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Cylindrical Tank Filled With a Liquid in a in Three-Dimensional Temperature Field

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Abstract—In the cylindrical coordinate system, we construct an exact solution of the three-dimensional thermoelasticity problem for a tank filled with a liquid. After determining the temperature field from the heat conduction equation, we solve the equations of the asymmetrical problem of the theory of elasticity. In doing so, the system of resolving equations is reduced to four separate equations with respect to the displacements of the construction. Several exact solutions of boundary-value problems are found. The results are presented in the form of rather simple formulas.

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In the cylindrical coordinate system, the most compact form of the equations of the three-dimensional elasticity theory, which are written in terms of the displacements and stresses, is given in the monographs [1, 2]. Analogous equations with allowance for the temperature terms, which were constructed using the Duhamel–Neumann relations, are presented in the monograph [3]. All these equations have a common disadvantage, namely, it is not possible to obtain their integrable combinations, which leads to insurmountable difficulties in constructing exact solutions. In the monograph [4], this problem was solved for equations that do not contain temperature terms by means of introducing into the system an additional equation with respect to the volume deformation. An exact solution to this equation was already published in the work [1], but the authors of the work [4] succeeded in using the equation to solve the entire system of resolving relations.

The aim of this work is to construct the system of equations of the elasticity theory, which are most convenient for integration, and to find the solution of these equations.

As initial equations, we take those for a circular cylinder [3]. Without taking into account the mass forces, they can be written with respect to the displacements as follows:

$$
\begin{gathered}
\Delta w + \frac{R}{(1-2\upsilon)}\left[\frac{1}{\varepsilon}\frac{\partial\theta}{\partial\gamma} - 2(1+\upsilon)\,\alpha_T\frac{1}{\varepsilon}\frac{\partial T}{\partial\gamma}\right] = 0,\\
\left(\Delta - \frac{1}{\alpha^2}\right)u + \frac{R}{(1-2\upsilon)}\left[\frac{\partial\theta}{\partial\alpha} - 2(1+\upsilon)\,\alpha_T\frac{\partial T}{\partial\alpha}\right] - \frac{2}{\alpha^2}\frac{\partial v}{\partial\beta} = 0,\\
\left(\Delta - \frac{1}{\alpha^2}\right)v + \frac{R}{(1-2\upsilon)}\left[\frac{1}{\alpha}\frac{\partial\theta}{\partial\beta} - 2(1+\upsilon)\,\alpha_T\frac{1}{\alpha}\frac{\partial T}{\partial\beta}\right] + \frac{2}{\alpha^2}\frac{\partial u}{\partial\beta} = 0,\\
\theta = \frac{1}{R}\left[\frac{1}{\alpha}\frac{\partial(\alpha\,u)}{\partial\alpha} + \frac{1}{\alpha}\frac{\partial v}{\partial\beta} + \frac{1}{\varepsilon}\frac{\partial w}{\partial\gamma}\right].
\end{gathered}
\tag{1}
$$

Here α, β, and γ are the dimensionless cylindrical coordinates (α is scaled by the outer radius R of the cylinder, γ is scaled by the height H of the cylinder, β is the angular coordinate along the guide); r is the radius of the inner lateral surface; u, v, and w are the displacements along the coordinate

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α, in the circumferential direction β and along γ, respectively; T is the body temperature; α_T is the temperature linear expansion coefficient; E and υ are the modulus of elasticity and the Poisson coefficient, respectively; θ is the volume deformation,

$$\varepsilon = \frac{H}{R}, \;\; t = \frac{r}{R}, \;\; \Delta \equiv \frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} + \frac{1}{\alpha^2}\frac{\partial^2}{\partial\beta^2} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\gamma^2}. \tag{2}$$

Applying to the first, second, and third equations of system (1) the operators $\frac{1}{\varepsilon}\frac{\partial}{\partial\gamma}$, $\frac{1}{\alpha}\frac{\partial}{\partial\alpha}$, and $\frac{1}{\alpha}\frac{\partial}{\partial\beta}$, respectively, and summing the obtained equations, we get

$$\left(1 + \frac{1}{(1-2\upsilon)}\right)\Delta\theta - \frac{2\,(1+\upsilon)\,\alpha_T}{(1-2\upsilon)}\Delta T = 0.$$

Taking into account the heat conduction equation for the unbound thermoelastic problem, we arrive at the equation with respect to the volume deformation

$$\Delta\theta = 0, \tag{3}$$

which coincides with the equation presented in the work [1] without allowance for temperature terms.

