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Review Article

Spectral Approach to Numerical Integration of the GKP or Belashov Class Equations in the Problems of Nonlinear Wave Dynamics Simulation

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Abstract

The original method for numerical integration of the generalized Kadomtsev-Petviashvili (KP) equation which includes the term proportional to the fifth derivative (so called the Belashov-Karpman equation) which enables to study the solution's evolution and the multidimensional soliton's interaction's dynamics is presented. This method is rather simple in its computer realization and not such cumbersome comparatively with other known methods for the numerical integration of the different equations of the KP-class. In the paper we consider spectral approach to the numerical integration of the equations of the KP-class describing the dynamics of the ion-acoustic and magnetosonic waves in a plasma on the basis of the generalized KP equation. The method is rather simple in its computer realization and doesn't such cumbersome comparatively with other methods for the numerical integration of the differential equations of the KP-class, and very effective, so it doesn't require big time and memory expenditures. This approach was first used by us for study of some problems of nonlinear evolution of the fast magnetosonic (FMS) wave beam in magnetized plasma and can be generalized easily for all equations of the KP class.

Keywords: Dynamics; Fast Magnetosonic (FMS); KP-Class

Introduction

The 2D and 3D nonlinear waves propagating in dispersive media (such as ion-acoustic (IA) and magnetosonic (MS) waves in space plasma, and also in hydrosphere, atmosphere and ionosphere) are described by the Belashov equations' class [1,2].

$$\partial_t u + \alpha u \partial_x u + \sum_{l-1}^{L} \beta_l \partial_x^{2l+1} u = \Re \quad \text{------------(1)}$$

Where $u = u(t, x, \mathbf{r}_\perp)$ is a function defining the wave field, and $\Re = \Re[u]$ is a some linear functional of u. The form of right-hand side of eq. (1) depends on the wave properties of medium and the dispersion sign, and the value of L is defined by the dispersion character. For example, in the cases of IA waves propagating in isotropic plasma and MS waves propagating in plasma near transverse direction to magnetic field when the dispersion law has form.

$$\omega \approx c_0 k \left[1 \pm \mathbf{k}_\perp^2 / 2k_x^2 + \sum_{l=1}^{L} (-1)^l \delta_l^2 k_x^{2l} \right]$$

Where signs '+' and '−' correspond to first and second cases, respectively, c_0 is a phase velocity of oscillations at $|k| \to 0$, and δ_l are the dispersion "scales", functional $\Re$ has the form $\Re = \kappa \nabla_\perp w$, $\partial_x w = \nabla_\perp u$. In this case for L=1, 2 eq. (1) is the Kadomtsev-Petviashvili (KP) or the generalized KP equation, respectively:

$$\partial_x \left[\partial_t u + \alpha u \partial_x u + \sum_{l-1}^{L} \beta_l \partial_x^{2l+1} u \right] = \kappa \Delta_\perp u, \quad \Delta_\perp = \partial_y^2 + \partial_z^2 \text{ , ------------ (2)}$$

And it can have the 1D, 2D and 3D wave solutions localized in space dependently on L value and signs of coefficients β_l and κ. In case of strong anisotropic media functional $\Re = \kappa \Delta_\perp \partial_x u$, and eq. (1) known as Zakharov-Kuznetsov equation has the soliton-like solutions too.

The 2D KP equation (L = 1, $\partial_z = 0$) can be integrated analytically using the IST method, but this method allows to obtain the exact solution only under certain initial conditions [2]. The technique of IST method for the integration of the 3D KP equation and the 2D and 3D generalized eq. (2) with L = 2 is not developed now. Therefore, development of the numerical technique for integration of the eq. (1) class is of indubitable interest for the nonlinear plasma physics.

The hard original method for numerical integration of the KP equation (eq. (2) with L = 1) was proposed in [3]. However, it doesn't enable to study the solution's evolution at initial stage and to consider the soliton interaction's dynamics. There are also some other methods for the numerical integration of the different equations of class (1) (see, for example, review in [2]), but they are rather cumbersome.

In this paper we consider spectral approach for the numerical integration of the equations of class (1) describing the dynamics of IA and MS waves in a plasma on the basis of eq. (2) with L=2 which is rather simple and very effective and doesn't require big time and memory expenditures. This approach was first used for study of some problems of nonlinear evolution of the FMS wave beam in magnetized plasma [4] and can be generalized easily for all equations of class (1).

