Acta Scientific Applied Physics

Volume 2 Issue 1 January 2022

Research Article

Structure, Stability and Dynamics of Multidimensional Solitons in Plasma

Vasily Yu Belashov*

Institute of Physics, Kazan Federal University, Kazan, Russia

***Corresponding Author:** Vasily Yu Belashov, Institute of Physics, Kazan Federal University, Kazan, Russia.

Received: October 07, 2021

Published: December 10, 2021

Abstract

The formation, structure, stability and dynamics of the multidimensional solitons forming on the low-frequency branch of oscillations in plasma for cases when $\beta \equiv 4\pi nT/B^2 << 1$ and $\beta > 1$ are studied. In first case, for $\omega < \omega_B = eB/Mc, k\lambda_D << 1$ the FMS waves are excited, and their dynamics under conditions $k_x^2 > k_\perp^2$, $v_x < c_A$ near the cone of $\theta = \arctan(M/m)^{1/2}$, is described by the GKP equation for magnetic field $h = B_\sim / B$ with due account of the high order dispersive correction defined by values of plasma parameters and angle $\theta = (\mathbf{B}, \mathbf{k})$. In another case, the dynamics of the finite-amplitude Alfvén waves propagating near-to-parallel to **B** is described by the 3-DNLS equation for $h = (B_y + B_z)/2B|1-\beta|$. To study the stability of multidimensional solitons in both cases the method of investigation of the Hamiltonian bounding with deformation conserving momentum by solving the variation problem is used. To study evolution of solitons and their collision dynamics the proper equations are integrated numerically using the codes specially developed. As a result, it was obtained that in both cases for single solitons on a level with wave spreading and collapse the formation of multidimensional solitons may be observed. These results may be also interpreted in terms of self-focusing phenomenon for the FMS and Alfvén waves' beam as stationary beam formation, scattering and self-focusing of beam. The soliton collisions on a level with known elastic interaction can lead to formation of complex structures including the multisoliton bound states. For all cases the problem of soliton dynamics including all stages of their evolution and collision dynamics is investigated in detail.

Keywords: Plasma; GKP Equation; 3-DNLS Equation; Dynamics; 2D Solutions; 3D Solutions; Structure; Stability; Multidimensional Solitons

Introduction

Basic equations

In this paper, following the technique developed in [1,2], we study the formation, structure, stability and dynamics of the multidimensional solitons forming on the low-frequency branch of oscillations in plasma. for cases $\beta \equiv 4\pi nT/B^2 << 1$ and $\beta > 1$. These oscillations are described by the BK[1] equation.

$$\partial_t u + \hat{A}(t,u)u = f, \qquad f = \kappa \int_{-\infty}^{x} \Delta_\perp u\, dx, \qquad \Delta_\perp = \partial_y^2 + \partial_z^2$$
---------- (1)

Which with operator

$$\hat{A}(t,u) = \alpha u \partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3)$$
----------------(2)

Turns into the GKP equation class and in case when $\beta \equiv 4\pi nT/B^2 << 1$ for $\omega < \omega_B = eB/Mc, k\lambda_D << 1$ describes propagation of the fast

[1]Belashov-Karpman (BK) equation

magnetosonic (FMS) waves in magnetized plasma with $k_x^2 > k_\perp^2$, $v_x < c_A$ near the cone of $= \arctan (M/m)^{1}$, [3]. In this case function u has a sense the dimensionless amplitude of the magnetic field of the wave, $h = B_\sim / B$, the coefficients at the terms describing nonlinearity, dissipation and dispersion effects, respectively, are defined by values of plasma parameters and angle $\theta = (\mathbf{B}, \mathbf{k})$. In opposite case, when operator

$$\hat{A}(t,u) = 3s\,|\,p|^2\, u^2 \partial_x - \partial_x^2 (i\lambda + \nu)$$ ------------ (3)

Eq. (1) turns into the 3-dimensional derivative nonlinear Schrödinger (3-DNLS) equation class and in case when $\beta > 1$ describes the dynamics of the finite-amplitude Alfvén waves propagating near-to-parallel to $\mathbf{B}$ for $u = h = (B_y + B_z)/2B\,|1-\beta|$, $\mathbf{h} = \mathbf{B}_\perp / B_0$ where $p = (1 + \dot{e})$, and e is "an eccentricity" of the polarization ellipse of the Alfvén wave [4]. The upper and lower signs of $_\smile = \pm$ correspond to the right and left circularly polarized wave, respectively; sign of nonlinearity is accounted by coefficient $s = \mathrm{sgn}(1 - p) = \pm 1$ in nonlinear term; $\kappa = -r_A/2$, $r_A = v_A / \omega_{0i}$.

