ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 8, pp. 2168–2175. © Pleiades Publishing, Ltd., 2022.

On the Inner Radius for Multiply Connected Domains

A. V. Kazantsev[1*] and M. I. Kinder[2]**

(Submitted by A. M. Elizarov)

[1]Institute of Computational Mathematics and Information Technologies, Kazan Federal University, Kazan, 420008 Russia

[2]Lobachevskii Institute of Mathematics and Mechanics, Kazan Federal University, Kazan, 420008 Russia

Received April 24, 2022; revised May 15, 2022; accepted May 25, 2022

Abstract—We study the properties of the inner radius of multiply connected domains. This conformally invariant quantity coincides with the conformal radius in the simply connected case. Using the vector field method, we prove that the number of critical points of the inner radius is not less than the connectivity order of the domain. The classification of critical points according to their indices is given. We also prove that these points can only be maxima, saddles or semi-saddles.

DOI: 10.1134/S1995080222110154

Keywords and phrases: *conformal map, multiply connected domain, canonical domain, inner radius of domains, generalized reduced module, Mityuk's radius.*

1. INTRODUCTION

Let D be a bounded n-connected domain in $\mathbb{C}$ with the boundary ∂D consisting of n Jordan curves $L_1, L_2, \dots, L_n$. We recall the definition of the inner radius of a multiply connected domain (see, for example, [1, 2], [3, p. 123]). Let $G(w, w_0)$ be Green's function of the domain D. The function $g(w, w_0) = G(w, w_0) + \ln|w - w_0|$ is harmonic at $w = w_0$, so there exists a limit

$$g(w_0, w_0) = \lim_{w \to w_0} \big[G(w, w_0) + \ln|\, w - w_0\,|\big].$$

The quantity $g(w_0, w_0)$ is also called the Robin constant of the domain D relative to the point w_0 ([4], p. 96; see also [5], pp. 26–27).

Definition 1. The function $R(w) = e^{g(w,w)}$ is called the *inner radius* of the region D at a point w.

If D is a simply connected domain whose boundary contains at least two points, then the inner radius of the domain D at the point $w_0 \in D$ coincides with the conformal radius of the domain. The properties of the function $R(w)$ for a simply connected domain D were studied in the paper of Haegi [6], see also [7]. In particular, it is shown in [6] that for convex domain the function $R(w)$ has exactly one critical point, which is the maximum of $R(w)$, except for the strip.This result gave rise to an important direction in the study of the inner radii $R(w)$ —to obtain the conditions for the set $M_R = \{w \in D : R_{\bar{w}}(w) = 0\}$ to be a singleton; see, for example, [8]. Elements of the set M_R are called critical, or stationary points of the function $R(w)$ (surface $R = R(w)$).

The inner radius does not decrease as the domain expands. For an arbitrary domain D its inner radius at the point $w_0 \in D$ is defined as the exact upper bound of the set of inner radii at the point w_0 of all domains contained in D and having the Green function.

The inner radius plays an essential role in the inverse boundary value problems. In 1952 F. D. Gakhov [9] obtained the condition for the solvability of the exterior inverse problem in the form of an equation for

*E-mail: avkazantsev63@gmail.com

**E-mail: detkinm@gmail.com

determining a stationary point of some real surface. As noted by L.A. Aksent'ev [7], this surface, in fact, coincides with the surface of the inner radius. The equation for finding the critical points of the function $R(w)$ is called the Gakhov equation. In the works of M.I. Kinder [10, 11] a classification of isolated critical points of the inner radius of a domain of finite connectivity was introduced, and, in particular, it was proved that the surface $R = R(w)$ in a neighborhood of such a point has a convex, saddle-shaped, or semi-saddle structure.

In this article we study the question about the number of stationary points of the inner radius surface for the case of an arbitrary finitely connected region. The research method is to study the rotation of a vector field associated with a surface. The use of such a method (see, for example, [12]) allows us not only to prove the existence of critical points of the function $R(w)$, but also to estimate their number from below.

