PAPER • OPEN ACCESS

The problem of electromagnetic wave transmission in a circular waveguide

To cite this article: I L Alexandrova and E A Osipov 2022 *J. Phys.: Conf. Ser.* **2388** 012094

The problem of electromagnetic wave transmission in a circular waveguide

I L Alexandrova and E A Osipov

Institute of Computational Mathematics and Information Technologies, Kazan Federal University, 35 Kremlyovskaya street, Kazan 420008, Russia

E-mail: iralexand@mail.ru

Abstract. The article deals with the problem of transmission in a circular waveguide, in the cross section of which there is a thin metal screen. The waveguide is bounded by a perfectly conducting surface. The transmission problem is an auxiliary problem that arose when solving inverse diffraction problems. In the transmission problem, it is required to find the electromagnetic field in the waveguide on one side of the screen by a given field on the other side of the screen: both by given waves incident on the screen and by given waves leaving the screen to infinity. It is shown how the transmission problem can be reduced to an infinite system of linear algebraic equations. To solve the resulting system, it is proposed to use the truncation method. We study under what conditions the numerical solution of the transmission problem will be stable to small perturbations of the input data.

1. Introduction

Currently, a lot of research is being carried out in the field of solving inverse problems of diffraction of electromagnetic waves [1-3]. We will talk about inverse problems of diffraction of electromagnetic waves in circular waveguides bounded by a perfectly conducting surface, in which information about the inhomogeneity is restored from the incident and measured reflected fields. As a rule, when solving inverse problems, to get a more accurate picture, you need to make many measurements with different input data. Thus, in inverse problems of diffraction in a waveguide, it will be necessary to generate various incident electromagnetic waves and measure the reflected field. Instead, to obtain additional information about the obstacle in the waveguide, we propose to use a thin ideally conducting screen located in the cross section of the waveguide between the source of electromagnetic waves and the inhomogeneity under study. Such a screen was called scanning in [4]. An electromagnetic field is incident on the screen, and by moving the screen, a large number of measurements of the reflected field can be made. When using a scanning screen, there is an additional task of recalculating the field on one side of the screen by the field on the other side of the screen. We called such a problem the transmission problem [5].

Having solved the transmission problem, it is possible to recover information about the inhomogeneity by a standard method for solving inverse problems, for example, by the method of minimizing the functional.

In this article, we consider a round waveguide, give a statement of the transmission problem, describe a method for solving it, and present the results of a computational experiment.

Journal of Physics: Conference Series **2388** (2022) 012094 doi:10.1088/1742-6596/2388/1/012094

2. Problem statement

Consider a waveguide with a circular cross section of radius *R* (figure 1) with a generatrix along the *z* axis, bounded by a perfectly conducting surface. We introduce a cylindrical coordinate system, the origin of which is placed at the centre of the cross-sectional circle. Let there be a thin perfectly conducting screen *M* in the section of the waveguide *S* with the plane *z*=0. Let N=S\M. Let us assume that in the left and right halves of the waveguide (to the left and to the right of the section *S*) the permittivity of the media is equal to ε^- and ε^+, respectively, the magnetic permeability is the same everywhere and equals μ.

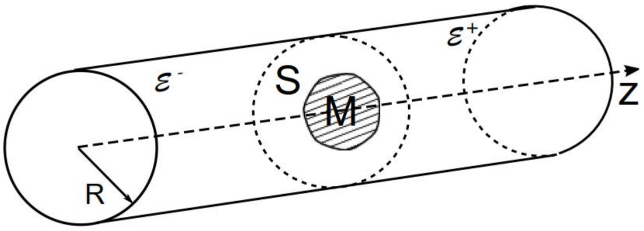

Figure 1. Circular waveguide.

Let us mark the waves that transfer energy or decay in the direction of the waveguide axis with the → symbol. We call such waves of positive orientation. Let us mark the waves that carry energy or decay in the opposite direction with the symbol ←. We will call them waves of negative orientation. The total field in a waveguide is a superposition of these two types of waves.

Let us consider an electromagnetic field harmonically dependent on time. The time dependence has the form $\exp(i\omega t)$. Denote by *Q* the side surface of the waveguide, by *v* the outer normal to it, by *n* the normal to the cross section.