Replacing the last equation of system (1) by Eq. (3) and including in the system the heat equation equation, we arrive at the system of resolving equations for the formulated problem:

$$\begin{gathered}
\Delta T = 0, \;\; \Delta\theta = 0, \\
\Delta w + \frac{R}{(1-2\upsilon)}\left[\frac{1}{\varepsilon}\frac{\partial\theta}{\partial\gamma} - 2\,(1+\upsilon)\,\alpha_T\frac{1}{\varepsilon}\frac{\partial T}{\partial\gamma}\right] = 0, \\
\left(\Delta - \frac{1}{\alpha^2}\right)u + \frac{R}{(1-2\upsilon)}\left[\frac{\partial\theta}{\partial\alpha} - 2\,(1+\upsilon)\,\alpha_T\frac{\partial T}{\partial\alpha}\right] - \frac{2}{\alpha^2}\frac{\partial v}{\partial\beta} = 0, \\
\left(\Delta - \frac{1}{\alpha^2}\right)v + \frac{R}{(1-2\upsilon)}\left[\frac{1}{\alpha}\frac{\partial\theta}{\partial\beta} - 2\,(1+\upsilon)\,\alpha_T\frac{1}{\alpha}\frac{\partial T}{\partial\beta}\right] + \frac{2}{\alpha^2}\frac{\partial u}{\partial\beta} = 0.
\end{gathered} \tag{4}$$

The order of system (4) (without the heat conduction equation) is higher than that of system (1) because the construction of Eq. (3) was carried out with additional differentiation. Obviously, the general solution to system (4) must contain extra integration constants.

In the monograph [4], it was shown that the attraction of the relation

$$\theta \equiv \frac{1}{R}\left[\frac{1}{\alpha}\frac{\partial\,(\alpha u)}{\partial\alpha} + \frac{1}{\alpha}\frac{\partial v}{\partial\beta} + \frac{1}{\varepsilon}\frac{\partial w}{\partial\gamma}\right], \tag{5}$$

as an additional condition, eliminates this deficiency, i.e., completely solves the problem.

An exact solution to system (4) is constructed under the assumption that the temperature and the displacements are linear with respect to the coordinate γ. This version has a practical application, in particular, in studying the deformation of tanks filled with a liquid.

The β-periodic solution we will seek in the form

$$\begin{gathered}
T\,(\alpha,\beta,\gamma) = T_m\,(\alpha,\gamma)\cos(m\beta)\,, \;\; \theta\,(\alpha,\beta,\gamma) = \theta_m\,(\alpha,\gamma)\cos(m\beta)\,, \\
w\,(\alpha,\beta,\gamma) = w_m\,(\alpha,\gamma)\cos(m\beta)\,, \\
u\,(\alpha,\beta,\gamma) = u_m\,(\alpha,\gamma)\cos(m\beta)\,, \;\; v\,(\alpha,\beta,\gamma) = v_m\,(\alpha,\gamma)\sin(m\beta)\,,
\end{gathered} \tag{6}$$

where m is any integer number.

Remark. a) In formula (6), it is possible to swap the cosines and sines and also to compose a combination of these functions.

b) The solution can be chosen in the form of finite or infinite sums over m.

The choice of one of the above forms of solutions depends only on the boundary conditions, and the forms coinciding with those of specifying the boundary conditions.

If the boundary conditions are given in the form of series, which often takes place under the static conditions, for the most cases there is no need to prove the convergence of the series representing the solution, due to the obvious convergence of the series on the boundary.

Substituting relations (6) into system (4) and calculating the sum and difference of the last two equations, we arrive at the following system:

$$\mathcal{L}_m T_m = 0, \quad \mathcal{L}_m \theta_m = 0,$$

$$\mathcal{L}_m w_m = -\frac{1}{(1-2\upsilon)\,\varepsilon}\frac{\partial}{\partial\gamma}\left[\theta_m - 2\,(1+\upsilon)\,\alpha_T T_m\right],$$

$$\mathcal{L}_{m+1}\,(u_m + v_m) = -\frac{1}{(1-2\upsilon)}\left(\frac{\partial}{\partial\alpha} - \frac{m}{\alpha}\right)\left[\theta_m - 2\,(1+\upsilon)\,\alpha_T T_m\right], \tag{7}$$

$$\mathcal{L}_{m-1}\,(u_m - v_m) = -\frac{1}{(1-2\upsilon)}\left(\frac{\partial}{\partial\alpha} + \frac{m}{\alpha}\right)\left[\theta_m - 2\,(1+\upsilon)\,\alpha_T T_m\right],$$

where $\mathcal{L}_m = \frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} - \frac{m^2}{\alpha^2} + \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\gamma^2}$.