The main result of the article is

Theorem 1. *The surface of the inner radius $R = R(w)$ in the case of an n-connected bounded domain D has at least n stationary points.*

2. KEY FACTS

We devote this section to the proof of Theorem 1.

First of all, let us establish an assertion on the boundary behavior of the function $R(w)$, which is needed to prove the existence of stationary points of the surface $R = R(w)$.

Lemma 1. *The function $R(w)$ is infinitely differentiable in the domain D and continuous in the closed domain $\overline{D}$, moreover* $\lim_{w\to t} R(w) = 0$, $t \in \partial D$.

Proof. We use the inequality

$$g(w_0, w_0) \leq 2g(w, w_0) - g(w, w), \quad w, w_0 \in D, \tag{1}$$

established by M. Schiffer [13, § 3] for plane multiply connected domains bounded by smooth, in particular, analytic curves. Any multiply connected domain bounded by closed Jordan curves can always be approximated by domains with analytic boundaries. Taking into account the continuity of the Green's function (and hence the function $g(w, w_0)$) as a function depending on the approximating domain, we conclude that the inequality (1) is valid for any multiply connected domain with Jordan boundary (see also [4, p. 96]). Further, inequality (1) yields the relation $\overline{\lim}_{w\to t} g(w, w) = -\infty$, meaning that the function $g(w, w)$ tends to $-\infty$, when the point w approaches to the boundary point $t \in \partial D$ along any path Γ_t. Indeed, assuming the opposite

$$\overline{\lim_{w\to t}}\, g(w, w) \geq -K, \tag{2}$$

and passing to the limit on the right-hand side (1) as $w \to t$ $(w \in \Gamma_t)$, we obtain the inequality $g(w_0, w_0) \leq K + 2g(t, w_0)$, $t \in \partial D$, that is

$$g(w_0, w_0) \leq K + 2\ln|\,t - w_0\,|, \tag{3}$$

valid for $w_0 \in D$. Remaining ourselves in the region D, we will approach the point w_0 to t along the path with "finishing" part lying on Γ_t. Then the value of $g(w_0, w_0)$, as we see from the right-hand side of (3), can be made less than any preassigned negative number. The latter contradicts (2), and the statement that $\lim_{w\to t} g(w, w) = -\infty$ is proved.

It follows that $R(w)$ vanishes on the boundary of the domain D, which implies the continuity of the function $R(w)$ in $\overline{D}$. Lemma 1 is proved. □

By the Weierstrass theorem $R(w)$ will attain its maximum inside the domain D. Geometrically, this means that the surface with the equation $R = R(w)$, located over the region D and attached to the boundary of this region, has at least one maximum point (vertex).

As noted above, the stationary points of the surface $R = R(u, v)$, $w = u + iv$, are found from the equations

$$\frac{\partial R}{\partial \overline{w}} = 0 \quad \Longleftrightarrow \quad \frac{\partial R}{\partial u} = \frac{\partial R}{\partial v} = 0. \tag{4}$$

In the domain D we introduce a plane vector field of gradients

$$\operatorname{grad} R(w) = \left(\frac{\partial R}{\partial u}, \frac{\partial R}{\partial v}\right), \tag{5}$$

the singular points of which are the roots of the equation (4).

Suppose now that the equation (4) has *finitely* many solutions. This requirement means that all singular points of the vector field (5) are isolated (if this is not the case, then the function $R = R(w)$ has an infinite number of stationary points and the main statement of the article is true).

The following statement describes the structure of the level lines of the function $R(w)$.