The sets of eqs. (1), (2) and (1), (3) are not completely integrable ones, and the well-known IST is not applicable for their solution. Therefore, in the analytical study of these sets we can use also the methods for investigation of stability and asymptotics of proper multidimensional solutions, and qualitative analysis of appropriate dynamic systems. To study evolution of solitons and their collision dynamics the proper equations should be integrated numerically using the special simulation codes. Let us consider these problems separately.

Stability of 2D and 3D solutions

In this paragraph we will consider the analytical approaches and the obtained with their help results of study of the problem of stability of the multidimensional solitons and nonlinear wave packets, which under the neglect of dissipative effects follow the GKP-class equations in form (1), (2) and the 3-DNLS equation in form (1), (3) with $\nu = 0$. At first, for the whole diapason of the dispersion coefficients' change we will give the estimations and formulate the sufficient conditions of stability of the GKP equation solutions in the 2D and 3D geometry on the basis of transformational properties of the Hamiltonian. Further, we will consider the same problem for the 3-DNLS equation in the 3D geometry.

GKP equation

To study the solutions stability, performing some coordinate transformation, rewrite eqs. (1), (2) in the Hamiltonian form

$$\partial_t u = \partial_x (\delta \mathrm{H} / \delta u)$$ ------------ (4)

With Hamiltonian

$$\mathrm{H} = \int \left[-\frac{\varepsilon}{2}(\partial_x u)^2 + \frac{\lambda}{2}\left(\partial_x^2 u\right)^2 + \frac{1}{2}(\nabla_\perp \partial_x v)^2 - u^3 \right] d\mathbf{r}$$ --------------- (5)

Where $\partial_x^2 v = u$, $\varepsilon = \beta\,|\,\gamma\,|^{-1/2}$, $\lambda = \mathrm{sgn}\,\gamma$. The stationary solutions of Eq. (4) are defined from the variation problem, δ (H + vP_x) = 0 where $P_x = \frac{1}{2}\int u^2 d\mathbf{r}$ is the momentum projection onto the x axis, v is a Lagrange's factor, which illustrates the fact that all finite solutions of Eq. (4) are the stationary points of the Hamiltonian for fixed P_x.

Let us consider the problem of stability. In conformity with the Lyapunov's theorem, in the dynamical system the stationary points which answer maximum or minimum of H are absolutely stable. If the present extremum is local that the locally stable solutions are possible. The unstable states correspond to monotonous dependence of H on its variables, i.e. to cases when the stationary point is a saddle point. In conformity with said above, it is needed to prove the Hamiltonian H boundedness (from below) for fixed P_x.

Let us consider in real vector space R the scale transformations $u(x, \mathbf{r}_\perp) \to \zeta^{-1/2} \eta^{(1-d)/2} u(x/\zeta, \mathbf{r}_\perp/\eta)$ (where d is the problem dimension, and $\zeta, \eta \in \mathrm{R}$) conserving the momentum projection P_x. The Hamiltonian as a function of parameters ζ, η assumes a form

$$\mathrm{H}(\zeta, \eta) = a\zeta^{-2} + b\zeta^2 \eta^{-2} - c\zeta^{-1/2} \eta^{(1-d)/2} + e\zeta^{-4}$$ ----------- (6)

Where

$a = -(\varepsilon/2)\int (\partial_x u)^2 d\mathbf{r}$, $b = (1/2)\int (\nabla_\perp \partial_x v)^2 d\mathbf{r}$, $c = \int u^3 d\mathbf{r}$, $e = (\lambda/2)\int (\partial_x^2 u)^2 d\mathbf{r}$.