Note that at $m = 0$, the displacement $v_0 \equiv 0$, and the last two equations coincide and define the displacement u_0.

Taking into account the assumptions about the linearity of the displacements with respect to γ, we introduce the notations

$$T_m\,(\alpha,\gamma) = T_{m1}\,(\alpha) + T_{m2}\,(\alpha)\,\gamma,\ \ \theta_m\,(\alpha,\gamma) = \theta_{m1}\,(\alpha) + \theta_{m2}\,(\alpha)\,\gamma,$$
$$w_m\,(\alpha,\gamma) = w_{m1}\,(\alpha) + w_{m2}\,(\alpha)\,\gamma,$$
$$u_m\,(\alpha,\gamma) = u_{m1}\,(\alpha) + u_{m2}\,(\alpha)\,\gamma,\ \ v_m\,(\alpha,\gamma) = v_{m1}\,(\alpha) + v_{m2}\,(\alpha)\,\gamma.$$

As a result, each of the first two equations of system (7) leads to a pair of identical equations:

$$\left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} - \frac{m^2}{\alpha^2}\right) T_{mn} = 0,\ \ \left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} - \frac{m^2}{\alpha^2}\right) \theta_{mn} = 0\ \ (n = 1, 2).$$

Their general solutions are:

$$T_{0n}\,(\alpha) = B^1_{0n} + B^2_{0n}\ln\alpha,\ \ \theta_{0n}\,(\alpha) = A^1_{0n} + A^2_{0n}\ln\alpha,$$
$$T_{mn}\,(\alpha) = B^1_{mn}\alpha^m + B^2_{mn}\alpha^{-m},\ \ \theta_{mn}\,(\alpha) = A^1_{mn}\alpha^m + A^2_{mn}\alpha^{-m}\ \ (m > 0),$$

where A^k_{mn}, B^k_{mn} are the constants of integration $(n, k = 1, 2)$.

The third equation of system (7) also splits into two equations. For $m = 0$,

$$\left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha}\right) w_{01} + \frac{R}{(1-2\upsilon)\,\varepsilon}\left(L^1_{02} + L^2_{02}\ln\alpha\right) = 0,\ \ \left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha}\right) w_{02} = 0$$

and for $m > 0$,

$$\left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} - \frac{m^2}{\alpha^2}\right) w_{m1} + \frac{R}{(1-2\upsilon)\,\varepsilon}\left(L^1_{m2}\alpha^m + L^2_{m2}\alpha^{-m}\right) = 0,$$

$$\left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\alpha}\frac{\partial}{\partial\alpha} - \frac{m^2}{\alpha^2}\right) w_{m2} = 0.$$

Here

$$L^k_{mn} = A^k_{mn} - 2\,(1+\upsilon)\,\alpha_T B^k_{mn}.$$

Their general solutions are of the form:

$$w_{01}\left(\alpha\right)=-\frac{R}{\left(1-2\upsilon\right)\varepsilon}\left[C_{01}^{1}+C_{01}^{2}\ln\alpha+\frac{1}{4}L_{02}^{1}\alpha^{2}+L_{02}^{2}\,\frac{\alpha^{2}}{4}\left(\ln\alpha-1\right)\right],$$

$$w_{02}\left(\alpha\right)=-\frac{R}{\left(1-2\upsilon\right)\varepsilon}\left(C_{02}^{1}+C_{02}^{2}\ln\alpha\right),$$

$$w_{11}=-\frac{R}{\left(1-2\upsilon\right)\varepsilon}\left\{C_{11}^{1}\alpha+C_{11}^{2}\frac{1}{\alpha}+\frac{1}{8}L_{12}^{1}\alpha^{3}+\frac{1}{2}L_{12}^{2}\alpha\ln\alpha\right\},$$

$$w_{12}=-\frac{R}{\left(1-2\upsilon\right)\varepsilon}\left(C_{12}^{1}\alpha+C_{12}^{2}\frac{1}{\alpha}\right),$$

$$w_{m1}=-\frac{R}{4\left(1-2\upsilon\right)\varepsilon}\left[C_{m1}^{1}\alpha^{m}+C_{m1}^{2}\alpha^{-m}+\frac{L_{m2}^{1}}{(m+1)}\alpha^{m+2}-\frac{L_{m2}^{2}}{(m-1)}\alpha^{-m+2}\right],$$

$$w_{m2}=-\frac{R}{\left(1-2\upsilon\right)\varepsilon}\left(C_{m2}^{1}\alpha^{m}+C_{m2}^{2}\alpha^{-m}\right)\ \ (m>1),$$

C_{mn}^{k} are the constants of integration.

