Characterization of certain traces on von Neumann algebras

Airat Bikchentaev[1]

N.I. Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kremlevskaya ul. 18, Kazan, Tatarstan, 420008 Russia,
Airat.Bikchentaev@ kpfu.ru,
WWW home page:

Abstract. Consider a unital C^*-algebra $\mathcal{A}$. Let $n \geq 2$ and let $P_1, \ldots, P_n$ be projections in $\mathcal{A}$ such that $P_1 + \ldots + P_n = I$. We costruct $\mathcal{P}_n \colon \mathcal{A} \to \mathcal{A}$ being a block projection operator given by the formula $\mathcal{P}_n(X) = \sum_{k=1}^{n} P_k X P_k$ for all $X \in \mathcal{A}$. For a weight φ on a von Neumann algebra $\mathcal{A}$, we prove that φ is a trace if and only if $\varphi(\mathcal{P}_2(A)) = \varphi(A)$ for all $A \in \mathcal{A}^+$. We also prove that if $\mathcal{A}$ is a von Neumann algebra then for a normal semifinite weight φ on $\mathcal{A}$ the following conditions are equivalent: (i) φ is a trace; (ii) $\varphi((A^{m/2}B^m A^{m/2})^k) \leq \varphi((A^{k/2}B^k A^{k/2})^m)$ for all $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ such that $k > m > 0$; (iii) $\varphi(|\mathcal{P}_n(A)|) \leq \varphi(|A|)$ for all $A \in \mathcal{A}$ and for all projections $P_1, \ldots, P_n \in \mathcal{A}$. As a consequence, we obtain a criterions for commutativity of von Neumann algebras and C^*-algebras.

Keywords: Hilbert space, linear operator, von Neumann algebra, C^*-algebra, block projection operator, weight, trace, tracial inequality, commutativity

1 Introduction

Traces and weights on C^*-algebras are basic tools in the operator theory and its applications. So it seems important to characterize traces in different classes of weights on C^*-algebras, see [1]–[10].

Consider a tracial positive normal linear functional φ on a von Neumann algebra $\mathcal{A}$, and positive numbers p, q such that $1/p + 1/q = 1$, then we have:

- Hölder's inequality [11, Chapter IX, Theorem 2.13], [10, Theorem 5]:

$$\varphi(|XY|) \leq \varphi(X^p)^{1/p} \varphi(Y^q)^{1/q} \text{ for all } X, Y \in \mathcal{A}^+;$$

- Cauchy–Schwarz–Buniakowski inequality [12, Theorem 4.21]:

$$\varphi(|XY|^{1/2}) \leq \varphi(X)^{1/2} \varphi(Y)^{1/2} \text{ for all } X, Y \in \mathcal{A}^+;$$

- Golden–Thompson inequality [13, Theorem 4]:

$$\varphi(e^{X+Y}) \leq \varphi(e^{X/2} e^Y e^{X/2}) \text{ for all } X, Y \in \mathcal{A}^{\mathrm{sa}};$$

- Peierls–Bogoliubov inequality [13, Theorem 7]:

$$\varphi(e^X)\exp\frac{\varphi(e^{X/2}Ye^{X/2})}{\varphi(e^X)} \leq \varphi(e^{X+Y}) \text{ for all } X, Y \in \mathcal{A}^+.$$

Also inequality

$$\operatorname{tr}((X^{1/2}YX^{1/2})^{rp}) \leq \operatorname{tr}((X^{r/2}Y^rX^{r/2})^p), \quad r \geq 1,\ p > 0,$$

holds true for positive operators X, Y on a Hilbert space $\mathcal{H}$ [14]. This inequality generalizes the inequalities of Lieb and Thirring, and resembles the Golden–Thompson inequality (see [15, §8]).

It can be said that any of the given trace inequalities is sharp in the following sense: only the trace of all positive linear functionals satisfies the inequality. It is known that if we limit ourselves only to projections of a von Neumann algebra $\mathcal{A}$ then each and every inequality of Hölder, Cauchy–Schwarz–Buniakowski, Golden–Tompson, Peierls–Bogoliubov, Araki–Lieb–Thirring etc., characterizes the tracial functionals among all positive normal functionals, see [16]–[22].

Gohberg and Krein had begun to study the block projection operators in [23]. These operators admit a natural extension to the setting of quasi-normed ideals and noncommutative integration. Consider a number $n \geq 2$ and let $P_1, \ldots, P_n$ be such projections in a unital C^*-algebra $\mathcal{A}$ that $P_1 + \ldots + P_n = I$. Introduce a block projection operator $\mathcal{P}_n \colon \mathcal{A} \to \mathcal{A}$ as follows: $\mathcal{P}_n(X) = \sum_{k=1}^n P_k X P_k$ for all $X \in \mathcal{A}$.

