then $(PA)^* = A^*P^*$ is also a non-commutator by Theorem 2.2. Recall that $P^* \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ and by the preceding part of the proof P^*A^* is a non-commutator. Therefore, $(P^*A^*)^* = AP$ is a non-commutator by Theorem 2.2. □

The condition $P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ is essential in Theorem 3.19. If $\mathcal{H}$ is separable and $\dim \mathcal{H} = +\infty$, then there exists a partial isometry $A \in \mathcal{B}(\mathcal{H})$ such that $A^*A = I$ and the operators AA^*, $I - AA^*$ are non compact (hence by Theorem 2.2 we conclude that A^*A is a non-commutator, but AA^* is a commutator).

Corollary 3.20 *Let* $\dim \mathcal{H} = n < +\infty$ *and matrices* $A, B \in \mathcal{B}(\mathcal{H})$ *be such that* $AB = \lambda BA$ *for some* $\lambda \in \mathbb{C}\setminus\{1\}$.

(i) *AB and BA are unitarily equivalent to matrices with zero diagonal.*
(ii) *We have* $\mathrm{tr}(|I + zAB|) \geq n$ *and* $\mathrm{tr}(|I + zBA|) \geq n$ *for all* $z \in \mathbb{C}$.

Proof (i) Follows by [25, Ch. II, Problem 209].
(ii) Follows by [11, Theorem 4.8]. □

Theorem 3.21 *Let $\mathcal{H}$ be a Hilbert space, $U \in \mathcal{B}(\mathcal{H})$ be an isometry.*

(i) *If $A \in \mathcal{B}(\mathcal{H})$ is a non-commutator, then the operator U^*AU is a non-commutator.*
(ii) *If $\mathcal{H}$ is separable and U is a non-commutator, then U is unitary.*

Proof If $\dim \mathcal{H} < +\infty$, then every isometry $U \in \mathcal{B}(\mathcal{H})$ is unitary. If $A \in \mathcal{B}(\mathcal{H})$ is a non-commutator, then

$$0 \neq \mathrm{tr}(A) = \mathrm{tr}(U^*AU)$$

and U^*AU is a non-commutator by [26, Ch. 24, Problem 230].

Assume that $\dim \mathcal{H} = +\infty$. Then, (i) follows by Theorem 2.2. For the proof of (ii), note that $U = xI + K$ for some $x \in \mathbb{C}\setminus\{0\}$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$, i.e., U is a thin operator. By Proposition of [2] via $U^*U = I$, we have $UU^* = I$. □

Theorem 3.22 *Let $\mathcal{A}$ be an algebra and $A, B \in \mathcal{A}$ be such that $A \sim B$. Let $n \in \mathbb{N}$ and $p_n(t) = \sum_{k=1}^{n} a_k t^k$ be a polynomial without a constant term, $q_{n+1}(t) = tp_n(t)$. Then,*

(i) *$p_n(A) \sim p_n(B)$ and $p_n(A) - p_n(B)$ is a commutator;*
(ii) *if $\mathbb{J}$ is an ideal in $\mathcal{A}$ and $p_n(A) \in \mathbb{J}$, then $q_{n+1}(B) \in \mathbb{J}$;*
(iii) *if $A^m = p_n(A)$ for some $m \in \mathbb{N}$, then $B^{m+1} = q_{n+1}(B)$.*

Proof Let $X, Y \in \mathcal{A}$ be such that $XY = A$ and $YX = B$.

(i) For $Z = a_n(XY)^{n-1}X + a_{n-1}(XY)^{n-2}X + \cdots + a_1X$, we have $p_n(A) = ZY$ and $p_n(B) = YZ$. Thus, $p_n(A) \sim p_n(B)$ and $p_n(A) - p_n(B) = [Z, Y]$.

(ii) Let $\mathbb{J}$ be an ideal in $\mathcal{A}$ and $p_n(A) \in \mathbb{J}$. Then,

$$\begin{aligned} q_{n+1}(B) &= \sum_{k=2}^{n+1} a_{k-1}B^k = \sum_{k=2}^{n+1} a_{k-1}(YX)^k = Y\left(\sum_{k=1}^{n} a_k(YX)^k\right)X \\ &= Yp_n(A)X \in \mathbb{J}. \end{aligned}$$

(iii) We have (see the proof of item (ii))

$$q_{n+1}(B) = Yp_n(A)X = YA^mX = Y(XY)^mX = (YX)^{m+1} = B^{m+1}.$$

□

Let $\mathcal{A}$ be a C^*-algebra and $A, B \in \mathcal{A}$ be such that $A \sim B$. By relation (3.3), we have $\sigma(A) \cup \{0\} = \sigma(B) \cup \{0\}$.

Theorem 3.23 *Let $\mathcal{A}$ be a $*$-algebra. Then, for $A \in \mathcal{A}$ and $B \in \mathcal{A}^{\mathrm{sa}}$ the following conditions are equivalent:*

(i) $A \sim B$;
(ii) $A^* \sim B$.

Under these conditions, we have $\sigma(A) \subset \mathbb{R}$.

Proof (i) $\Rightarrow$(ii). Let $X, Y \in \mathcal{A}$ be such that $XY = A$ and $YX = B$. Then, $A^* = Y^*X^*$ and $B = YX = (YX)^* = X^*Y^*$.

(ii) $\Rightarrow$(i). Since $(A^*)^* = A$, we can repeat the proof of the implication (i)$\Rightarrow$(ii) for the pair $\{A^*, B\}$.

Via (3.3) and the relation $\sigma(B) = \sigma(YX) \subset \mathbb{R}$ we infer that $\sigma(A) = \sigma(XY) \subset \mathbb{R}$. □

Theorem 3.24 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. If $A, B \in \mathcal{B}(\mathcal{H})$ are non-commutators and $A \sim B$, then $A - B \in \mathcal{J}$.*

Proof Let $A = aI + J$, $B = bI + J_1$ for some $a, b \in \mathbb{C}\backslash\{0\}$ and certain operators $J, J_1 \in \mathcal{J}$, see Theorem 2.2. Let $X, Y \in \mathcal{B}(\mathcal{H})$ be such that $A = XY$, $B = YX$. We have

$$\begin{aligned} X \cdot YX \cdot Y = X(bI + J_1)Y = bXY + XJ_1Y = abI + bJ + XJ_1Y \\ = (XY)^2 = (aI + J)^2 = a^2I + 2aJ + J^2. \end{aligned}$$

Note that the operators $bJ + XJ_1Y$ and $2aJ + J^2$ lie in $\mathcal{J}$. Therefore, $a = b$ and $A - B \in \mathcal{J}$. □

Example If $A, B \in \mathbb{M}_n(\mathbb{C})$ and $A \sim B$, then $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, $\dim \mathcal{H} = \infty$. Then, there exist operators $A \in \mathcal{A}^+$ and $B \in \mathcal{A}$ such that $A \sim B$, $A \in \mathfrak{S}_1(\mathcal{H})$, but $B \notin \mathfrak{S}_1(\mathcal{H})$. Hint: for some projections $P, Q \in \mathcal{B}(\mathcal{H})$, we have $PQP \in \mathfrak{S}_1(\mathcal{H})$, but $QP \notin \mathfrak{S}_1(\mathcal{H})$, see [3, Remark 1]. Put $X = P$ and $Y = QP$.

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