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Conformal Radius Has Unique Critical Point when Pre-Schwarzian Derivative is Subordinate to Classical Majorants

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Abstract—We obtain a number of the uniqueness criteria for the critical point of the conformal radius in the form of the subordination of pre-Schwarzian derivative both to functions of well-known subclasses in the geometric function theory and to the images of such functions under the action of some classical operators.

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Remembering Professor L. A. Aksent'ev (1932–2020)

1. INTRODUCTION

Soviet La Belle Époque had simple, but effective slogan: "developed socialism is the unity of what has already been done with what remains to be done". Professor L.A. Aksent'ev wrote articles in the style of developed socialism: formulations were chased, words were perceived as formulas, and constants were allowed not to be sharp—success was determined by the taking of new forms, concepts, effects, etc., and their further development could be left to the followers.

So, this article is a unity of what has already been done in the author's works [1, 2], with the results that will now be presented, and the emphasis will be placed on some updating of already known results about the uniqueness of the critical point of the conformal radius. As Professor Aksent'ev said about himself in such cases, the author's contribution to this direction is modest—he owns only some initiative and some symmetry. And—let's add from ourselves—some ordering.

The very last commandment of Professor Aksent'ev was a warning—not to seek for the functional analysis when working with uniqueness conditions. As the papers [1] and (especially) [3] show, this is really a problem, because the functional-analytic context can be very convenient. Nevertheless, in formulating the results of present article we follow the verified vintage patterns, leaving a wide scope for imagination, which was always appreciated by Professor Aksent'ev.

In this note we obtain new sufficient conditions for the uniqueness of a (zero) critical point of conformal radius

$$R_f(\zeta) = |f'(\zeta)|(1 - |\zeta|^2) \tag{1}$$

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for a function f holomorphic in the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ (concerning the history and current state of topics related to extreme behavior of (1), we refer the reader, e.g., to [4]). We assume that the function f has a series expansion

$$f(\zeta) = \zeta + \sum_{k=n+2}^{\infty} a_k \zeta^k, \quad n \geq 1, \tag{2}$$

where the natural number n, which determines the length of the first gap in the series, is a controlling parameter in the uniqueness criteria announced above.

These criteria have the form of conditions for the subordination of the pre-Schwarzian derivative to the values of a simple structure, namely,

$$f''(\zeta)/f'(\zeta) \prec aF(\zeta), \quad \zeta \in \mathbb{D}, \tag{3}$$

or

$$f''(\zeta)/f'(\zeta) \prec aP[F](\zeta), \quad \zeta \in \mathbb{D}. \tag{4}$$

Here P is an injective operator acting on subclasses of holomorphic functions in the unit disc. Note that conditions of the form (3) or (4) will ensure the local univalence of the function f in $\mathbb{D}$, i.e. relation $f'(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}$.

The numerical factors $a = a(n)$ at the right-hand sides of conditions (3) or (4) are positive constants of a specific type; as a rule, $a(n) = n$ in the case when a function F, holomorphic in $\mathbb{D}$, has an expansion

$$F(\zeta) = \zeta + \dots, \tag{5}$$

and $a(n) = 2n$, when F, moreover, satisfies the condition

$$F''(0) = 0. \tag{6}$$

The functional factors at the right-hand sides of conditions (3) or (4) are precisely the classical majorants from the title of this article. The mentioned majorants will run over the functions from the following classes.