Lemma 2. *The level lines $R(w) = \varepsilon$ for sufficiently small $\varepsilon > 0$ define the domain D_ε with the boundary of n closed smooth Jordan curves approximating the boundary of the domain D.*

Proof. Let's choose the number $\delta > 0$ less than the distance of all zeros of the function $\partial R/\partial w$ to the boundary ∂D, and also less than half the distance between different boundary curves included in ∂D. On the set D_δ, consisting of all points in D whose distance from the boundary ∂D is at least δ, the function $R(w)$ has the minimum $\varepsilon_0 > 0$. Let Γ_ε be the set of points in D defined by the equation $R(u, v) = \varepsilon$, $0 < \varepsilon < \varepsilon_0$. It consists of a finite number of non-intersecting smooth curves that cannot end on the boundary contours $D \setminus D_\delta$ (where $R(w) = 0$ or $R(w) \geq \varepsilon_0$). Every such curve is closed and does not have self-intersections. All these statements follow from the theorem on the existence of an implicit function applied to the equation $R(u, v) - \varepsilon = 0$, and from the condition that on the set $D \setminus D_\delta$ at least one of the derivatives $\partial R/\partial u$ or $\partial R/\partial v$ is nonzero.

Further, the set $\Gamma_{k\varepsilon} \subset \Gamma_\varepsilon$ of points that lie at a distance at most δ from the boundary curve L_k consists of only one closed smooth Jordan curve. Indeed, if $\Gamma_{k\varepsilon}$ would consist of several curves, then in the part $D \setminus D_\delta$ adjacent to L_k there would be a region Δ such that $R(u, v)$ is a continuously differentiable function on Δ and $R(u, v) = \varepsilon$ on $\partial\Delta$. But from this, by the Weierstrass theorem, it would follow that the function $R(u, v)$ reaches its maximum (or minimum) inside the region Δ, and hence, at some point of the region $D \setminus D_\delta$ holds (4), which contradicts the choice of δ.

Based on the obtained conclusion about the structure of Γ_ε, we get that the level lines $R(u, v) = \varepsilon$ define the domain $D_\varepsilon \subset D$ with the boundary consisting of n closed smooth Jordan curves $\Gamma_{1\varepsilon}, \ldots, \Gamma_{n\varepsilon}$ and approximating the boundary of the domain D. □

Let us define the rotation of the vector field grad R along any curve Γ lying in the domain D:

$$\gamma(\Gamma) = \frac{1}{2\pi} \operatorname{var} \arg \frac{\partial R}{\partial \overline{w}} = -\frac{1}{2\pi i} \int\limits_{\Gamma} d \ln \frac{\partial R}{\partial w}.$$

Geometrically, the value of $\gamma(\Gamma)$ is equal to the winding number of the field vector under the single circuit of the curve Γ. Let us take as Γ the system $\Gamma_\varepsilon = \partial D_\varepsilon$ of closed curves $\Gamma_{1\varepsilon}, \ldots, \Gamma_{n\varepsilon}$, each of which is the ε-level line of the function $R(u, v)$. Let's calculate the rotation of the gradient field on $\Gamma_{n\varepsilon}$. To do this, note that on the level line we have the equality

$$\frac{\partial R}{\partial u} du + \frac{\partial R}{\partial v} dv = 0.$$

Hence, the tangent vector (du, dv) is orthogonal to the vector-gradient (5), that is, $\overrightarrow{\operatorname{grad}} R$ is directed along the normal to level line. The rotation of the normal field on a smooth simple closed curve Γ is 1 if Γ is traversed counterclockwise, and is equal to -1 if the traversal is clockwise. Therefore, the rotation of the gradient field along the boundary of the region D_ε is equal to $1 - (n - 1) = 2 - n$.

Let us denote the critical points of the inner radius $R(w)$ by $w_0, w_1, \ldots, w_m$. For an isolated singular point w_i of the vector field (4), we define the index by the formula

$$\gamma(w_i) = -\frac{1}{2\pi i} \int\limits_{l_i} d \ln \frac{\partial R}{\partial w}.$$