Consider a weight φ on a von Neumann algebra $\mathcal{A}$. Here we prove that the following conditions are equivalent: (i) φ is a trace; (ii) $\varphi(\mathcal{P}_2(A)) = \varphi(A)$ for all $A \in \mathcal{A}^+$ (Theorem 1). Note that the block projection operators on certain algebras (von Neumann algebras and algebras of operators measurable with respect to semifinite normal traces) already appeared in [24], [25]. We also proved several uniform submajorization inequalities for block projection operators [26]. Here we show that the following conditions are equivalent for a normal semifinite weight φ on a von Neumann algebra $\mathcal{A}$: (i) φ is a trace; (ii) $\varphi((A^{m/2}B^mA^{m/2})^k) \leq \varphi((A^{k/2}B^kA^{k/2})^m)$ for all $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ such that $k > m > 0$; (iii) $\varphi(|\mathcal{P}_n(A)|) \leq \varphi(|A|)$ for all $A \in \mathcal{A}$ and for all $P_1, \ldots, P_n \in \mathcal{A}^{\mathrm{pr}}$ with $P_1 + \ldots + P_n = I$, where $\mathcal{P}_n(A) = \sum_{k=1}^n P_k A P_k$ (Theorem 2). As a consequence, we obtain certain criterions for commutativity of von Neumann algebras and C^*-algebras (Corollaries 1, 3).

2 Definitions and notation

The basic notion here is a C^*-algebra, being a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for every $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ we denote its subsets of projections ($A = A^* = A^2$), self-adjoint elements ($A^* = A$) and positive elements, respectively. For any $A \in \mathcal{A}$ we have $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. If I is the unit of the algebra $\mathcal{A}$ and $P \in \mathcal{A}^{\mathrm{pr}}$, then $P^\perp = I - P$.

We say that a mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is *a weight* on a C^*-algebra $\mathcal{A}$, if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$). Introduce the set

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ \colon \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+$$

for a weight φ.

We can always extend by linearity the restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ to a functional on $\mathfrak{M}_\varphi$. This extension is denoted by the same letter φ. Such an extension tells us that finite weights (i.e., $\varphi(X) < +\infty$ for all $X \in \mathcal{A}^+$) are virtually the same with positive functionals on $\mathcal{A}$. We call a positive linear functional φ on $\mathcal{A}$ with $\|\varphi\| = 1$ *a state*. A weight φ is said to be *faithful*, if $\varphi(X) = 0$ $(X \in \mathcal{A}^+)$ implies that $X = 0$; *a trace*, if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for every $Z \in \mathcal{A}$. If $\varphi(A) = \sup\{\varphi(B) : B \in \mathcal{A}^+, B \leq A, \varphi(B) < +\infty\}$, for every $A \in \mathcal{A}^+$ then a trace φ on a C^*-algebra $\mathcal{A}$ is *semifinite*. *A subadditive weight* on a C^*-algebra $\mathcal{A}$ is a mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ such that $\varphi(X+Y) \leq \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$), see [27]–[30]. A subadditive weight φ is called *a subadditive trace*, if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$.

Let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ over the field $\mathbb{C}$. Gelfand–Naimark theorem states that every C^*-algebra is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [31, II.6.4.10]. For any set $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$ we construct the commutant

$$\mathcal{X}' = \{Y \in \mathcal{B}(\mathcal{H}) \colon XY = YX \ \text{ forall } \ X \in \mathcal{X}\}.$$

A von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is a $*$-subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}''$. For $P, Q \in \mathcal{A}^{\mathrm{pr}}$ we write $P \sim Q$ (the Murray–von Neumann equivalence), if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{A}$.

A *normal* weight φ on von Neumann algebra $\mathcal{A}$ is a weight such that $\varphi(\sup X_i) = \sup \varphi(X_i)$ for every bounded increasing net $\{X_i\}$ in $\mathcal{A}^+$; a weight φ is *semifinite*, if the set $\mathfrak{M}_\varphi$ is ultraweakly dense in $\mathcal{A}$ (see [32, Definition VII.1.1]).

Consider a von Neumann algebra $\mathcal{A}$, let $U \in \mathcal{A}$ be a unitary operator, i.e., $U^*U = UU^* = I$. There exists an automorphism α of a von Neumann algebra $\mathcal{A}$, defined by the formula $\alpha(A) = U^*AU$ for all $A \in \mathcal{A}$. By [33, Theorem 1.4] it follows that the automorphism α can be represented as a finite product of involutions $U_S : A \mapsto SAS$, here S is a symmetry in $\mathcal{A}$, i.e., $U^*AU = S_1 \cdots S_m A S_m \cdots S_1$ with unitaries $S_1, \ldots, S_m \in \mathcal{A}^{\mathrm{sa}}$. Moreover, if $\mathcal{A}$ possesses no type I_{fin} direct summands then the unitary operator U by itself is a finite product of symmetries from $\mathcal{A}^{\mathrm{sa}}$ [33, Theorem 1.6].