Among all of these classes the central position is occupied by the class S^0 of all convex univalent functions F of the form (5) in the disc $\mathbb{D}$; recall that the convexity condition for F has the form $\mathrm{Re}\,(1 + \zeta F''(\zeta)/F'(\zeta)) > 0$, $\zeta \in \mathbb{D}$. Class $S^*(b)$ of starlike functions of order $b \in [0, 1)$ consists of all holomorphic functions (5) in $\mathbb{D}$ with the condition $\mathrm{Re}\,\zeta F'(\zeta)/F(\zeta) > b$, $\zeta \in \mathbb{D}$. The inclusion of the class S^0 in the class $S^*(b)$ takes place when $0 \leq b \leq 1/2$; we distinguish two classes corresponding to the ends of this interval of parameters, the class $S^*(0) = S^*$ of starlike functions and the second extension $S^*(1/2)$ of the class of convex functions in the sense of Marx–Strohhäcker. The first extension of the class S^0 in the sense of Marx–Strohhäcker is the closed convex hull, $\overline{co}S^0$, of the class S^0, consisting of exactly those functions F that are holomorphic in $\mathbb{D}$ and satisfy the condition $\mathrm{Re}\, F(\zeta)/\zeta > 0$, $\zeta \in \mathbb{D}$; both extensions were introduced and studied in [5, 6].

Along with the scale of starlikeness, parametrized by its order, we consider the scale of the Alexander classes. Alexander's class, $\mathcal{A}(\alpha)$, of the order $\alpha \in [0, 1)$ unites all holomorphic functions (5) in $\mathbb{D}$ satisfying the condition $\mathrm{Re}\, F'(\zeta) > \alpha$, $\zeta \in \mathbb{D}$; $\mathcal{A}(0) = \mathcal{A}$ [7].

The belonging of function (5) to classes S^0, $S^*(1/2)$, $\overline{co}S^0$, $\mathcal{A}$ implies the estimate

$$|F(\zeta)| \leq |\zeta|/(1 - |\zeta|), \quad \zeta \in \mathbb{D}, \tag{7}$$

which, under the conditions (3) or (4), implies the absence of nonzero roots of the Gakhov equation

$$f''(\zeta)/f'(\zeta) = 2\bar{\zeta}/(1 - |\zeta|^2), \tag{8}$$

acting as a necessary condition for the extremum of function (1) (see [8]). By adding condition (6) to (5) we expand the list of introduced classes with new objects that ensure the uniqueness of the zero critical point of conformal radius (1) due to the strengthening the inequality (7) to

$$|F(\zeta)| \leq |\zeta|/(1 - |\zeta|^2), \quad \zeta \in \mathbb{D}. \tag{9}$$

New objects of this list will be, in addition to the classes S^* and $\mathcal{A}(1/2)$ defined above, the Nehari class, N, consisting of all holomorphic functions (5) with the estimate $|\{F, \zeta\}| \leq 2/(1 - |\zeta|^2)^2$, $\zeta \in \mathbb{D}$, of the

Schwarzian derivative $\{F,\zeta\} = (F''/F')'(\zeta) - (F''/F')^2(\zeta)/2$ of the function F [9], and also the class $S^{(2)}$ of odd univalent functions (see, e.g., [10], p. 8). For the fulfillment of conditions (7) and (9) in the corresponding classes of functions, as well as for the presence of inclusions between some of these classes, one can see [1] and the bibliography contained there.

By P we mean the following operators.

Miller–Mocanu operator [11]

$$P_{\beta,\gamma}[F](\zeta) = \left((\beta+\gamma)\zeta^{-\gamma}\int\limits_0^\zeta F^\beta(t)t^{\gamma-1}dt\right)^{1/\beta}. \tag{10}$$

As in [11], the domain of $P_{\beta,\gamma}$ consists of functions F with $F'(0) = 1$ and $F(\zeta)/\zeta \neq 0$ in $\mathbb{D}$. However, we take $\beta > 0$, $\beta+\gamma > 0$, in contrast to [11], where complex values for β, γ are allowed. Note that $B_c = P_{1,c}$ is the Bernardi operator [12] and that the inverse to (10) will be considered for $\beta \geq 1$.

Biernacki type operator (see, e.g., [13], p. 48)

$$J_c[F](\zeta) = \int\limits_0^\zeta (F(t)/t)^c dt, \tag{11}$$

which will be considered for nonnegative values of the parameter, as well as

Hornich operator (see, e.g., [3])

$$G_b[F](\zeta) = \int\limits_0^\zeta F'(t)^b dt. \tag{12}$$

The next section deals with conditions of the form (3), the rest of the article is devoted to conditions of the form (4). In third and fourth sections we study the case $P = P_{\beta,\gamma}$, in the fifth section the uniqueness conditions are given with $P = P^{-1}_{\beta,\gamma}$, $p = J_c$ and $P = G_b$.