The quantity $\gamma(w_i)$ does not depend on the closed curve l_i with only one singularity w_i in its interior. By the main theorem in the theory of plane vector fields [12, p. 28] the algebraic number of (the indices of) the singular points $\sum_{i=0}^{m}\gamma(w_i)$ is equal to the rotation of the field on $\Gamma_\varepsilon = \partial D_\varepsilon$, that is

$$\sum_{i=0}^{m}\gamma(w_i) = 2 - n. \tag{6}$$

One of the terms on the left side of (6) is known: the index of an isolated maximum point, say w_0, of the function $R(w)$ is equal to 1 [14, p. 210]. Therefore, the equality (6) can be rewritten as

$$\sum_{i=1}^{m}\gamma(w_i) = 1 - n. \tag{7}$$

The index of each of the remaining points w_i $(1 \le i \le m)$ depends on the order of the zero w_i of the function $\partial R(w)/\partial\overline{w}$. To clarify this question, consider the behavior of the second differential form of the function $R(u, v)$ in a neighborhood of singular points:

$$R_{uu}du^2 + 2R_{uv}du\,dv + R_{vv}dv^2. \tag{8}$$

Let us prove that at singular points of the vector field (5) the differential form (8) is nondegenerate, that is, at least one of the elements R_{uu}, R_{uv}, R_{vv} is different from zero.

Lemma 3. *The function* $\ln R(w) = g(w, w)$ *is strictly superharmonic in the domain D, that is,*

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} < 0 \quad \Longleftrightarrow \quad \frac{\partial^2 g(w, w)}{\partial w\,\partial\overline{w}} < 0. \tag{9}$$

Proof. We transform the expression on the right-hand side of (9), namely, we prove the equality

$$\frac{\partial^2 g(w, w)}{\partial w\,\partial\overline{w}} = 2\left[\frac{\partial^2 g(w, z)}{\partial w\,\partial\overline{z}}\right]_{z=w}. \tag{10}$$

To do this, we introduce new notation: instead of $g(w, z)$ we will write $g(w, \overline{w}; z, \overline{z})$, thus emphasizing the dependence of the function g on the variables $\overline{w}$ and $\overline{z}$. It follows from the symmetry, $G(w, z) = G(z, w)$, of the Green function that the function $g(w, z)$ has the same symmetry property; in the new notation this property is written as

$$g(w, \overline{w}; z, \overline{z}) = g(z, \overline{z}; w, \overline{w}). \tag{11}$$

Differentiating (11) with respect to w, we obtain

$$g'_1(w, \overline{w}; z, \overline{z}) = g'_3(z, \overline{z}; w, \overline{w}), \tag{12}$$

where the subscript (1,2,3 or 4) indicates by what argument the function $g(w, \overline{w}; z, \overline{z})$ is differentiated. The right-hand side of the equality (10) in the new notation coincides with $2g''_{14}(w, \overline{w}; w, \overline{w})$. Let's treat the left-hand side of (10). By virtue of (12) we have

$$\partial g(w, w)/\partial w = g'_1(w, \overline{w}; w, \overline{w}) + g'_3(w, \overline{w}; w, \overline{w}) = 2g'_1(w, \overline{w}; w, \overline{w}),$$

and therefore

$$\partial^2 g(w, w)/\partial w\,\partial\overline{w} = 2g''_{12}(w, \overline{w}; w, \overline{w}) + 2g''_{14}(w, \overline{w}; w, \overline{w}).$$

But $g''_{12}(w, \overline{w}; w, \overline{w}) \equiv 0$ because the function $g(w, \overline{w}; z, \overline{z})$ is harmonic in w in the domain D. This yields the required equality (10).

In 1946 M. Schiffer [15] (see also [13, p. 252]) showed that the function

$$K(w, \overline{z}) = -\frac{2}{\pi}\,\frac{\partial^2 G(w, w_0)}{\partial w\,\partial\overline{w}_0} = -\frac{2}{\pi}\,\frac{\partial^2 g(w, z)}{\partial w\,\partial\overline{z}}, \tag{13}$$

which plays an important role in the theory of conformal mappings, is closely related to complete orthonormal systems of functions regular in the domain D. Let us introduce the necessary definitions (see, for example, [13]).