The universal representation of a C^*-algebra $\mathcal{A}$ is the pair

$$\{\pi, \mathfrak{H}\} = \sum_{\varphi \in \mathcal{S}(\mathcal{A})}^{\oplus} \{\pi_\varphi, \mathfrak{H}_\varphi\},$$

where $\mathcal{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$, $(\pi_\varphi, \mathfrak{H}_\varphi)$ is the Gelfand–Naimark–Segal representation of a C^*-algebra $\mathcal{A}$, assosiated with φ. Here we say that the von

Neumann algebra $\mathcal{M} = \pi(\mathcal{A})''$, generated by $\pi(\mathcal{A})$, is *the universal enveloping von Neumann algebra* of C^*-algebra $\mathcal{A}$ [11, Chap. III, Definition 2.3].

Consider a C^*-algebra $\mathcal{A}$. Let φ be a positive linear functional on $\mathcal{A}$ and π be the universal representation of $\mathcal{A}$. Then arbitrary state on $\mathcal{A}$ by construction of π turns into a vector state on $\pi(\mathcal{A})$, hence it extends to a normal state on the universal enveloping algebra $\mathcal{M} = \pi(\mathcal{A})''$. Then φ yields such a positive normal functional $\widehat{\varphi}$ on the universal enveloping von Neumann algebra that $\widehat{\varphi}(\pi(A)) = \varphi(A)\ \ (A \in \mathcal{A}^+)$.

Recall the finite dimensional Spectral Theorem: every normal matrix $A \in \mathbb{M}_n(\mathbb{C})$ reduces to the sum $A = \sum_{i=1}^m \lambda_i P_i$, where $\lambda_i \in \mathbb{C}$, and the projections $P_i \in \mathbb{M}_n(\mathbb{C})^{\mathrm{pr}}$ with $P_i P_j = 0$ for $i \neq j$ and $i, j = 1, \dots, m$, $m \leq n$.

3 Trace characterization on C^*- algebras

Lemma 1. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $n \geq 2$ and let $P_1, \dots, P_n \in \mathcal{A}^{\mathrm{pr}}$ be such that $P_1 + \ldots + P_n = I$, $\mathcal{P}_n(A) = \sum_{k=1}^n P_k A P_k$ for $A \in \mathcal{A}$. Then*

(i) *If φ is a trace on $\mathcal{A}$, then $\varphi(\mathcal{P}_n(A)) = \varphi(A)$ for all $A \in \mathcal{A}^+$.*

(ii) *If φ is a subadditive trace on $\mathcal{A}$, then $\varphi(\mathcal{P}_n(A)) \leq \varphi(A)$ for all $A \in \mathcal{A}^+$.*

Proof. (i). For any $A \in \mathcal{A}^+$ and $n \geq 2$ we have

$$\varphi(\mathcal{P}_n(A)) = \sum_{k=1}^n \varphi(P_k A P_k) = \sum_{k=1}^n \varphi(P_k A^{1/2} \cdot A^{1/2} P_k) = \sum_{k=1}^n \varphi((A^{1/2} P_k)^* A^{1/2} P_k) =$$

$$= \sum_{k=1}^n \varphi(A^{1/2} P_k (A^{1/2} P_k)^*) = \sum_{k=1}^n \varphi(A^{1/2} P_k A^{1/2}) = \varphi\Big(A^{1/2} \Big(\sum_{k=1}^n P_k\Big) A^{1/2}\Big) = \varphi(A).$$

(ii). For any $A \in \mathcal{A}$ and $n \geq 2$ by [25, Lemma 2] we have the representation

$$\mathcal{P}_n(A) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} S_k A S_k,$$

where the unitaries $S_k \in \mathcal{A}^{\mathrm{sa}}$, $k = 1, \dots, 2^{n-1}$, have the form

$$P_1 \pm P_2 \pm \dots \pm P_n.$$

Consider $A \in \mathcal{A}^+$. We have $\varphi(S_k A S_k) = \varphi(A)$ for all $k = 1, \dots, 2^{n-1}$ and

$$\varphi(\mathcal{P}_n(A)) = \varphi\Big(\frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} S_k A S_k\Big) = \frac{1}{2^{n-1}} \varphi\Big(\sum_{k=1}^{2^{n-1}} S_k A S_k\Big) \leq$$

$$\leq \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} \varphi(S_k A S_k) = \varphi(A).$$

Theorem 1. *For a weight φ on a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *φ is a trace;*

(ii) *$\varphi(\mathcal{P}_2(A)) = \varphi(A)$ for all $A \in \mathcal{A}^+$.*

Proof. For (i)⇒(ii) see item (i) of Lemma 1.