2. SUBORDINATIONS TO CLASSICAL MAJORANTS

By the routine calculations we get the following useful statement.

Lemma 1. *Maximal constant $a = A(m,n)$, ensuring the fulfillment of the inequality*

$$ar^n/(1-r^{mn}) \leq 2r/(1-r^2) \tag{13}$$

for all $r \in [0,1)$ and $n \in \mathbb{N}$, is equal to $A(m,n) = mn$ only for $m = 1$ and $m = 2$. Here the inequality (13) is strict on the interval $r \in (0,1)$, except for the case $m = 2$, $n = 1$, where the equality holds in (13) for all $r \in (0,1)$. For $m \geq 3$ strict inequalities $n < A(m,n) < mn$, $n \in \mathbb{N}$, hold.

The following statement defines the pattern of our research.

Theorem 1. *Suppose the function (2) is holomorphic in the disc $\mathbb{D}$ and satisfies the subordination (3) with $a = n$. If the function F in (3) belongs to one of the subclasses S^0, $S^*(1/2)$, $\overline{co}S^0$ or $\mathcal{A}$, then the conformal radius (1) has a unique critical point (maximum) $\zeta = 0$. The constant $a = n$ is sharp for the first three of these subclasses.*

Proof. If we rewrite (3) with the help of Schwarz's lemma, then due to representation (2) and functional membership of F, that ensures the fulfillment of estimate (7), we obtain the inequality

$$|f''(\zeta)/f'(\zeta)| \leq a|\zeta|^n/(1-|\zeta|^n), \quad \zeta \in \mathbb{D}.$$

By Lemma 1, for $a = n$ it is sharply completed to the strict estimate

$$|f''(\zeta)/f'(\zeta)| < 2|\zeta|/(1-|\zeta|^2), \quad \zeta \in \mathbb{D}\setminus\{0\},$$

which excludes the presence of nonzero roots of the Gakhov equation (8). The function that loses the uniqueness of the zero critical point (8) for $a > n$ solves the differential equation $f''(\zeta)/f'(\zeta) = a\zeta^n/(1-\zeta^n)$ $(n \geq 1)$.

Remark 1. For $n = 1$ Theorem 1 was essentially proved in [2] for the first three of the indicated subclasses. The absence of sharpness of the constant $a = n$, when the class of majorants in (3) coincides with $\mathcal{A}$, is justified similarly to how it was done in [1] for the pre-Schwarzian derivative with multiplication, $\zeta f''(\zeta)/f'(\zeta)$.

As noted above, the fulfillment of the condition (6) for function (5) expands the list of classical majorants. The following is true.

Theorem 2. *Suppose the holomorphic functions (2) and (5) in the disc $\mathbb{D}$ are related by the subordination (3), where $a = 2n$ and the function (5) satisfies (6) and belongs to the class S^0 or one of its extensions, $\overline{co}S^0$, $S^*(1/2)$, S^*, N, or one of the classes $S^{(2)}$, $\mathcal{A}(1/2)$, $\mathcal{A}$. Then the conformal radius (1) has a unique critical point (maximum) $\zeta = 0$ unless $f(\mathbb{D})$ is a strip which is possible only for simultaneous fulfillment of the equality $n = 1$ and the inclusion of (5) into one of the classes S^0, $\overline{co}S^0$, S^*, $S^{(2)}$. For the latter four classes the constant $a = 2n$ in (3) is sharp for any n. For $n = 1$ it will be sharp for all of the above eight classes.*

The proof passes in exactly the same way as in the case of Theorem 1. The appearance of a new (in comparison with Theorem 1) conclusion on sharpness for $n = 1$ is ensured by condition (9), which allows the presence of bifurcations of the zero root of (8) for $a = 2$.