Let the electromagnetic field be given to the right of the screen at *z*>0: a positively oriented wave $(\overrightarrow{E}^+, \overrightarrow{H}^+)$ and a negatively oriented wave $(\overleftarrow{E}^+, \overleftarrow{H}^+)$, and the following condition is satisfied:

$$\left[n, \overleftarrow{E}^+\right] + \left[n, \overrightarrow{E}^+\right] = 0, \quad (x, y) \in M.$$

In the transmission problem, it is required to find the waves $(\overrightarrow{E}^-, \overrightarrow{H}^-)$ and $(\overleftarrow{E}^-, \overleftarrow{H}^-)$, satisfying the system of Maxwell equations for *z*<0

$$rotH = i\omega\varepsilon_0\varepsilon E, \quad rotE = -i\omega\mu_0\mu H, \tag{1}$$

boundary conditions

$$[v, E]|_Q = 0, \tag{2}$$

boundary conditions and conjugation conditions on the section *S*

$$[n, \overleftarrow{E}^-] + [n, \overrightarrow{E}^-] = 0, \qquad (x, y) \in M, \tag{3}$$

$$[n, \overleftarrow{E}^-] + [n, \overrightarrow{E}^-] = [n, \overleftarrow{E}^+] + [n, \overrightarrow{E}^+], \quad (x, y) \in N, \tag{4}$$

$$[n, \overleftarrow{H}^-] + [n, \overrightarrow{H}^-] = [n, \overleftarrow{H}^+] + [n, \overrightarrow{H}^+], \quad (x, y) \in N. \tag{5}$$

3. Solution of the transmission problem

3.1. Electromagnetic field in a circular waveguide

It is known [6] that the solution to problem (1),(2) can be represented as the following series in terms of TE and TM eigenwaves

$$\begin{pmatrix} E \\ H \end{pmatrix} = \sum_{p,q} \overleftarrow{A}_{pq} \begin{pmatrix} \overleftarrow{E}^{\varphi}_{pq} \\ \overleftarrow{H}^{\varphi}_{pq} \end{pmatrix} + \sum_{p,q} \overleftarrow{B}_{pq} \begin{pmatrix} \overleftarrow{E}^{\psi}_{pq} \\ \overleftarrow{H}^{\psi}_{pq} \end{pmatrix} + \sum_{p,q} \overrightarrow{A}_{pq} \begin{pmatrix} \overrightarrow{E}^{\varphi}_{pq} \\ \overrightarrow{H}^{\varphi}_{pq} \end{pmatrix} + \sum_{p,q} \overrightarrow{B}_{pq} \begin{pmatrix} \overrightarrow{E}^{\psi}_{pq} \\ \overrightarrow{H}^{\psi}_{pq} \end{pmatrix} \quad (6)$$

Let us write down the components of the electric and magnetic field components for positively oriented TE waves

$$\overrightarrow{E}^{\varphi}_{pq}(r,\alpha,z) = i\omega\mu_0\mu\left(-\frac{1}{r}\frac{\partial\varphi_{pq}}{\partial\alpha}, \frac{\partial\varphi_{pq}}{\partial r}, 0\right)e^{-i\gamma_{pq}z}, \quad (7)$$

$$\overrightarrow{H}^{\varphi}_{pq}(r,\alpha,z) = \left(-i\gamma_{pq}\frac{\partial\varphi_{pq}}{\partial r}, -i\gamma_{pq}\frac{1}{r}\frac{\partial\varphi_{pq}}{\partial\alpha}, \frac{K_{pq}^2}{R^2}\varphi_{pq}\right)e^{-i\gamma_{pq}z} \quad (8)$$

and TM waves

$$\overrightarrow{E}^{\psi}_{pq}(r,\alpha,z) = \left(-i\delta_{pq}\frac{\partial\psi_{pq}}{\partial r}, -i\delta_{pq}\frac{1}{r}\frac{\partial\psi_{pq}}{\partial\alpha}, \frac{L_{pq}^2}{R^2}\psi_{pq}\right)e^{-i\delta_{pq}z}, \quad (9)$$

$$\overrightarrow{H}^{\psi}_{pq}(r,\alpha,z) = i\omega\varepsilon_0\varepsilon\left(\frac{1}{r}\frac{\partial\psi_{pq}}{\partial\alpha}, -\frac{\partial\psi_{pq}}{\partial r}, 0\right)e^{-i\delta_{pq}z}. \quad (10)$$