(ii)⇒(i). By applying Upmeier's results [33], one can see that [34, Theorem 1.4.2] actually shows us that a weight φ on a von Neumann algebra $\mathcal{A}$ is a trace if and only if $\varphi(SAS) = \varphi(A)$ for any positive operator $A \in \mathcal{A}^+$ and any symmetry $S \in \mathcal{A}^{\mathrm{sa}}$, see Section 2. Let $A \in \mathcal{A}^+$, a symmetry $S \in \mathcal{A}^{\mathrm{sa}}$ and $P_1 \in \mathcal{A}^{\mathrm{pr}}$ be such that $S = 2P_1 - I$, i.e., $P_1 = (S+I)/2$. For $P_2 = P_1^{\perp}$ we have $\mathcal{P}_2(A) = \frac{1}{2}(A + SAS)$ and

$$\varphi(A) = \varphi(\mathcal{P}_2(A)) = \varphi\Big(\frac{1}{2}(A + SAS)\Big) = \frac{1}{2}\varphi(A) + \frac{1}{2}\varphi(SAS).$$

For $\varphi(A) < +\infty$, or $\varphi(SAS) = \varphi(A) = +\infty$ we obtain $\varphi(SAS) = \varphi(A)$, the theorem is thereby established. Assume that $\varphi(SAS) < \varphi(A) = +\infty$. By repeating the above argument for the operator $A_1 = SAS$ instead of A (then $SA_1S = A$), we conclude that

$$+\infty > \varphi(SAS) = \frac{1}{2}\varphi(SAS) + \frac{1}{2}\varphi(A) = +\infty.$$

This is a contradiction, hence the theorem holds.

Recall Taylor's formula with Peano's remainder. Then we have

Lemma 2. *If $a \in \mathbb{R}$, then*

$$(1+t)^a = 1 + at + \frac{1}{2!}a(a-1)t^2 + \ldots + \frac{1}{n!}a(a-1)\cdots(a-n+1)t^n + o(t^n) \quad as \quad t \to 0.$$

Theorem 2. *For a normal semifinite weight φ on a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *φ is a trace;*

(ii) *$\varphi((A^{m/2}B^m A^{m/2})^k) \le \varphi((A^{k/2}B^k A^{k/2})^m)$ for all $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ with $k > m > 0$;*

(iii) *$\varphi(|\mathcal{P}_n(A)|) \le \varphi(|A|)$ for all $A \in \mathcal{A}$ and for all $P_1, \ldots, P_n \in \mathcal{A}^{\mathrm{pr}}$ with $P_1 + \ldots + P_n = I$, where $\mathcal{P}_n(A) = \sum_{k=1}^n P_k A P_k$.*

Proof. (i)⇒(ii). Consider $A, B \in \mathcal{A}^+$ and $k > m > 0$. Put

$$X = A^m, \;\; Y = B^m, \;\; r = \frac{k}{m}, \;\; p = m. \tag{1}$$

Then $r > 1$ and

$$(A^{m/2}B^m A^{m/2})^k = (X^{1/2}YX^{1/2})^{rp}, \;\; (A^{k/2}B^k A^{k/2})^m = (X^{r/2}Y^r X^{r/2})^p.$$

The inequality

$$\varphi((X^{1/2}YX^{1/2})^{rp}) \le \varphi((X^{r/2}Y^r X^{r/2})^p), \quad r \ge 1, \; p > 0 \tag{2}$$

was proved in [35].

(ii)⇒(i). Again we rewrite the inequality of item (ii) in the form of relation (2) applying transformations (1). It follows by Lemma 2 of [20] that for every projection $P \in \mathcal{A}^{\mathrm{pr}}$ with $\varphi(P) < +\infty$ the reduced weight $\varphi|_P$ on the reduced von Neumann algebra $P\mathcal{A}P$ is a trace. Therefore φ is a trace by Lemma 2 of [36].

(i)⇒(iii). For $n = 2$ see [37, Lemma 13]; for the general case see [24, Lemma 3.2].

(iii)⇒(i). *Step 1.* Consider a positive normal functional φ on a von Neumann algebra $\mathcal{A}$. Then the proof of implication (iii)⇒(i) for arbitrary von Neumann algebra is reducible to the case of the algebra $\mathbb{M}_2(\mathbb{C})$ as in similar situations (see [7] or [38]).

A positive normal linear functional φ on a von Neumann algebra $\mathcal{A}$ is tracial if and only if $\varphi(P) = \varphi(Q)$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $PQ = 0$ and $P \sim Q$ (see [7], [38, Lemma 2]). Consider a *-algebra $\mathcal{N}$ in the reduced algebra $(P + Q)\mathcal{A}(P+Q)$ generated by a partial isometry $V \in \mathcal{A}$ that realizes the equivalence between P and Q. The algebra $\mathcal{N}$ is *-isomorphic to $\mathbb{M}_2(\mathbb{C})$. Inequality (iii) holds for operators of $\mathcal{N}$ and the restricted functional $\varphi|\mathcal{N}$. Let us show that this restriction is tracial on $\mathcal{N}$; henceforce, $\varphi(P) = \varphi(Q)$.