Spectral approach

Performing the Fourier transform F

$$U(t,\xi,\zeta,\eta) = (2\pi)^{-3}\iiint u(t,x,y,z)e^{-i(x\xi+y\zeta+z\eta)}dxdydz, \quad \text{-----------} (3)$$
$$u(t,x,y,z) = \iiint U(t,\xi,\zeta,\eta)e^{i(x\xi+y\zeta+z\eta)}d\xi d\zeta d\eta.$$

We write eq. (2) in the form

$$\partial_t U + fW + gU = 0 \quad \text{---------------}(4)$$

Where $f = i\alpha\xi/2$, $W = U*U$, and assumed that $f' = -if$ $g' = -ig$, $U = X + iY$, $W = Z_1 + iZ_2$, we rewrite equation as the set

$$\partial_t X - f' Z_2 - g'Y = 0,$$
$$\partial_t Y + f' Z_1 + g' X = 0. \quad \text{-----------------} (5)$$

The X, Y values at t=0 are defined by the Fourier transform of the initial condition of the Cauchy problem u (0, x , y , z) = $\psi(x, y, z)$ for eq. (2) and $W|_{t=0}$ = F [ψ^2]. The convolution W values on the next temporal layers can be obtained by using the convolution theorem according to the scheme

$$\{U\} \to \{F[U]\} \to \{V\}=\{F[U]F[U]\} \to \{W\}=\{F^{-1}[V]\}.$$

Let us note that the coefficient g' has a singularity on the plane of $\xi = 0$, that must be accounted for in the computations. But, one can show [2] that the Fourier-image U (t,0,0,0) = F [u (t,x,y,z)] = 0 if function u satisfies eq. (2).

The finite sizes of the numerical integration region lead to the spectral components' infiltration in the spectrum of function u (that are so-called Gibbs oscillations), this infiltration is connected with the existence of discontinuities of the function u periodic continuation at the integration region boundaries. Therefore, to compute the Fourier transform numerically, with the purpose of decreasing of discontinuity order it should be introducing to (3) the multiplicative weight function that coordinate as much of derivatives of function u weighted. Approximating the integrals in (3) by finite differences one can write the transformation (3) in the form

$$U_{\sigma_1\sigma_2\sigma_3}(t,\xi_p,\zeta_q,\eta_r) = \frac{1}{MNK}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{K-1}\sigma_1(m\Delta x)\sigma_2(n\Delta y)\sigma_3(k\Delta z)\times$$
$$\times u(t,m\Delta x,n\Delta y,k\Delta z)\exp\left[-i\left(\xi_p m\Delta x + \zeta_q n\Delta y + \eta_r k\Delta z\right)\right],$$

Where

$$\sigma_1(m\Delta x) = \sigma_1[(M-m)\Delta x],$$

$$\sigma_2(n\Delta y) = \sigma_2[(N-n)\Delta y],$$

$$\sigma_3(k\Delta z) = \sigma_3[(K-k)\Delta z];$$

$$\xi_p = 2\pi p / M\Delta x,\ \zeta_q = 2\pi q / N\Delta y,\ \eta_r = 2\pi r / K\Delta z,$$

p = 0,1,2, ..., M -1; q = 0,1,2, ..., N -1; r = 0,1,2, ..., K -1.

The periodic continuation is a continuous function up to the differential coefficients of higher order if only it is a success to obtain the graded tending of function u to zero on boundaries under the weight functions giving rather small perturbation of the spectrum in the integration region's center.

The different windows applied widely in the spectral analysis can be used as weight functions $\sigma_{1,2,3}$ [1]. The numerical experiments showed that the weight functions in the form of Blackman-Harris windows

$$\sigma(j) = c_0 - c_1\cos(2\pi j/N) + c_2\cos(4\pi j/N) - c_3\cos(6\pi j/N),$$

$j = 0, 1, 2, ..., N - 1.$

Were the most acceptable for the equation (1) class [1,2].

The set (5) can be solved easily by the Runge-Kutta method for the difference equations' set corresponding (5), namely:

$$X^{n+1} - X^n = \Delta t\left(k_{11} - \frac{1}{2\gamma}(k_{11} - k_{12})\right), \quad Y^{n+1} - Y^n = -\Delta t\left(k_{21} - \frac{1}{2\gamma}(k_{21} - k_{22})\right) \text{------------ (6)}$$

Where

$$k_{11} = f' Z_2^n + g' Y^n, \quad k_{12} = k_{11}(1 + \gamma\xi), \; k_{21} = f' Z_1^n + g' X^n, \; k_{22} = k_{21}(1 + \gamma\xi).$$

The convergence of the difference problem (6) to the solution of the set (5) was proved in [2].