In the formulas presented above, the eigenfunctions φ_{pq}, ψ_{pq} are calculated by the formulas:

$$\varphi_{pq}(r,\alpha) = T^{\varphi}_{pq}J_p\left(\frac{K_{pq}}{R}r\right)e^{ip\alpha}. \quad \psi_{pq}(r,\alpha) = T^{\psi}_{pq}J_p\left(\frac{L_{pq}}{R}r\right)e^{ip\alpha},$$

where $J_p(x)$ are the pth-order Bessel functions, K_{pq} are the roots of the derivative of the Bessel function $J'_p(x)$, numbered in ascending order, L_{pq} are the roots of the Bessel function $J_p(x)$, T^{φ}_{pq}, numbered in ascending order, and T^{ψ}_{pq} are normalizing factors. Here $p = 0, \pm1, \pm2, \dots$, $q = 0,1,2,$ ...and

$$\left(T^{\varphi}_{pq}\right)^2 = \frac{1}{\pi R^2}\frac{1}{\left(1-\frac{p^2}{K_{pq}^2}\right)J_p^2(K_{pq})}, \qquad \left(T^{\psi}_{pq}\right)^2 = \frac{1}{\pi R^2}\frac{1}{\left(J'_p(L_{pq})\right)^2}.$$

The longitudinal propagation constants γ_{pq} and δ_{pq} are found by the formulas:

$$\gamma_{pq} = \sqrt{k^2 - \left(\frac{K_{pq}}{R}\right)^2}, \qquad \delta_{pq} = \sqrt{k^2 - \left(\frac{L_{pq}}{R}\right)^2}.$$

The sign of the quantities γ_{pq} and δ_{pq} is chosen as follows: $Re\,\gamma_{pq} > 0$ or $Im\,\gamma_{pq} < 0$, $Re\,\delta_{pq}$ >0 or Im $Im\,\delta_{pq} < 0$. Then for waves of negative orientation in formulas (6)-(9) before the values γ_{pq} and δ_{pq} it is necessary to change the sign.

3.2. Solution of the transmission problem in a circular waveguide

In the transmission problem, to the right of the screen for $z > 0$, the total field is given: a positively oriented wave ($\overrightarrow{E}^{+}, \overrightarrow{H}^{+}$) and a negatively oriented wave ($\overleftarrow{E}^{+}, \overleftarrow{H}^{+}$). The task of transmission is divided into two subtasks. In the first subtask, for negatively oriented waves incident on the screen, it is necessary to restore the field at $z < 0$, that is, on the other side of the screen. In the second subtask, for positively oriented waves leaving the screen, it is necessary to restore the field at $z < 0$. The first subproblem is the classical problem of diffraction by a perfectly conducting screen in a circular waveguide, the solution of which is well studied. The solution of the transmission problem is the sum of the solutions of the first and second subtasks, so we will consider a simplified transmission problem - the second subtask, in which the field ($\overrightarrow{E}^{+}, \overrightarrow{H}^{+}$) given at $z > 0$must be restored at $z < 0$. In other words, it is necessary to find the functions $(\overrightarrow{E}^{-}, \overrightarrow{H}^{-})$ and $(\overleftarrow{E}^{-}, \overleftarrow{H}^{-})$ as a solution to equation (1) with boundary conditions and conjugation conditions (2)-(5).

A detailed solution of the transmission problem in the case of a cylindrical waveguide is described in [7]. For a circular waveguide, similar proofs are carried out only in a cylindrical coordinate system. We present here only the main points.

To solve the transmission problem, it is convenient to first solve the diffraction problem, and then use the resulting solution in the transmission problem. In our case, for this it is necessary to solve system (1)-(5), in which the field $(\overrightarrow{E}^{-}, \overrightarrow{H}^{-})$, is given, it is required to find the field $(\overrightarrow{E}^{+}, \overrightarrow{H}^{+})$ and field $(\overleftarrow{E}^{-}, \overleftarrow{H}^{-})$. The solution of problem (1)-(5) will be sought in the form of series (6). We will consider only TE waves.

Lemma. The problem of diffraction in a circular waveguide is reduced in an infinite system of linear algebraic equations (ISLAE) with respect to the coefficients $\overrightarrow{A}^{+}_{pq}$ of the form

$$-\overrightarrow{A}^{+}_{pq}\left(\frac{K_{pq}}{R}\right)^2 + \sum_{k,l}\overrightarrow{A}^{+}_{kl}(\gamma^{+}_{kl}+\gamma^{-}_{kl})\left(\frac{K_{kl}}{R}\right)^2\sum_{s,v}\frac{1}{\gamma^{+}_{sv}+\gamma^{-}_{sv}}I^{\varphi}_{kl,sv}J^{\varphi}_{sv,pq} = \sum_{k,l}\overrightarrow{A}^{-}_{kl}\frac{2\gamma^{-}_{kl}}{\gamma^{+}_{kl}+\gamma^{-}_{kl}}\left(\frac{K_{kl}}{R}\right)^2 I^{\varphi}_{kl,pq},$$
$$p = 0, \pm 1, \pm 2, \dots,\ q = 0,1,2,\dots, \tag{11}$$