The common solving of the equalities having sense of the necessary conditions of the Hamiltonian extremum existence and the inequalities that have sense of the sufficient conditions of the H (local) minimum existence enables us to obtain the following results.

In 2D case [d=2 in expression (6)] one can obtain that for $\lambda = 1, \varepsilon \le 0$ the Hamiltonian at fixed P_x is bounded from below, and, hence, the 2D solitons are absolutely stable in this case. In the cases $\lambda = 1, \varepsilon > 0$ and $\lambda = -1, \varepsilon < 0$ the Hamiltonian H has a local minima, and Eq. (4) may have the locally stable solutions for some values of parameters (see [3] for details). All another cases correspond to unstable 2D solutions.

In 3D case the solution of appropriate set of equalities and inequalities enables to obtain that the absolutely stable 3D solutions take place at $\lambda = 1, \varepsilon > 0$, and the locally stable solutions may be observed at $\lambda = 1, \varepsilon \le 0$ if condition on integral coefficients $b^2 e / c^4 < 9/512$ is satisfied.

This is a noteworthy fact that the GKP equation accounting, unlike the usual KP equation, the next order dispersive correction has stable 3D solutions. The application of this analysis to the problem of the FMS waves beam's propagation in magnetized plasma enables us to prove [3], for example, that the 3D beam propagating at θ angle to magnetic field doesn't focused and becomes stationary and stable in the cone of $\theta < \arctan (M/m)^{1/2}$ when inequality $(m/M - \cot^2\theta)^2[\cot^4\theta(1+\cot^2\theta)]^{-1} > 4/3$ is satisfied. Let us note also that obtained here results give us the possibility to interpret correctly some our numerical and theoretical results on the dynamics of the internal gravity waves' solitons induced by the pulse-type sources which propagate at heights of the ionosphere F region [5] from the point of view of such solitons stability.

DNLS equation

Let us rewrite the 3-DNLS equation in form (4) performing formal change $u \to h$ with the Hamiltonian [6]

$$\mathrm{H} = \int_{-\infty}^{\infty}\left[\frac{1}{2}|h|^4 + \lambda s h h^* \partial_x \varphi + \frac{1}{2}\kappa(\nabla_\perp \partial_x w)^2\right] d\mathbf{r}, \quad \partial_x^2 w = h, \ \varphi = \arg(h)$$

-------- (7)

And, by analogy with above (see also [7]), will investigate the boundedness of H under its deformations conserving $P_x = \frac{1}{2}\int |h|^2\, d\mathbf{r}$, when variation equation considered above takes place. Considering in complex vector space C the scale transformations $h(x, \mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{-1} h(x/\zeta, \mathbf{r}_\perp/\eta)$ ($\zeta, \eta \in$ C) conserving P_x. The Hamiltonian as a function of parameters ζ, η takes form

$$\mathrm{H}(\zeta,\eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} + c\zeta^2\eta^{-2}$$ ----------- (8)

Where

$a = (1/2)\int |h|^4\, d\mathbf{r}, \quad b = \lambda s \int \boldsymbol{h}^* \partial_x \varphi\, d\mathbf{r}, \quad c = (\sigma/2)\int (\nabla_\perp \partial_x w)^2 d\mathbf{r}$.

Solving, by analogy with the GKP equation, the extremum problem for functional (8) one can obtain that the Hamiltonian (7) is bounded from below,

$$\mathrm{H} > -3bd/(1+2d^2) \qquad b < 0$$,-------------(9)