We denote by $L^2(D)$ the family of all functions $f(w)$, analytic in the domain D and satisfying the condition

$$\iint_D |f(w)|^2\,du\,dv < \infty, \quad w = u + iv.$$

We say that a system of functions $\{\varphi_i(w)\}_1^\infty$ from the class $L^2(D)$ is orthonormal with respect to the domain D if

$$\iint_D \varphi_i(w)\,\overline{\varphi_k(w)}\,du\,dv = \delta_{ik}.$$

In the Hilbert space $L^2(D)$ the function $K(w,\overline{z})$ has the so-called *reproducing property*: for any function $f(w)$ in $L^2(D)$ the following integral identity holds

$$f(w) \equiv \iint_D K(w,\overline{z})\,f(z)\,dx\,dy, \quad z = x + iy.$$

The kernel $K(w,\overline{z})$ as an element of $L^2(D)$ can be expanded in a Fourier series in the orthonormal system $\{\varphi_i(w)\}_1^\infty \subset L^2(D)$, and the reproducing property implies the expansion [13, p. 274]

$$K(w,\overline{z}) = \sum_{i=1}^{\infty} \varphi_i(w)\,\overline{\varphi_i(z)}. \tag{14}$$

The infinite series in (14) converges absolutely and uniformly in each closed subdomain D. Using the equalities (13), (14), we can rewrite the formula (10) as

$$\frac{\partial^2 g(w,w)}{\partial w\,\partial\overline{w}} = -\pi K(w,\overline{w}) = -\pi \sum_{i=1}^{\infty} |\,\varphi_i(w)\,|^2.$$

This implies the inequality

$$\frac{\partial^2 g(w,w)}{\partial w\,\partial\overline{w}} \le 0. \tag{15}$$

Equality in (15) would lead to the existence of a point z where all functions $\varphi_i(w)$ of the orthonormal system vanish. Function $f(w) \equiv 1$ belongs to the space $L^2(D)$ due to the boundedness of the domain D, so we can expand it in a series in the system $\{\varphi_i(w)\}_1^\infty$:

$$1 = \sum_{i=1}^{\infty} c_i \varphi_i(w). \tag{16}$$

Substituting the point $w = z \in D$ into (16), we obtain a contradiction. This proves the inequality (9), and completes the proof of the Lemma 3. □

Lemma 4. *At the singular points w_i of the vector field* (5) *at least one of the numbers R_{uu}, R_{vv} is nonzero.*

Proof. From the definition of the function $R(w)$ we have

$$\frac{\partial^2 g(w)}{\partial w\,\partial\overline{w}} = \frac{1}{R(w)}\,\frac{\partial^2 R(w)}{\partial w\,\partial\overline{w}} - \frac{1}{R^2(w)}\left|\frac{\partial R(w)}{\partial w}\right|^2.$$

Since at the singular points of the field we have $\partial R(w)/\partial w = 0$, then by Lemma 3

$$\frac{\partial^2 g(w_i)}{\partial w\,\partial\overline{w}} = \frac{1}{R(w_i)}\,\frac{\partial^2 R(w_i)}{\partial w\,\partial\overline{w}} < 0,$$

that is, $R_{uu}(w_i) + R_{vv}(w_i) < 0$. This implies the assertion of Lemma 4. □

We continue the proof of the Theorem 1. By virtue of Lemma 4 at the singular point w_i one of the numbers R_{uu}, R_{vv} is nonzero, and therefore the second differential form (8) of the function $R(u,v)$ is nondegenerate. As shown in [12, pp. 69–70], in this situation the index $\gamma(w_i)$ can take only three values: -1, 0 or $+1$. Since the indices of the singular points in the equality (7) are at least -1, we conclude that the number of such points is $m \geq n-1$.

Thus, with regard to the maximum point w_0 the total number of critical points of the inner radius $R(w)$ will be at least n. Theorem 1 is proved.

3. CLASSIFICATION OF CRITICAL POINTS OF THE INNER RADIUS

Let us investigate the surface of the inner radius $R = R(u,v)$, $w = u + iv$, near its critical points. Assuming, as before, that the number of such points is finite, we denote them by $w_0, w_1, \ldots, w_m$.