Recall that every linear functional φ on $\mathbb{M}_2(\mathbb{C})$ possesses the form $\varphi(\cdot) = \mathrm{tr}(S_\varphi\, \cdot)$. The matrix $S_\varphi \in \mathbb{M}_2(\mathbb{C})$ is the so-called density matrix of φ. Without loss of generality assume that

$$S_\varphi = \mathrm{diag}\Big(\frac{1}{2} - s, \frac{1}{2} + s\Big), \quad 0 \le s \le \frac{1}{2}.$$

Thus $\varphi(X)$ equals $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2 \in \mathbb{M}_2(\mathbb{C})$.

Consider a complex $\sigma \in \mathbb{C}$ with $|\sigma| = 1$ and a real $t \in [0, 1]$. These numbers define the projection

$$R^{(t,\sigma)} = \begin{pmatrix} t & \sigma\sqrt{t - t^2} \\ \bar{\sigma}\sqrt{t - t^2} & 1 - t \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Put $P_1 = R^{(1/2,1)}$, $P_2 = P_1^\perp = R^{(1/2,-1)}$ and

$$A = \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & \varepsilon^2 \end{pmatrix}$$

for $0 < \varepsilon < 1/2$. Then $\mathcal{P}_2(A) = \frac{1+\varepsilon^2}{2} I$ and $\varphi(|\mathcal{P}_2(A)|) = \mathrm{tr}(S_\varphi \mathcal{P}_2(A)) = \frac{1+\varepsilon^2}{2}$.

The matrix

$$|A|^2 = \begin{pmatrix} 1 + \varepsilon^2 & -\varepsilon + \varepsilon^3 \\ -\varepsilon + \varepsilon^3 & \varepsilon^2 + \varepsilon^4 \end{pmatrix}$$

has the characteristic equation $\lambda^2 - (1 + \varepsilon^2)^2\lambda + 4\varepsilon^4 = 0$. Therefore,

$$\lambda_1 = \frac{(1 + \varepsilon^2)^2 + \sqrt{(1 + \varepsilon^2)^4 - 16\varepsilon^4}}{2} \quad \text{and} \quad \lambda_2 = \frac{(1 + \varepsilon^2)^2 - \sqrt{(1 + \varepsilon^2)^4 - 16\varepsilon^4}}{2}.$$

Taylor's formula with Peano remainder (cf Lemma 2 with $a = 1/2$) implies that

$$\sqrt{(1+\varepsilon^2)^4 - 16\varepsilon^4} = 1 + 2\varepsilon^2 - 7\varepsilon^4 + o(\varepsilon^5), \quad (\varepsilon \to 0+).$$

Hence

$$\lambda_1 = 1 + 2\varepsilon^2 - 3\varepsilon^4 + o(\varepsilon^5), \quad \lambda_2 = 4\varepsilon^4 + o(\varepsilon^5), \quad (\varepsilon \to 0+).$$

The finite-dimensional Spectral Theorem yields the representation

$$|A|^2 = \lambda_1 R^{(t,1)} + \lambda_2 R^{(t,1)\perp} = \lambda_1 R^{(t,1)} + \lambda_2 R^{(1-t,-1)}.$$

We determine the parameter $t \in [0,1]$ from the equation

$$1 + \varepsilon^2 = \lambda_1 t + \lambda_2 (1-t), \text{ i.e.,}$$

$$1 + \varepsilon^2 = (1 + 2\varepsilon^2 - 3\varepsilon^4 + o(\varepsilon^5))t + (4\varepsilon^4 + o(\varepsilon^5))(1-t), \quad (\varepsilon \to 0+).$$

Hence

$$t = \frac{1 + \varepsilon^2 - 4\varepsilon^4}{1 + 2\varepsilon^2 - 7\varepsilon^4 + o(\varepsilon^5)}, \quad (\varepsilon \to 0+).$$

By Lemma 2 with $a = -1$ we find

$$t = (1 + \varepsilon^2 - 4\varepsilon^4)(1 - 2\varepsilon^2 + 11\varepsilon^4 + o(\varepsilon^5)) = 1 - \varepsilon^2 + 5\varepsilon^4 + o(\varepsilon^5), \qquad (\varepsilon \to 0+).$$