If condition $ac^{-1} < d = \left(2\sqrt{2}\right)^{-1}\sqrt{13 + \sqrt{185}}$ is carried out, and in this case the 3D solutions of the 3-DNLS equation are stable, and they are unstable in opposite case, $ac^{-1} \ge d,\ b < 0$. Condition $b < 0$ corresponds to the right circularly polarized wave propagating in plasma with p=4πnT/B²>1, i.e. when $\lambda = 1, s = -1$ in the eqs. (1), (3), and to the left circularly polarized wave when $\lambda = -1,\ s = 1$. But it is necessary to note that the sign change $\lambda = 1 \to -1,\ s = -1 \to 1$ is equivalent to change $t \to -t, \kappa \to -\kappa$ and for negative κ the Hamiltonian becomes negative in the area "occupied" by the 3D wave weakly limited in the $\mathbf{k}_\perp$-direction, in this case condition (9) is not satisfied. Change of sign of b to positive [when $\lambda = 1,\ s = 1$ or $\lambda = -1,\ s = -1$ in eqs. (1), (3)] is equivalent to analytical extension of solution from real y, z to pure imaginary values: $y \to -iy$, $z \to -iz$ and, therefore, equivalent to change of sign of κ in basic equations. In this case instead of inequality (9) the opposite inequality will take place. From the physical point of view, it means that if this opposite inequality is satisfied, the right polarized waves with positive nonlinearity and the left polarized waves with negative nonlinearity will be stable. Note, that in particular case when $\kappa = 0$ in eqs. (1), (3) (1D approximation), with using of accepted approach, instead of inequality (9) and opposite one, it is easy to obtain the conditions $\mathrm{H} > 0$ and $\mathrm{H} < 0$, respectively, that is completely in an agreement with the results obtained in [8] for the 1-DNLS equation.

So, the analysis of the transformation properties of the Hamiltonian of the 3-DNLS equation enable us the ranges of values of the coefficients and H (that has a sense of energy) corresponding with the stable and unstable 3D solutions.

Structure and dynamics of 2D and 3D solutions

The structure, evolution and the interaction dynamics of multidimensional solitons were studied numerically using the numerical codes especially developed for the classes of eqs. (1), (2) and (1), (3) [3,9-12]. Let us now consider the results of numerical simulation for the GKP and 3-DNLS equations separately.

GKP equation

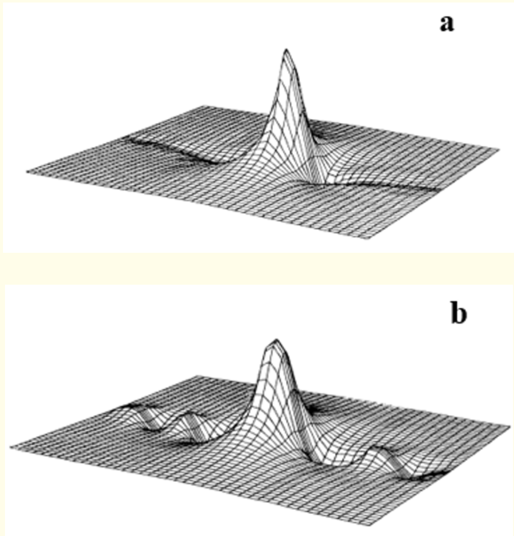

Figure 1: General view of 2D soliton of eqs. (1), (2) with at ν=0: (a) λ=1, ε=-0.6 (t=0.2); (b) λ=1, ε=3.16 (t=0.5).

Let us first consider the results for 2D case when $\partial_z = 0$ in Eq. (1). Initial conditions have been taken in form of exact 2D soliton solution of usual 2D KP equation [11]. The results obtained are the following. For $\lambda = 1$, $\varepsilon \leq 0$ we observed formation of stable lump solutions with asymptotics which is very close to that of algebraic KP soliton (figure 1, a). In case $\varepsilon > 0$ the formation of solitons oscillating in the direction of propagation and monotonic in transverse direction was observed (figure 1,b), their asymptotics were investigated in [13] in detail.

For $\lambda = -1$ and $\varepsilon \geq 0$, and small $\varepsilon < 0$ the evolution of initial condition leads to spreading of wave packet which is formed at first stage. At big $\varepsilon < 0$, however, we observed the formation of stable soliton solution with oscillating asymptotics, that corresponds to above mentioned analytical results. It is interesting that the 2D soliton interaction dynamics is not trivial for the GKP equation unlike usual KP equation [11,13]. So, for example, for $\lambda = 1$, $\varepsilon > 0$ the formation of stable two-soliton structure (so-called "bisoliton") may be observed as a final result of interaction of two initial pulses (Figure 2). Let us note that for all cases the analysis of the H deformations on numerical solutions confirmed the stability of solutions considered above.