Let k_1 and k_2 be the principal curvatures of the surface $R(u,v)$ at the point $w = u + iv$. Let us recall some definitions from surface theory (see, for example, [16]). *The total* (or *Gaussian*) curvature of the surface at a given point is found by the formula $K = k_1k_2$. The half-sum of principal curvatures $H = \frac{1}{2}(k_1 + k_2)$ is called the *mean curvature* of the surface at the point w.

The sign of the total curvature characterizes the structure of the surface in the vicinity of a given point. In the case when the surface is given in the form of a graph $R = R(u,v)$, the value of the total curvature K at the singular points w_i is

$$K(w_i) = R_{uu}(w_i)\,R_{vv}(w_i) - R_{uv}^2(w_i). \tag{17}$$

To determine the principal curvatures of k_1 and k_2 at the points w_i, we use the quadratic equation

$$k^2 - [R_{uu}(w_i) + R_{vv}(w_i)]k + K(w_i) = 0. \tag{18}$$

Lemma 5. *The mean curvature H of the inner radius surface in a neighborhood of critical points is negative.*

Proof. The sum of the roots of the equation (18) is $k_1 + k_2 = R_{uu}(w_i) + R_{vv}(w_i)$. Due to the strict superharmonicity of the function $R(w)$ near singular points (Lemma 4), this yields the inequality $k_1 + k_2 < 0$, and hence $H < 0$. This inequality, in particular, implies that in a neighborhood of the points w_i at least one of the principal curvatures is negative. Therefore, on the surface with the equation $R = R(u,v)$, there are no *flattening points*, that is, the points at which $k_1 = k_2 = 0$. □

Remark. It also follows from the lemma that on the surface of the inner radius there are no points of relative minima at which $k_1 > 0$ and $k_2 > 0$.

Let us denote by r_i the number of branches of the level line $R(w) = R(w_i)$, converging at the point w_i. Since the critical points are isolated, there is a sufficiently small neighborhood w_i in which grad $R \neq 0$. As proved in [17, p. 76] (see also [14, p. 223]), under these assumptions the number r_i is always finite and even. The quantity $m(w_i) = r_i/2$ is called the *saddle-like order* at the point w_i.

The saddle-like order characterizes exactly the structure of the surface $R = R(w)$ in a neighborhood of a given point. For example, to the point w_i of the local convexity of the surface there corresponds a saddle-like order equal to zero. If the intersection line of the surface $R = R(w)$ with the tangent plane at the point $R(w_i)$ has only one branch passing through w_i, then $m(w_i) = 1$. An ordinary saddle with apex in w_i corresponds to a saddle-like order equal to 2.

There is a simple relationship between the saddle-like order $m(w_i)$ and the index $\gamma(w_i)$ of the point w_i, $\gamma(w_i) = 1 - m(w_i)$, installed in [17, p. 80] (see also [14, p. 225]).

It is more convenient to classify the critical points $w_i \in D$ of the surface $R = R(w)$ geometrically according to the values of their indices $\gamma(w_i)$, and to calculate using the curvature $K(w_i)$ (cf. [14, p. 224]). Taking into account the admissible values of the indices of points, we obtain

Theorem 2. *The inner radius surface $R = R(w)$ in the neighborhood $U(w_i)$ of an isolated critical point w_i admits only three types of structure:*

1) *if $\gamma(w_k) = +1$, then U_k is convex upwards;*

2) *if $\gamma(w_k) = 0$, then U_k has a “semi-saddle” structure;*

3) *if $\gamma(w_k) = -1$, then U_k is similar to an ordinary saddle.*

From the proof of Theorem 1 it is easy to obtain a connection between the number of the points of positive and negative indices. If by M we denote the number of critical points with index $+1$, and by S the number of saddle points (that is, points with index -1), then the formula (6) can be rewrite as

$$M - S = 2 - n. \tag{19}$$

Other quantitative information can be obtained from the formula (19). Suppose that the function $R(w)$, in addition to the maximum point w_0, has at least one more stationary point. Then (19) implies that there must be at least n saddle points.