Finite-dimensional Spectral Theorem yields $|A| = \sqrt{\lambda_1} R^{(t,1)} + \sqrt{\lambda_2} R^{(1-t,-1)}$, where

$$\sqrt{\lambda_1} = 1 + \varepsilon^2 - 2\varepsilon^4 + o(\varepsilon^5), \quad \sqrt{\lambda_2} = 2\varepsilon^2 \sqrt{1 + o(\varepsilon)} = 2\varepsilon^2 + o(\varepsilon^3), \qquad (\varepsilon \to 0+)$$

for $a = 1/2$ in Lemma 2 and thanks to the relation $\varepsilon^k o(\varepsilon^m) = o(\varepsilon^{k+m})$, $(\varepsilon \to 0+)$ for all $k, m \in \mathbb{N}$. Hence

$$\begin{aligned}\varphi(|A|) = \operatorname{tr}(S_\varphi |A|) = \sqrt{\lambda_1} \operatorname{tr}(S_\varphi R^{(t,1)}) + \sqrt{\lambda_2} \operatorname{tr}(S_\varphi R^{(1-t,-1)}) = \\ = \sqrt{\lambda_1}\Big(\Big(\frac{1}{2} - s\Big)t + \Big(\frac{1}{2} + s\Big)(1-t)\Big) + \sqrt{\lambda_2}\Big(\Big(\frac{1}{2} - s\Big)(1-t) + \Big(\frac{1}{2} + s\Big)t\Big) = \\ = \frac{1}{2} - s + \frac{3}{2}\varepsilon^2 + 3s\varepsilon^2 + o(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0+.\end{aligned}$$

The inequality $\varphi(|\mathcal{P}_2(A)|) \le \varphi(|A|)$ then takes the form

$$\frac{1}{2} + \frac{\varepsilon^2}{2} \le \frac{1}{2} - s + \frac{3}{2}\varepsilon^2 + 3s\varepsilon^2 + o(\varepsilon^3), \qquad (\varepsilon \to 0+).$$

It holds true only for $s = 0$ for all ε, $0 < \varepsilon < \frac{1}{2}$.

Step 2. Let a normal semifinite weight φ on a von Neumann algebra $\mathcal{A}$ meet condition (iii). It follows by Step 1 that for every projection $P \in \mathcal{A}^{\mathrm{pr}}$ with $\varphi(P) < +\infty$ the reduced weight $\varphi|_P$ on the reduced von Neumann algebra $P\mathcal{A}P$ is a trace. Therefore φ is a trace by Lemma 2 of [36]. The assertion is proved.

Corollary 1. *For a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *the algebra $\mathcal{A}$ is commutative;*

(ii) $\varphi(\mathcal{P}_2(A)) = \varphi(A)$ *for all normal states φ on $\mathcal{A}$, for all operators $A \in \mathcal{A}^+$, for all $P_1 \in \mathcal{A}^{\mathrm{pr}}$ and $P_2 = P_1^{\perp}$;*

(iii) $\varphi(|\mathcal{P}_2(A)|) \leq \varphi(|A|)$ *for all normal states φ on $\mathcal{A}$, for all operators $A \in \mathcal{A}$, for all $P_1 \in \mathcal{A}^{\mathrm{pr}}$ and $P_2 = P_1^{\perp}$;*

(iv) $\varphi((A^{m/2}B^m A^{m/2})^k) \leq \varphi((A^{k/2}B^k A^{k/2})^m)$ *for all normal states φ on $\mathcal{A}$, for all operators $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ with $k > m > 0$.*

Proof. (iv)⇒(i). Every normal state on $\mathcal{A}$ is tracial by Theorem 2, i.e., $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathcal{A}$. The set of all normal states separates elements of the algebra $\mathcal{A}$ [31, Chap. III, Theorem 2.4.5]. This fact implies that $XY = YX$ $(X, Y \in \mathcal{A})$. So the von Neumann algebra $\mathcal{A}$ is commutative.

Corollary 2. *Let φ be a positive functional on C^*-algebra $\mathcal{A}$ such that the inequality $\varphi((A^{m/2}B^m A^{m/2})^k) \leq \varphi((A^{k/2}B^k A^{k/2})^m)$ holds for any $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ such that $k > m > 0$. Then φ is a tracial functional.*

Proof. Consider the universal enveloping von Neumann algebra $\mathcal{M}$ [11, III.2]. Assume that π is the corresponding universal representation of the C^*-algebra $\mathcal{A}$, and $\widehat{\varphi}$ is the positive normal functional on $\mathcal{M}$ with $\widehat{\varphi}(\pi(A)) = \varphi(A)$ for $A \in \mathcal{A}$, see Section 2. Fix positive operators $\widehat{A}, \widehat{B} \in \mathcal{M}$. Then by Kaplansky density theorem there exist bounded positive nets $\{A_\alpha\}$ and $\{B_\alpha\}$ in $\mathcal{A}$ such that $\pi(A_\alpha) \to \widehat{A}$ and $\pi(B_\alpha) \to \widehat{B}$ in the strong operator topology. Fix $k, m \in \mathbb{R}$ such that $k > m > 0$. We take into account inequality (ii) of Theorem 1, apply continuity of the operations in the strong operator topology and conclude that $\widehat{\varphi}((\widehat{A}^{m/2}\widehat{B}^m \widehat{A}^{m/2})^k) \leq \widehat{\varphi}((\widehat{A}^{k/2}\widehat{B}^k \widehat{A}^{k/2})^m)$. By Theorem 2 $\widehat{\varphi}$ is a tracial functional on $\mathcal{M}$, hence φ is a tracial functional on $\mathcal{A}$.