3. SUBORDINATIONS TO THE OPERATOR $P_{\beta,\gamma}$

Let us establish analogs of Theorems 1 and 2 for the subordinations of the form (4). We have the following result.

Theorem 3. *Suppose the function (2) is holomorphic in the disc $\mathbb{D}$ and satisfies the subordination (4) with $a = n$ and $P = P_{\beta,\gamma}$, where $\beta + \gamma > 0$, $\beta > 0$. If the function F in (4) belongs to one of the subclasses S^0, $S^*(1/2)$, $\overline{co}S^0$ or $\mathcal{A}$, then the conformal radius (1) has a unique critical point (maximum) $\zeta = 0$.*

The proof is obvious in the light of the first part of the following assertion (cf. Lemma 1 in [2]).

Lemma 2. 1) *Let $U(r)$ be a real-valued, continuous and strictly increasing function on the half-interval $[0, 1)$, and let $\alpha > 0$. Operator $P_{\beta,\gamma}$ strengthens the non-strict estimate $|F(\zeta)| \leq \alpha|\zeta|U(|\zeta|)$, $\zeta \in \mathbb{D}$, for the holomorphic function (5) in $\mathbb{D}$ to the strict inequality $|P_{\beta,\gamma}[F](\zeta)| < \alpha|\zeta|U(|\zeta|)$, $\zeta \in \mathbb{D} \setminus \{0\}$.*

2) *For any $\alpha \in \mathbb{C}$ the function $F_\alpha(\zeta) = \alpha\zeta$ is the fixed point of the operator $P_{\beta,\gamma}$.*

Due to the condition (9) the second part of the above lemma allows us to demonstrate the sharpness in the following

Theorem 4. *Suppose the holomorphic functions (2) and (5) in the disc $\mathbb{D}$ are related by the subordination (4), where $a = 2n$ and $P = P_{\beta,\gamma}$ for $\beta + \gamma > 0$, $\beta > 0$. Let the function (5) in (4) satisfies the condition (6) and belongs to the class S^0 or one of its extensions, $\overline{co}S^0$, $S^*(1/2)$, S^*, N, or one of the classes $S^{(2)}$, $\mathcal{A}(1/2)$, $\mathcal{A}$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$. The constant $a = 2n$ in (4) is sharp when $n = 1$ for all of the above classes.*

Remark 2. As a rule, the sharpness in the uniqueness conditions for the critical points (1) is connected with the bifurcation of the zero root of (8) if $n = 1$, and with the bifurcation of the nonzero root of (8) or with the boundary bifurcation if $n \geq 2$. In our situation, only first and third variants are possible. Moreover, the first variant corresponds to the case $a = 2$ and therefore disappears in Theorem 1 and, generally speaking, in Theorem 3, and is realized in Theorems 2 and 4 for any F with expansion (5), hence, for all classes considered there.

The sharpness for arbitrary n in individual classes depends on whether the functions

$$F_k(\zeta) = \zeta/(1-\zeta^k), \quad k = 1, 2,$$

lie in the images of these classes under the mapping by the operator $P_{\beta,\gamma}$. According to the first part of Lemma 2, the answer to this question is negative. The calculations of the following example allow us not only to confirm this answer in a different way, but also to formulate the uniqueness conditions of a new type.