or to the ISLAE with respect to the coefficients $\overleftarrow{A}^{-}_{pq}$ of the form

$$-\overleftarrow{A}^{-}_{pq}\left(\frac{K_{pq}}{R}\right)^2 + \sum_{k,l}\overleftarrow{A}^{-}_{k,l}(\gamma^{+}_{kl}+\gamma^{-}_{kl})\left(\frac{K_{kl}}{R}\right)^2\sum_{s,v}\frac{1}{\gamma^{+}_{sv}+\gamma^{-}_{sv}}I^{\varphi}_{kl,sv}J^{\varphi}_{sv,pq} = \sum_{k,l}\overrightarrow{A}^{-}_{kl}\frac{2\gamma^{-}_{kl}}{\gamma^{+}_{kl}+\gamma^{-}_{kl}}\left(\frac{K_{kl}}{R}\right)^2 I^{\varphi}_{kl,pq},$$
$$p = 0, \pm 1, \pm 2, \dots,\ q = 0,1,2,\dots, \tag{12}$$

where

$$I^{\varphi}_{pq,kl} = \iint_M \xi T^{\varphi}_{pq}J_p\left(\frac{K_{pq}}{R}\xi\right)T^{\varphi}_{kl}J_k\left(\frac{K_{kl}}{R}\xi\right)e^{i(p-k)\theta}\,d\xi d\theta,$$
$$J^{\varphi}_{pq,kl} = \iint_M \xi T^{\varphi}_{pq}J_p\left(\frac{K_{pq}}{R}\xi\right)T^{\varphi}_{kl}J_k\left(\frac{K_{kl}}{R}\xi\right)e^{i(p-k)\theta}\,d\xi d\theta.$$

Proof. From the boundary conditions and conjugation conditions, by solving the overdetermined boundary value problem [8], a paired summation functional equation (PSFE) is derived, which are converted to ISLAE by the method of integral-summator identities.

Theorem. The problem of transmission in a circular waveguide is reduced to ISLAE with respect to the coefficients $\overleftarrow{A}^{-}_{pq}$.

$$-\overleftarrow{A}^{-}_{pq}\left(\frac{K_{pq}}{R}\right)^2 + \sum_{k,l}\overleftarrow{A}^{-}_{k,l}(\gamma^{+}_{kl}+\gamma^{-}_{kl})\left(\frac{K_{kl}}{R}\right)^2\sum_{s,v}\frac{1}{\gamma^{+}_{sv}+\gamma^{-}_{sv}}I^{\varphi}_{kl,sv}J^{\varphi}_{sv,pq} + \sum_{k,l}\overleftarrow{A}^{-}_{k,l}\left(\frac{K_{kl}}{R}\right)^2 I^{\varphi}_{kl,pq}$$
$$= \sum_{k,l}\overrightarrow{A}^{+}_{kl}\left(\frac{K_{kl}}{R}\right)^2 I^{\varphi}_{kl,pq}, \qquad p = 0, \pm 1, \pm 2, \dots,\ q = 0,1,2,\dots \tag{13}$$

Proof. ISLAE (13) is derived from ISLAE (12), in which the replacement is used on the right side

$$\frac{2\gamma^{-}_{kl}}{\gamma^{+}_{kl}+\gamma^{-}_{kl}}\overrightarrow{A}^{-}_{kl} = \overrightarrow{A}^{+}_{kl} - \overleftarrow{A}^{-}_{k,l}.$$

4. Computational experiment

Let's simplify the problem: we will assume that the field does not depend on the coordinate α; the scanning screen M is a thin metal round disk $M = \{r: 0 \le r < R_1\}$.

An approximate solution of ISLAE (13) is sought by the truncation method. We pass from infinite sums to finite ones, and we obtain a finite-dimensional SLAE. We leave the number of unknown coefficients N_l, respectively, in the external sums (by k, l) we leave N_l terms, and in the internal sums (by s, v) we leave a certain number of M_l terms. The computational experiment showed that for the stability of the solution, the condition that $M_1 \geq N_1$ must be satisfied. When $M_1 < N_1$, the solution is not stable with respect to changing M_l.

If the result of solving the diffraction problem is used as the initial data in the transmission problem (coefficients $\overrightarrow{A}^{+}_{kl}$then the field ($\overrightarrow{E}^{-}$, $\overrightarrow{H}^{-}$) and ($\overleftarrow{E}^{-}$, $\overleftarrow{H}^{-}$)is restored with sufficient accuracy.