Corollary 3. *Let $\mathcal{A}$ be a C^*-algebra such that $(A^{m/2}B^m A^{m/2})^k \leq (A^{k/2}B^k A^{k/2})^m$ for all $A, B \in \mathcal{A}^+$ and some numbers $k, m \in \mathbb{R}$ with $k > m > 0$. Then $\mathcal{A}$ is commutative.*

Proof. The inequality of Corollary 2 holds for every positive functional on $\mathcal{A}$. Then every positive functional on $\mathcal{A}$ is tracial, and $\mathcal{A}$ is commutative.

For other trace characterizations see also [38]–[44].

Acknowledgments. The author was supported by the development program of Volga Region Mathematical Center (agreement no. 075-02-2021-1393).

References

1. Abed, S.A.: An inequality for projections and convex functions. *Lobachevskii J. Math.* **39**(9), 1287–1292 (2018). doi:10.1134/S1995080218090214

2. Bikchentaev, A.M.: Commutation of projections and characterization of traces on von Neumann algebras. III. Internat. J. Theor. Physics **54**(12), 4482–4493 (2015). doi:10.1007/s10773-015-2639-6
3. Bikchentaev, A.M.: Inequality for a trace on a unital C^*-algebra. Math. Notes **99**(4), 487–491 (2016). doi:10.1134/S0001434616030214
4. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras. Sib. Math. J. **58**(2), 183–189 (2017). doi:10.1134/S003744661702001X
5. Bikchentaev, A.M.: Differences of idempotents in C^*-algebras and the quantum Hall effect. Theoret. and Math. Phys. **195**(1), 557–562 (2018). doi:10.1134/S0040577918040074
6. Bikchentaev, A.M.: Trace and differences of idempotents in C^*-algebras. Math. Notes **105**(5-6), 641–648 (2019). doi:10.1134/S0001434619050018
7. Gardner, L.T.: An inequality characterizes the trace. Canad. J. Math. **31**(6), 1322–1328 (1979). https://doi.org/10.4153/CJM-1979-109-9
8. Hoa, D.T., Tikhonov, O.E.: Weighted monotonicity inequalities for traces on operator algebras. Math. Notes **88**(1-2), 177–182 (2010). doi:10.1134/S0001434610070175
9. Hoa, D.T., Osaka, H., Toan, H.M.: On generalized Powers-Størmer's inequality. Linear Algebra Appl. **438**(1), 242–249 (2013). doi:10.1016/j.laa.2012.07.053
10. Petz, D., Zemánek, J.: Characterizations of the trace. Linear Algebra Appl. **111**, 43–52 (1988). doi:10.1016/0024-3795(88)90050-X
11. Takesaki, M.: *Theory of operator algebras. Vol. I.* Operator Algebras and Noncommutative Geometry, 5 (Springer-Verlag, Berlin, 2002).
12. Manjegani, S.M.: *Inequalities in operator algebras. A thesis for the PhD in mathematics degree of Regina University.* Canada, Regina, 2004. 95 pp.
13. Ruskai, M.: Inequalities for traces on von Neumann algebras. Commun. Math. Physics **26**(4) , 280–289 (1972). doi:10.1007/BF01645523
14. Araki, H.: On an inequality of Lieb and Thirring. Lett. Math. Phys. **19**(2), 167–170 (1990). doi:10.1007/BF01045887
15. Simon, B.: *Trace ideals and their applications.* Second edition. Mathematical Surveys and Monographs, 120 (American Mathematical Society, Providence, RI, 2005).
16. Bikchentaev, A.M., Tikhonov, O.E.: Characterization of the trace by Young's inequality. JIPAM. J. Inequal. Pure Appl. Math. **6**(2), no. 2, Article 49, 3 pp. (2005).
17. Bikchentaev, A.M., Tikhonov, O.E.: Characterization of the trace by monotonicity inequalities. Linear Algebra Appl. **422**(1), 274–278 (2007). doi:10.1016/j.laa.2006.10.005
18. Bikchentaev, A.M.: Commutativity of projections and characterization of traces on von Neumann algebras. Sib. Math. J. **51**(6), 971–977 (2010). doi:10.1007/s11202-010-0096-2
19. Bikchentaev, A.M.: Commutation of projections and trace characterization on von Neumann algebras. II. Math. Notes **89**(3-4), 461–471 (2011). doi:10.1134/S0001434611030175
20. Bikchentaev, A.M.: The Peierls-Bogoliubov inequality in C^*-algebras and characterization of tracial functionals. Lobachevskii J. Math. **32**(3), 175–179 (2011). doi:10.1134/S1995080211030061
21. Bikchentaev, A.M.: Commutativity of operators and characterization of traces on C^*-algebras. Dokl. Math. **87**(1), 79–82 (2013). doi:10.1134/S1064562413010298
22. Cho, K., Sano, T.: Young's inequality and trace. Linear Algebra Appl. **431**(8), 1218–1222 (2009). doi:10.1016/j.laa.2009.04.016
23. Gohberg, I.C., Kreĭn, M.G.: *Introduction to the theory of linear nonselfadjoint operators.* (Transl. Mathem. Monographs, v. 18, Amer. Math. Soc., Providence, R.I., 1969).