Example 1. Routine but uncomplicated counting shows that the relations $F_k = P_{\beta,\gamma}[F_{k,\beta,\gamma}]$, $\beta, \gamma > 0$, are true for the functions $F_{k,\beta,\gamma}(\zeta) = F_k(\zeta)V_{k,\beta,\gamma}(\zeta)$, where

$$V_{k,\beta,\gamma}(\zeta) = \left\{1 + \frac{\beta}{\beta+\gamma}\frac{k\zeta^k}{1-\zeta^k}\right\}^{1/\beta}, \quad V_{k,\beta,\gamma}(0) = 1, \quad k = 1, 2. \tag{14}$$

It is clear that for $r \in (0, 1)$ the following conditions hold: $|F_{k,\beta,\gamma}(r)| > |F_k(r)|$, $k = 1, 2$. Therefore, the function $F_{k,\beta,\gamma}$ does not belong to any class in Theorem k, $k = 1, 2$. This implies that for any such class X there will be $F_k = P_{\beta,\gamma}[F_{k,\beta,\gamma}] \notin P_{\beta,\gamma}[X]$, $k = 1, 2$, as was stated above. We devote the next section to the application of this topic to the construction of new uniqueness conditions.

4. CLASSES $\mathcal{U}_{k,\beta,\gamma}$

In this section we additionally assume that $\gamma > 0$ in order to exclude the vanishing of the function $V_{k,\beta,\gamma}$ on $\mathbb{D} \setminus \{0\}$; in this case the function $F_{k,\beta,\gamma}$ will lie in the domain of the operator $P_{\beta,\gamma}$, i.e. $F_{k,\beta,\gamma}(\zeta)/\zeta \neq 0$, $\zeta \in \mathbb{D}$.

Let X be any class of functions F of the form (5) holomorphic in $\mathbb{D}$ and non-zero on $\mathbb{D} \setminus \{0\}$ such that one of two following conditions is satisfied:

1) for any $F \in X$ the estimate (7) is valid, and it is attainable for $\zeta \neq 0$ only in the case of functions of the form $F(\zeta) = \bar{\varepsilon}F_1(\varepsilon\zeta)$, $|\varepsilon| = 1$, also belonging to X;

2) for any $F \in X$ the relation (6) and the estimate (9) hold, where equality for $\zeta \neq 0$ can be attained only on functions of the form $F(\zeta) = \bar{\varepsilon}F_2(\varepsilon\zeta)$, $|\varepsilon| = 1$, containing in the class X.

More briefly, we define such a class X as a subclass of functions F of the form (5) holomorphic in $\mathbb{D}$ with the estimate

$$|F(\zeta)| \leq F_k(|\zeta|), \quad \zeta \in \mathbb{D}, \tag{15}$$

where the number $k = k(X)$ is 1 or 2, the equality in (15) with $\zeta \neq 0$ is possible only for functions in X of the form $F(\zeta) = \bar{\varepsilon}F_k(\varepsilon\zeta)$ with $|\varepsilon| = 1$, and the condition (6) is additionally imposed when $k = 2$.

Definition 1. Such a class X will be called a (k-)class with sharp S^0-*majorant*.

In Theorem 1, where $k = 1$, the classes with sharp S^0-majorant are S^0, $S^*(1/2)$, $\overline{co}S^0$, in Theorem 2, i.e. for $k = 2$ and under the condition (6), such classes are S^0, $\overline{co}S^0$, S^*, $S^{(2)}$. The functions of these classes are non-vanishing on $\mathbb{D} \setminus \{0\}$, just like those of the classes from Corollaries 1 and 2 below.

Definition 2. Let k be one of the numbers 1 or 2, X be the k-class with sharp S^0-majorant, $\beta, \gamma > 0$ and let $\mathcal{U}_{k,\beta,\gamma}(X)$ be the class of holomorphic functions F in $\mathbb{D}$ of the form (5), each of which admits a representation as the product of two functions $F = UV_{k,\beta,\gamma}$, where $V_{k,\beta,\gamma}$ is the function (14), and U belongs to X. The notation $F \in \mathcal{U}_{k,\beta,\gamma}$ will mean that $F \in \mathcal{U}_{k,\beta,\gamma}(X)$ for some k-class X with sharp S^0-majorant.

The following version of the Schwarz lemma is valid for the operator $P_{\beta,\gamma}$.