4. SHARPNESS OF AN ESTIMATE FOR THE NUMBER OF CRITICAL POINTS

The question of the sharpness of the estimate for the number of critical points of the inner radius $R(w)$ has been fully solved only in the one- and doubly connected cases.

1°. For simply connected domains, the corresponding example can be easily constructed if we take as D, for example, the unit disc $|w| < 1$. Then

$$g(w, w_0) = G(w, w_0) + \ln|w - w_0| = \ln(1 - w\overline{w}_0),$$

and the inner (conformal) radius $R(w) = 1 - |w|^2$ has a single critical point $w = 0$. It is clear that its index is $+1$, and $M = 1$, $S = 0$.

2°. Consider an example of a doubly connected domain D for which the inner radius surface $R = R(w)$ has an infinite set of stationary points. Let D be a concentric ring $E_q = \{\, w \mid q < |w| < 1\}$. Green's function for the region E_q is known (see, for example, [18, p. 171]):

$$G(w, w_0) = \frac{\ln|w| \cdot \ln|w_0|}{\ln q} - \ln|F(w, w_0)|,$$

where $F(w, w_0)$ is a function that conformally and univalently maps E_q onto the unit disc with a circular concentric cut of radius $|w_0|$:

$$F(w, w_0) = \frac{w - w_0}{1 - \overline{w}_0 w} \prod_{k=1}^{\infty} \left[\frac{(1 - q^{2k} w/w_0)(1 - q^{2k} w_0/w)}{(1 - q^{2k} w\overline{w}_0)(1 - q^{2k}/(w\overline{w}_0))}\right].$$

The inner radius at the point w, up to a constant factor, will be equal to

$$R(w) = r^{\frac{\ln r}{\ln q}} (1 - r^2) \prod_{k=1}^{\infty} (1 - q^{2k} r^2)(1 - q^{2k} r^{-2}), \quad r = |w|.$$

The critical points of $R(w)$ are found from the equation (4):

$$\frac{\ln r}{\ln q} = \frac{r^2}{1 - r^2} + \sum_{k=1}^{\infty} \left[\frac{q^{2k} r^2}{1 - q^{2k} r^2} - \frac{q^{2k} r^{-2}}{1 - q^{2k} r^{-2}}\right]. \tag{20}$$

The function on the left-hand side of (20) decreases, and on the right-hand side it increases. Since for $r \to 1$ ($r \to q$) the right-hand side tends to $+\infty$ ($-\infty$), we conclude that the equation (20) has a unique solution $r = r_0$.

Changing the order of summation of the terms, we represent (20) in the equivalent form

$$\frac{\ln r}{\ln q} = \sum_{k=0}^{\infty} \frac{q^{2k}(r^4 - q^2)}{(1 - q^{2k} r^2)(r^2 - q^{2k+2})}.$$

For $r < \sqrt{q}$ all terms of the series are negative, while the function on the left side is positive for all r, $q < r < 1$. It follows that the critical points of the inner radius of the ring E_q are all points of some circle of radius r_0, $\sqrt{q} < r_0 < 1$. The above example also shows that in the general case it is impossible to indicate an upper bound for the number of critical points of the inner radius.

3°. Consider an example of a doubly connected domain D, for which the inner radius surface has exactly two critical points. Let $f(w)$ be a conformal mapping of the domain D. Then the inner radius of the domain $f(D)$ at the point $f(w)$ is $R(f(w)) = R(w)|f'(w)|$. This relation is easily deduced from the corresponding property $G(f(w), f(w_0)) = G(w, w_0)$ for Green's function of the domain D. The equation (4) for finding the critical points of the function $R(f(w))$ is equivalent to the following:

$$\frac{f''(w)}{f'(w)} = -2\,\frac{\partial}{\partial w}\ln R(w) = -2\,\frac{\partial g(w,w)}{\partial w}. \tag{21}$$