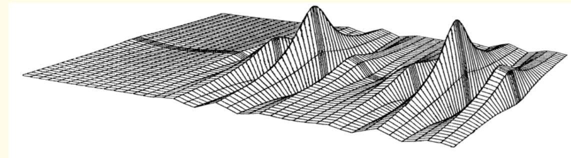

Figure 2: 2D bisoliton solution formed from initial pulses with $u_1(0)$=1.35, $u_2(0)$=1.3, △x(0)=6 for λ=1, ε=1.9, ν=0 (t=1.3).

In 3D case we established three stages of evolution of forming soliton-like structures [12]. In first stage the self-focusing type instability is developed (Figure 3), and the pulse "wings" fall behind its center and the amplitude increases sharply. Then, in the instability saturation stage, the equation term being proportional to the fifth derivative begins to play a dominant role owing to decreasing of the pulse characteristic dimensions. In the next stage the defocusing of wave field is observed. At this, for $\varepsilon \leq 0$ it leads to the pulse spreading, and for $\varepsilon > 0$ the evolution ends with the 3D soliton structure formation (Figure 3). For $\lambda = -1$ with any values of ε the solutions are the 3D wave packets which spread in due course. These results are confirmed by analysis of H bounding with its deformation on numerical solutions. So, unlike the 3D KP equation, there are not the collapsing solutions for the FMS waves propagating near the cone of $\theta < \arctan\,(M/m)^{1/2}$ but the stable 3D solutions may be observed. Let us note that more detail results on structure classification of 2D and 3D solutions may be found in [13], and the applications to the FMS waves including the FMS wave beam dynamics are discussed in detail in [3,13].

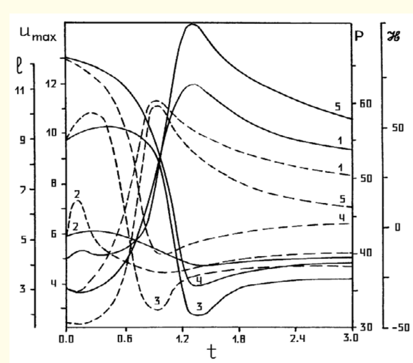

Figure 3: Changing with time of the solution parameters, momentum and H of the system for λ= 1 (solid lines - ε=-0.45, dashed lines - ε =1.34): (1) u_{max} ; (2) lx; (3) l_x ; (4) P; (5) H.

DNLS equation

For simulation the initial conditions have been chosen in two different forms: soliton-like solution, and modulated “plane” wave (see [4,10]). The results for different signs of integral parameter b and different initial values of H which were given by initial amplitudes of wave and its characteristic lengths along the axes are following. For $\lambda = 1$ and $s = -1$ with big $\kappa > 0$, ands initial pulse weakly bounded in the transverse direction (when Ineq. (9) is satisfied) we observed the formation of stable 3D soliton-like solution (Figure 4). For opposite signs of λ and s [that is equivalent to change $t \to -t$, $\kappa \to -\kappa$ in Eq. (1)] the Hamiltonian becomes negative, and the 3D wave is spread. For $\lambda = 1$ and $s = -1$ with small $\kappa > 0$, and initial pulse rather strong bounded in the transverse direction, we have observed formation of the 3D collapsing solutions. This effect is typical for all nonlinear systems where there are both non-limitation of H for fixed “junior” integrals and positive-defined quadratic terms in (7). The series of numerical experiments for $b > 0$ when $\lambda = 1$, $s = 1$ and $\lambda = -1$, $s = -1$ in the 3-DNLS equation showed that the initial 3D pulse is unstable and spreads with time. These results are well confirmed by the analysis of H bounding on the numerical solutions.

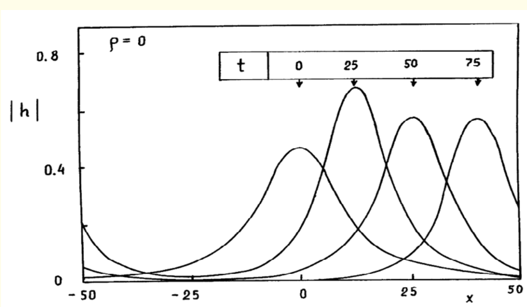

Figure 4: Evolution of a 3D right circularly polarized nonlinear pulse for λ=1, s=-1, κ =1; H>-3bd/(1+2d^2)>0.