Scheme testing

Scheme (6) has been tested in two stages. At first, the scheme's characteristics related to integration on x were investigated, coefficient κ in the right-hand side of eq. (2) was supposed as equal to zero for that. At this, eq. (2) was being transformed to the KdV equation (L=1) or the generalized KdV equation (L=2). The initial condition has been chosen in form of the KdV exact solution.

$$u(0,x) = (3v/\alpha)\,\text{sech}^2\left[\left(v^{1/2}/2\beta\right)(x - x_0)\right], \quad v = \alpha = 6, \quad \beta = 1$$

In both cases. In case L=1 the accuracy control has been carried out comparing the numerical solution with the exact analytic one for all time layers. At this, the mean value relative declination ε and the mean square declination s, namely:

$$\varepsilon = \left| u_\tau^{num} - u_\tau^{ex} \right| / u_\tau^{ex}, \; s = \left[(MNK)^{-1} \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k=1}^{K} \left| (u_{mnk}^{num})^2 - (u_{mnk}^{ex})^2 \right| \right]^{1/2}$$

Were being computed. For example, for t=1 it was obtained that ε and s were no more than 10^{-3} and 10^{-4}, respectively, and it is more better than, for example, the results obtained for the schemes proposed in [5] for the KdV equation. In case L=2 the accuracy control has been carried out in comparison with the results obtained by another methods based on explicit and implicit difference schemes [2], and it was obtained that the result's declinations were rather small.

On the second stage the derivatives on y and z in the right-hand side of eq. (2) were being "switched", and the scheme was tested on the exact KP equation solution for $\partial_{\ddot{y}} = 0$ and with initial condition.

$$u(0,x,y,z) = 2\partial_x^2 \ln\left\{ 4(\nu + \nu^*)^{-2} + \left| x - i\nu(y+z) - \phi - 3\nu^2 t \right|^2 \right\}$$

For $\partial_{\ddot{y}} \neq 0$. On a level with control parameters ε and s, we checked the conservation of integrals which are invariants for the KP equation (eq. (2) with L=1) in case when $\partial_{\ddot{y}} = 0$:

$$\Im_1 = \int u d\mathbf{r}, \; \Im_1 = \int u d\mathbf{r}, \; \Im_3 = \int \left[\tfrac{1}{2}\beta_1(\partial_x u)^2 - \tfrac{1}{2}\beta_2(\partial_x^2 u)^2 + \tfrac{1}{2}(\nabla_\perp \partial_x w)^2 - u^3 \right] d\mathbf{r}.$$

At this, these integrals were being computed with $O(h^4)$ approximation using the Newton-Kothes formulae on each time layer. Our calculations showed that the accuracy of the results is rather high comparatively with another methods considered in [2], and the spectral scheme is rather economical on its temporal characteristics because the solution accuracy doesn't depend much on the choice of both space and time steps. Moreover, the weight functions σ introduce automatically the effective absorption near the region boundaries when the region of the disturbance localization is approached boundary with evolution.

Some Simulation Results, Discussion and Conclusion

The technique considered above was used for simulation of the dynamics of 2D and 3D nonlinear IA and FMS waves in a plasma. In this case the coefficients of eq. (2) are the following:

(a) For the IA waves [1]

$$\alpha = \tfrac{3}{2} c_0 / n_i, \quad \kappa = c_0/2, \quad \beta_1 = c_0 D^2/2, \quad \beta_2 = 0,$$
$$c_0 = \sqrt{T_e/M}, \quad D^2 = T/4\pi n_0 e^2;$$

(b) For the FMS waves [4]

$$\alpha = \tfrac{3}{2} v_A \sin\theta, \quad \kappa = -v_A/2, \; \beta_1 = v_A \frac{c^2}{2\omega_{0i}^2}\left(\frac{m}{M} - \cot^2\theta\right),$$

$$\beta_2 = v_A \frac{c^4}{8\omega_{0i}^4}\left[3\left(\frac{m}{M} - \cot^2\theta\right)^2 - 4\cot^4\theta\left(1 + \cot^2\theta\right)\right]$$

Where D is Debye radius, v_A is Alfven velocity, ω_{0i} is the ion Lengmure frequency, m and M are the masses of electrons and ions, respectively, and θ is the angle between wave vector $\mathbf{k}_x$ and magnetic field B.

In case (a) the dispersion in eq. (2) is negative and the solution in 2D space has form of 1D IA soliton (Figure 1).

Propagating in isotropic plasma with $v = \mathrm{const}$ and integrals $\Im_1, \Im_2, \Im_3 = \mathrm{const}$. That corresponds the

analytical results for the KP equation with negative dispersion obtained in [6] by use of Krylov-Bogolubov method and in [7] by use of the IST technique.