We choose $D = f(E_q)$—the image of the ring E_q mapped by the function $f(w) = e^w$. In this case, the equality (21) has the form

$$w = -\frac{2\ln r}{\ln q} + \frac{2r^2}{1-r^2} + 2\sum_{k=1}^{\infty}\left[\frac{q^{2k}r^2}{1-q^{2k}r^2} - \frac{q^{2k}r^{-2}}{1-q^{2k}r^{-2}}\right], \quad w = re^{i\varphi}. \tag{22}$$

The right-hand side takes only real values, so from (22) we find the arguments of the critical points of the function $R(w)$: $\varphi = 0, \pi$. After substituting these values φ into (22), we obtain the condition on the modules of critical points:

$$\pm 1 = -\frac{2\ln r}{r\ln q} + \frac{2r}{1-r^2} + 2\sum_{k=1}^{\infty}\left[\frac{q^{2k}r}{1-q^{2k}r^2} - \frac{q^{2k}r^{-3}}{1-q^{2k}r^{-2}}\right]. \tag{23}$$

The plus sign here corresponds to the value $\varphi = 0$, the minus sign—to the value $\varphi = \pi$. Since the right-hand side is strictly monotonic (its derivative with respect to r is greater than zero), each of the two equations (23) has a single root.

Thus, the equation (21) for the critical points of the function $R(w)$ has exactly two roots.

FUNDING

This paper has been supported by the Kazan Federal University Strategic Academic Leadership Program ("PRIORITY-2030").

REFERENCES

1. G. Szegö, "On the capacity of a condenser," Bull. Am. Math. Soc. **51**, 325–350 (1945).
2. M. Schiffer, "Hadamard's formula and variation of domain-functions," Am. J. Math. **68**, 417–448 (1946).
3. W. K. Hayman, *Multivalent Functions,* 2nd ed. (Cambridge Univ. Press, Cambridge, 1996).
4. S. Bergman, *The Kernel Function and Conformal Mapping* (Am. Math. Soc., New York, 1951), Vol. 5.
5. V. N. Dubinin, *Condenser Capacities and Symmetrization in Geometric Function Theory* (Birkhauser/Springer, Basel, 2014).
6. H. R. Haegi, "Extremalprobleme und Ungleichungen konformer Gebietsgrößen," Compos. Math. **8**, 81–111 (1950).
7. L. A. Aksent'ev, "The connection of the exterior inverse boundary value problem with the inner radius of the domain," Sov. Math. **28** (2), 1–13 (1984).
8. A. V. Kazantsev, "Conformal radius: At the interface of traditions," Lobachevskii J. Math. **38**, 469–475 (2017).
9. F. D. Gakhov, "On inverse boundary value problems," Uch. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki **113** (10), 9–20 (1953).
10. M. I. Kinder, "The number of solutions of F. D. Gakhov's equation in the case of a multiply connected domain," Sov. Math. **28** (8), 69–72 (1984).
11. M. I. Kinder, "Investigation of F. D. Gakhov's equation in the case of multiply connected domains," Tr. Semin. Kraev. Zad. **22**, 104–116 (1985).
12. M. A. Krasnosel'skii, A. I. Perov, A. I. Povolockii, and P. P. Zabreiko, *Plane Vector Fields* (Academic, New York, 1966).
13. M. Schiffer, "Some recent developments in the theory of conformal mappings," in *Dirichlet's Principle, Conformal Mappings and Minimal surfaces,* Ed. by R. Courant (Interscience, New York, 1950), Appendix.
14. Ya. Bakelman, A. L. Verner, and B. E. Kantor, *Introduction in Differential Geometry 'in the Large'* (Nauka, Moscow, 1973) [in Russian].
15. M. Schiffer, "The kernel function of an orthonormal system," Duke Math. J. **13**, 529–540 (1946).
16. P. K. Rashevski, *Course of Differential Geometry* (GITTL, Moscow, 1956) [in Russian].
17. N. V. Efimov, "Qualitative problems of the theory of deformations of surfaces 'in the small'," Tr. Mat. Inst. Steklov **30**, 1–128 (1949).
18. N. I. Akhiezer, *Elements of the Theory of Elliptic Functions* (Am. Math. Soc., New York, 1990).