Conclusion

In conclusion, we have considered two types of the low-frequency oscillations in plasma with $\beta < 1$ and $\beta > 1$ which correspond to two types of waves and can lead to the formation of the multidimensional solitary wave structures. As a result, we have obtained that for the FMS waves on a level with sound scattering the 2D and 3D soliton formation may be observed. In particular, in 3D case for the FMS wave beam having the close angular distribution the stationary propagation may be observed as a result of nonlinear beam stabilization. In case of Alfvén waves propagating along magnetic field we have obtained that the 3D stable solutions may be observed on a level with 3D spreading and collapsing ones. These results may be also interpreted in terms of self-focusing phenomenon for the Alfvén waves’ beam as stationary beam formation, scattering and self-focusing of beam. Let us note that we observed the dynamics of the Alfvén waves’ beam propagating in plasma with $\beta > 1$ to magnetic field at angles near 0° looking like the dynamics of the FMS wave beam propagating in plasma with $\beta < 1$ to magnetic field at angles near $\pi / 2$.

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

Bibliography

1. Belashov VYu. “On stability of two-dimensional and three-dimensional solitons in weakly dispersive media”. Dokl. Acad. Nauk SSSR, 320 (1991): 85-89.
2. Belashov VYu and Vladimirov SV. “Dynamics and stability of multidimensional solitons in a plasma”. Plasma Physics: 11th Int. Congress on Plasma Physics: ICPP 2002. Ed. I.S. Falconer, R.L. Dewar, and J. Khachan. American Institute of Physics, Melville, New York, 2003. AIP Conf. Proceedings (2003): 669, 670-674.
3. Belashov VYu. “Nonlinear effects for FMS waves propagating in magnetized plasma”. *Plasma Physics and Controlled Fusion* 36 (1994): 1661-1669.
4. Belashov VYu. “Structure and evolution of two-dimensional electron density solitary waves caused by internal gravity waves solitons at heights of ionosphere F layer”. URSI and STEP/GAPS Workshop, Warsaw, Space Research Center, Poland 9.3 (1995).
5. Belashov VYu. “Solitary electron density waves induced by the IGW’s solitons in the ionosphere”. Proc. 1989 Intern. Symp. on EMC, Nagoya, Japan, New York, IEEE, Inc., 1 (1989): 228-231.
6. Belashov VYu. “2D and 3D Solitons in Plasma: Structure, Stability, Dynamics. Double layers. Potential Formation and Related Nonlinear Phenomena in Plasmas”. Ed. Sendai «Plasma Forum», Tohoku Univ. World Scientific Publishing Co. Pte. Ltd. Singapore (1997): 337-342.

7. Belashov VYu. "Dynamics of the 3D FMS Waves Beam in Plasma with Stochastic Fluctuations of Magnetic Field". Proc. ICPP 1996, Contributed Papers, Nagoya, Japan, 1 (1997): 954-957.
8. Dawson SP and Fontán CF. "Soliton decay of nonlinear Alfven waves: numerical studies". *Physics of Fluids* 31.1 (1988): 83-89.
9. Belashov VYu. "Numerical Study of Dynamics of 3D Ion-acoustic and FMS Nonlinear Waves in Plasma Using Spectral Approach". Proc. ISSS-5, Kyoto, Japan, RASC, Kyoto Univ. (1997): 118-122.
10. Belashov VYu and Suschenko OV. "Numerical integration of the 3-DNLS class equations". NEISRI FER RAS, Dep. in VINITI 02.08.00, N 2163-B00, Magadan (2000).
11. Karpman VI and Belashov VYu. "Dynamics of two-dimensional solitons in weakly dispersive media". *Physics Letters* 154A (1991): 3-4, 131-139.
12. Karpman VI and Belashov VYu. "Evolution of three-dimensional nonlinear pulses in weakly dispersive media". *Physics Letters* 154A (1991): 3-4, 140-144.
13. Belashov VYu. "The KP equation and its generalizations". Theory and applications, NEISRI FEB RAS, Magadan (1997).