Lemma 3. *Let $k = 1$ or $k = 2$, and $\beta, \gamma > 0$. If $F \in \mathcal{U}_{k,\beta,\gamma}$, then $|P_{\beta,\gamma}[F](\zeta)| \leq F_k(|\zeta|)$, $\zeta \in \mathbb{D}$. Equality in this estimate for $\zeta \neq 0$ is possible only in the case $F = F_{k,\beta,\gamma}$, whence $P_{\beta,\gamma}[F] \equiv F_k$.*

Proof of the inequality is quite simple and consists of several steps. Analysis of an equality is a careful step-by-step treatment of the resulting chain of inequalities. Lemma 3 allows us to establish the following statements.

Proposition 1. *Suppose the function (2) is holomorphic in the disc $\mathbb{D}$ and satisfies the subordination (4) with $a = n$, $P = P_{\beta,\gamma}$ and $F \in \mathcal{U}_{1,\beta,\gamma}$, where $\beta, \gamma > 0$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$, and the constant $a = n$ in (4) is sharp for any n.*

Proposition 2. *Suppose the holomorphic functions (2) and (5) in the disc $\mathbb{D}$ are related by the subordination (4), where $a = 2n$ and $P = P_{\beta,\gamma}$ for $\beta, \gamma > 0$. Let the function (5) in (4) satisfies the conditions (6) and $F \in \mathcal{U}_{2,\beta,\gamma}$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$ unless $f(\mathbb{D})$ is a strip which is possible only for $n = 1$. The constant $a = 2n$ in (4) is sharp for any n.*

Proposition 1 allows us to extend the action of Theorem 3 to the level of generality of Theorem 1. We have the following

Corollary 1. *Suppose the holomorphic function (2) in the disc $\mathbb{D}$ satisfies the subordination (4) with $a = n$ and $P = P_{\beta,\gamma}$, where $\beta, \gamma > 0$. If the majorant $F \in \mathcal{U}_{1,\beta,\gamma}(X)$, where X is one of the subclasses S^0, $S^*(1/2)$, $\overline{co}S^0$ or $\mathcal{A}$, then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$. The constant $a = n$ is sharp for the first three of these subclasses.*

In the same way, Proposition 2 modifies Theorem 4:

Corollary 2. *Suppose the holomorphic functions (2) and (5) in the disc $\mathbb{D}$ are related by the subordination (4), where $a = 2n$ and $P = P_{\beta,\gamma}$ for $\beta, \gamma > 0$. Let the function (5) in (4) satisfies the conditions (6) and $F \in \mathcal{U}_{2,\beta,\gamma}(X)$, where X is the class S^0 or one of its extensions, $\overline{co}S^0$, $S^*(1/2)$, S^*, N, or one of the classes $S^{(2)}$, $\mathcal{A}(1/2)$, $\mathcal{A}$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$ unless $f(\mathbb{D})$ is a strip which is possible only for simultaneous fulfillment of the equality $n = 1$ and one of the alternatives $X = S^0$, $\overline{co}S^0$, S^*, or $S^{(2)}$. For the latter four classes the constant $a = 2n$ in (4) is sharp for any n. For $n = 1$ it will be sharp for all of the above eight initial variants for X.*

5. OPERATORS $P^{-1}_{\beta,\gamma}$, J_c AND G_b

Let's go back to our "line of subordinations". Let us pass from the problem of sharpness to the search of new forms of uniqueness conditions. The following is true.

Lemma 4. 1) *Let $U(r)$ be a real-valued, continuous and strictly increasing function on the half-interval $[0, 1)$, and let $\alpha > 0$. Operator $P^{-1}_{\beta,\gamma}$ strengthens the non-strict estimate $|F'(\zeta)| \le \alpha U(|\zeta|)$, $\zeta \in \mathbb{D}$, for the holomorphic function (5) in $\mathbb{D}$ to the strict inequality $|P^{-1}_{\beta,\gamma}[F](\zeta)| < \alpha|\zeta|U(|\zeta|)$, $\zeta \in \mathbb{D} \setminus \{0\}$.*