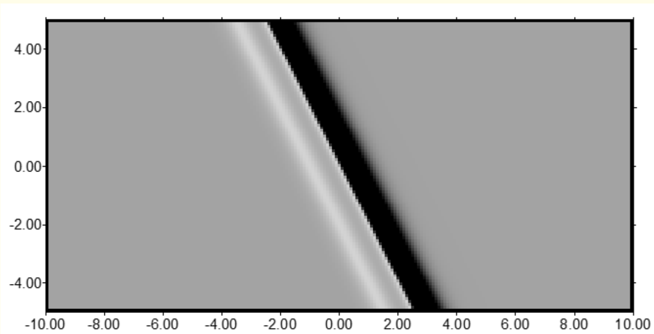

Figure 1: General view of 2D IA soliton of eq. (2) with L=1, α=6, β_1=1, β_2=0.

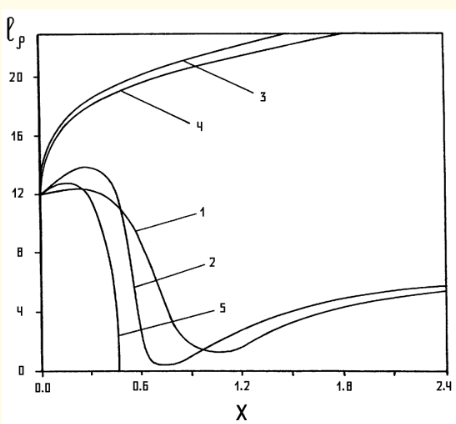

Figure 2: Changing cross-section of beam propagating along x-axis: 1- λ =1 , ε =1.34; 2- λ =1, ε =2.24 ; 3- λ =-1 ε =1.34 ; 4- λ =-1, ε =-1.34 ; 5- λ =0, ε =-1.34.

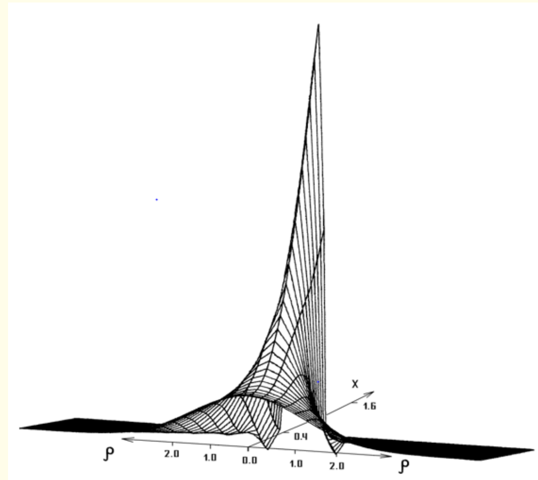

Figure 3: Central part of the FMS wave beam h(x, ρ) for its intensity at h_0 =4 on the sub-focusing stage of evolution: λ =1, ε =1.34.

The case (b) is more complicate because the dispersion sign is defined by correlation of signs of dispersive coefficients β_1 and $\beta_2 \neq 0$. Therefore, the form of the FMS wave essentially depends on angle θ between vector k and magnetic field B. The results obtained were discussed in detail in [2,4], and we show here only the most interest case when in 3D case the nonlinear stabilization of the FMS wave beam propagating in plasma near the cone $|\pi/2-\theta| \leq (m/M)^{1/2}$ takes place after the stages of its initial sub-focusing and nonlinear saturation (Figure 2, 3) [2,4]. In this case we studied the problem for eq. (2) with L=2 rewritten, by use of transition to new variables $x \to -st$, $y \to -s\kappa^{1/2} y$, $z \to -s\kappa^{1/2} z$, $t \to sx$, $h \to -(6/\alpha)h$, $s = |\gamma_2|^{1/4}$, $\kappa = v_A/2$, in the form $\partial_t(\partial_x h + 6h\partial_t h - \varepsilon\partial_t^3 h - \lambda\partial_t^5 h) = \Delta_\perp h$ (we assumed that $\Delta_\perp = \partial_\rho^2 + (1/\rho)\partial_\rho$) and describing the propagation of the FMS wave beam along the x-axis from boundary x=0 with initial condition $h_0 = h(t,0,\rho) = \cos(t\pi)\exp(-\rho^2)$.

The results presented show that spectral approach considered above can be successively used for study of the problems of 2D and 3D nonlinear wave and soliton dynamics in space plasma and other dispersive media.

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