For small screen sizes (if $R_1 \leq R/4$), the solution of the transmission problem is stable to small perturbations of the right side $\overrightarrow{A}^{+,\delta}_{kl}$ satisfying the condition $\left\| \overrightarrow{A}^{+,\delta} - \overrightarrow{A}^{+} \right\| \leq \delta$. In this case, the wavelength must be less than the radius of the waveguide $\lambda^-, \lambda^+ < R$ otherwise the energy flux through the section z=0 is close to zero.

If $R/4 < R_1 \leq R/3$, then the solution of the transmission problem will be stable to small perturbations of the right side only when $\lambda^-, \lambda^+ < R_1$. These conclusions were made on the basis of the counting results, some of which are shown in table 1.

Table 1. Stability of the solution to the transmission problem for various parameters $R, R_1, \lambda^-, \lambda^+$..

R	R_1	λ^-	λ^+	Resilience to change $\overrightarrow{A}^{+}$
1	0.3	1	1	No
1	0.3	0.75	0.75	No
1	0.3	0.5	0.5	No
1	0.3	0.3	0.3	No
1	0.3	0.25	0.25	Yes
1	0.3	0.2	0.2	Yes
1	0.3	0.1	0.1	Yes
1	0.5	from 0.1 to 2	from 0.1 to 2	No
2	0.9	0.5	0.5	No
2	0.8	1.2	1.2	No
2	0.5	0.5	0.5	Yes
2	0.5	1	0.5	Yes
2	0.5	1.5	1	Yes
3	0.8	1.8	1.5	Yes
3	0.9	1	1	No
3	1	1	1	No

In cases where the solution is unstable to small perturbations of the right-hand side $\overrightarrow{A}^{+,\delta}_{kl}$, Tikhonov's regularization scheme was applied. The regularized solution turned out to be stable to the change $\overrightarrow{A}^{+,\delta}_{kl}$ .However, the minimum relative error that was achieved by choosing the regularization parameter was 26%.

Thus, the transmission problem has a stable solution if $\lambda^-, \lambda^+ < R$ and $R_1 \leq R/4$ or $\lambda^-, \lambda^+ < R_1$ and $R/4 < R_1 \leq R/3$.

5. Conclusion

To solve the inverse problem of electromagnetic wave diffraction, it was proposed to use an additional scanning screen, which allows obtaining more information about the inhomogeneity in the waveguide. Along the way, an additional task of transmission arose. The solution of the problem of transmission in a circular waveguide is given, the results of calculations are presented, and at what ratios of wavelengths

Journal of Physics: Conference Series **2388** (2022) 012094 doi:10.1088/1742-6596/2388/1/012094

and sizes of the scanning screen the numerical solution is investigated for which the numerical solution will be stable for small perturbations of the input data.

Acknowledgement
This paper has been supported by the Kazan Federal University Strategic Academic Leadership Program ("PRIORITY-2030").

References

[1] Colton D and Kress R 2013 *Inverse acoustic and electromagnetic scattering theory* (New-York: Springer).
[2] Isakov V 2005 *Inverse problems for partial differential equations* (Berlin: Springer).
[3] Smirnov Yu G, Shestopalov Yu V and Derevyanchuk E D 2013 Permittivity reconstruction of layered dielectrics in a rectangular waveguide from the transmission coefficients at different frequencies *Inverse Problems and Large-Scale (Springer Proc. in Mathematics & Statistics)* **52** 169-81
[4] Pleshchinskii N B 2003 On reconstruction of the reflection matrix in the plane waveguide by the scanning screen method *Research in Applied Mathematics and Computer Science* **24** 115-21
[5] Aleksandrova I L and Pleshchinskii N B 2013 Thin conducting screen in a waveguide structure: the problem of diffraction and the problem of transmission *Physics and mathematics sciences. Mathematics University proc. Volga region* **1(25)** 45-60
[6] Ilyinskii A S, Kravtsov V V and Sveshnikov A G 1991 *Mathematical models of electrodynamics* (Moscow: High School)
[7] Alexandrova I L 2015 Electromagnetic wave transmission issue in cylindrical waveguide *International Journal of Applied Engineering Research* **10(12)** 31349-62
[8] Pleshchinskaya I E and N B Pleshchinskii 1999 The Cauchy problem and potentials for elliptic partial differential equations and some of their applications *Advances in Equations and Inequalities* 127-46