24. Chilin, V., Krygin, A., Sukochev, F.: Extreme points of convex fully symmetric sets of measurable operators. Integr. Equ. Oper. Theory **15** (2), 186–226 (1992). doi:10.1007/BF01204237
25. Bikchentaev, A.M.: Block projection operator in normed ideal spaces of measurable operators. Russian Math. (Iz. VUZ) **56** (2), 75–79 (2012). doi:10.3103/S1066369X12020107
26. Bikchentaev, A.M., Sukochev, F.: Inequalities for the block projection operators. J. Funct. Anal. **280** (7), article 108851, 18 pp (2021). doi:10.1016/j.jfa.2020.108851
27. Haagerup, U.: Normal weights on W^*-algebras. J. Functional Analysis **19** (3), 302–317 (1975). doi: 10.1016/0022-1236(75)90060-9
28. Bikchentaev, A.M.: On the Haagerup problem on subadditive weights on W^*-algebras. Russian Math. (Iz. VUZ) **55** (10), 82–85 (2011). doi:10.3103/S1066369X11100112
29. Bikchentaev, A.M.: The Haagerup problem on subadditive weights on W^*-algebras. II. Russian Math. (Iz. VUZ) **57** (12), 66–69 (2013). doi:10.3103/S1066369X13120074
30. Bikchentaev, A.M.: Seminorms Associated with Subadditive Weights on C^*-Algebras. Mathematical Notes **107** (3-4), 383–391 (2020). doi:10.1134/S0001434620030025
31. Blackadar, B.: *Operator algebras, theory of C^*-algebras and von Neumann algebras*, in: Encyclopaedia of Mathematical Sciences, in: Operator Algebras and Noncommutative Geometry, III, vol. 122 (Springer-Verlag, Berlin, 2006).
32. Takesaki, M.: *Theory of Operator Algebras, Operator Algebras and Non-Commutative Geometry, 6. v. II*, Encyclopaedia Math. Sci., 125 (Springer-Verlag, New York, 2003).
33. Upmeier, H.: Automorphism groups of Jordan C^*-algebras. Math. Z. **176**(1), 21–34 (1981). doi:10.1007/BF01258901
34. Ayupov, Sh.A.: *Classification and representation of ordered Jordan algebras* (Russian) ("Fan", Tashkent, 1986).
35. Kosaki, H.: On an inequality of Araki-Lieb-Thirring (von Neumann algebra case). Proc. Amer. Math. Soc. **114**(2), 477–481 (1992). doi:10.1090/S0002-9939-1992-1065951-1
36. Stolyarov, A.I., Tikhonov, O.E., Sherstnev, A.N.: Characterization of normal traces on von Neumann algebras by inequalities for the modulus. Math. Notes **72**(3-4), 411–416 (2002). doi:10.1023/A:1020559623287
37. Kaftal, V., Weiss, G.: Compact derivations relative to semifinite von Neumann algebras. J. Funct. Anal. **62** (2), 202–220 (1985). doi:10.1016/0022-1236(85)90003-5
38. Tikhonov, O.E.: Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals. Positivity **9**(2), 259–264 (2005). doi:10.1007/s11117-005-2711-1
39. Bikchentaev, A.M.: On a property of L_p-spaces on semifinite von Neumann algebras. Math. Notes **64**(1-2), 159–163 (1998). doi:10.1007/bf02310299
40. Bikchentaev, A.M.: Metrics on projections of the von Neumann algebra associated with tracial functionals. Sib. Math. J. **60**(6), 952–956 (2019). doi:10.1134/S003744661906003X
41. Bikchentaev, A.M., Abed, S.A.: Projections and traces on von Neumann algebras. Lobachevskii J. Math. **40**(9), 1260–1267 (2019). doi:10.1134/S1995080219090051
42. Bikchentaev, A.M.: Inequalities for determinants and characterization of the trace. Sib. Math. J. **61**(2), 248–254 (2020). doi:10.1134/S0037446620020068
43. Bikchentaev, A.M., Sherstnev, A.N.: Studies on Noncommutative Measure Theory in Kazan University (1968-2018). Internat. J. Theor. Phys. **60**(2), 585–596 (2021). doi:10.1007/s10773-019-04156-x

44. Hoa, D.T., Tikhonov, O.E.: Weighted trace inequalities of monotonicity. Lobachevskii J. Math. **26**, 63–67 (2007).