2) *For any $\alpha \in \mathbb{C}$ the function $F_\alpha(\zeta) = \alpha\zeta$ is the fixed point of the operator $P^{-1}_{\beta,\gamma}$.*

In contrast to the "direct" operator $P_{\beta,\gamma}$, which acts on functions, the inverse one, $P^{-1}_{\beta,\gamma}$, acts both on functions and on their derivatives. Due to the latter fact, the natural domain of definition of the operator $P^{-1}_{\beta,\gamma}$ is the unit pre-ball in the Bloch space [3], i.e. the class of holomorphic functions F in $\mathbb{D}$ with the condition

$$|F'(\zeta)| \le 1/(1 - |\zeta|^2), \quad \zeta \in \mathbb{D}. \tag{16}$$

The next assertion contains the constant $a = 2n$, but not $a = n$, since it is the condition (16), but not (6), as before, that ensures the fulfillment of the (strict in $\mathbb{D} \setminus \{0\}$) estimate (9). Equality (6) is not postulated in this assertion.

Theorem 5. *Suppose the functions (2) and (5) are holomorphic in the disc $\mathbb{D}$ and satisfy the subordination (4) with $a = 2n$ and $P = P^{-1}_{\beta,\gamma}$, where $\beta + \gamma > 0$, $\beta \ge 1$. Assume that the holomorphic majorant F in (4) satisfies the condition (16). Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$. The constant $a = 2n$ in (4) is sharp for $n = 1$.*

In the following statement, which includes condition (6), we return to the classical majorants; their number has clearly decreased.

Theorem 6. *Suppose the functions (2) and (5) are holomorphic in the disc $\mathbb{D}$ and satisfy the subordination (4) with $a = 2n$ and $P = P^{-1}_{\beta,\gamma}$, where $\beta + \gamma > 0$, $\beta \ge 1$. Assume that the holomorphic majorant F in (4) satisfies the condition (6) and belongs to one of the classes S^0, N or $\mathcal{A}(1/2)$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$. The constant $a = 2n$ in (4) is sharp for $n = 1$.*

The statement below corresponds to the case $a = n$; condition (6) provides one of the possibilities here (see also [3], discussion after Corollary 3).

Theorem 7. *Suppose the functions (2) and (5) are holomorphic in the disc $\mathbb{D}$ and satisfy the subordination (4) with $a = n$ and $P = P^{-1}_{\beta,\gamma}$, where $\beta + \gamma > 0$, $\beta \geq 1$. Assume that the function F in (4) belongs to the class of the functions holomorphic in $\mathbb{D}$ with the condition*

$$|F'(\zeta)| \leq 2/(1 - |\zeta|^2), \quad \zeta \in \mathbb{D},$$

or—under the additional restriction (6)—to the class $\mathcal{A}$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$. The constant $a = n$ in (4) is sharp for $n = 1$.

In conclusion, we present two statements concerning operators (11) and (12).

Theorem 8. *Suppose the function (2) is holomorphic in the disc $\mathbb{D}$ and satisfies the subordination (4) with $P = J_c$, where $0 < c \leq 1$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$ under one of two following alternatives: function F in (4) 1) belongs to one of the subclasses in Theorem 3 when $a = n$, or 2) includes into one of the subclasses in Theorem 4 and satisfies the condition (6) when $a = 2n$.*

Theorem 9. *Suppose the functions (2) and (5) are holomorphic in the disc $\mathbb{D}$ and satisfy the subordination (4) with $P = G_b$, where $0 < b \leq 1$. Then the conformal radius (1) has the unique critical point (maximum) $\zeta = 0$ under one of the following conditions: 1) $a = 2n$, and the function F in (4) satisfies the condition (16); 2) $a = n$, and the majorant F in (4) belongs to the class of holomorphic functions satisfying the conditions (6) and*

$$F'(\zeta) \prec \frac{1 + \beta\zeta}{1 - \alpha\zeta}, \quad \zeta \in \mathbb{D},$$

where $|\alpha| < 1$, $|\beta| < 1$.

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