In the case of axisymmetric deformation of the cylinder ($m=0$), the equation with respect to u_0 leads to:

$$\left(\frac{\partial^{2}}{\partial\alpha^{2}}+\frac{1}{\alpha}\frac{\partial}{\partial\alpha}-\frac{1}{\alpha^{2}}\right)u_{0n}=-\frac{R}{(1-2\upsilon)}L_{0n}^{2}\frac{1}{\alpha}\ \ (n=1,2).$$

The general solutions of the above equations are:

$$u_{0n}=-\frac{R}{2\left(1-2\upsilon\right)}\left[D_{0n}^{1}\alpha+D_{0n}^{2}\frac{1}{\alpha}+L_{0n}^{2}\alpha\ln\alpha\right]\ \ (n=1,2).$$

At $m=1$, from the two last equations of system (7) we get two pair of equations:

$$\left[\frac{d^{2}}{d\alpha^{2}}+\frac{1}{\alpha}\frac{d}{d\alpha}-\frac{4}{\alpha^{2}}\right](u_{1n}+v_{1n})=\frac{2R\,L_{1n}^{2}}{(1-2\upsilon)}\frac{1}{\alpha^{2}},$$

$$\left[\frac{d^{2}}{d\alpha^{2}}+\frac{1}{\alpha}\frac{d}{d\alpha}\right](u_{1n}-v_{1n})=-\frac{2R\,L_{1n}^{1}}{(1-2\upsilon)}.$$

Their solutions at $A=-R/[2(1-2\upsilon)]$ are:

$$u_{11}+v_{11}=A\left[D_{11}^{1}\alpha^{2}+D_{11}^{2}\alpha^{-2}+L_{11}^{2}\right],$$

$$u_{11}-v_{11}=A\left\{D_{11}^{3}+D_{11}^{4}\ln\alpha+L_{11}^{1}\alpha^{2}\right\},$$

$$u_{12}+v_{12}=A\left[D_{12}^{1}\alpha^{2}+D_{12}^{2}\alpha^{-2}+L_{12}^{2}\right],$$

$$u_{12}-v_{12}=A\left\{D_{12}^{3}+D_{12}^{4}\ln\alpha+L_{12}^{1}\alpha^{2}\right\}.$$

For $m>1$, we come to the equations

$$\mathcal{L}_{m+1}\left(u_{mn}+v_{mn}\right)=\frac{2R\,m}{(1-2\upsilon)}L_{mn}^{2}\alpha^{-(m+1)},$$

$$\mathcal{L}_{m-1}\left(u_{mn}-v_{mn}\right)=-\frac{2Rm}{(1-2\upsilon)}L_{mn}^{1}\alpha^{m-1}.$$

There solutions are:

$$u_{m1}+v_{m1}=A\left[D_{m1}^{1}\alpha^{(m+1)}+D_{m1}^{2}\alpha^{-(m+1)}+L_{m1}^{2}\alpha^{-(m-1)}\right],$$
$$u_{m2}+v_{m2}=A\left[D_{m2}^{1}\alpha^{(m+1)}+D_{m2}^{2}\alpha^{-(m+1)}+L_{m2}^{2}\alpha^{-(m-1)}\right],$$
$$u_{m1}-v_{m1}=A\left[D_{m1}^{3}\alpha^{(m-1)}+D_{m1}^{4}\alpha^{-(m-1)}+L_{m1}^{1}\alpha^{(m+1)}\right],$$
$$u_{m2}-v_{m2}=A\left[D_{m2}^{3}\alpha^{(m-1)}+D_{m2}^{4}\alpha^{-(m-1)}+L_{m2}^{1}\alpha^{(m+1)}\right].$$

Here D_{mn}^{k} is a constant of integration.

As a result, for $m=0$ we obtain

$$\theta_{01}(\alpha)=A_{01}^{1}+A_{01}^{2}\ln\alpha,\ \ \theta_{02}(\alpha)=A_{02}^{1}+A_{02}^{2}\ln\alpha,$$

$$w_{01}(\alpha)=\frac{2A}{\varepsilon}\left[C_{01}^{1}+C_{01}^{2}\ln\alpha+\frac{1}{4}L_{02}^{1}\alpha^{2}+L_{02}^{2}\frac{\alpha^{2}}{4}(\ln\alpha-1)\right],$$

$$w_{02}(\alpha)=\frac{2A}{\varepsilon}\left(C_{02}^{1}+C_{02}^{2}\ln\alpha\right),$$

$$u_{01}=A\left[D_{01}^{1}\alpha+D_{01}^{2}\frac{1}{\alpha}+L_{01}^{2}\alpha\ln\alpha\right],$$

$$u_{02}=A\left[D_{02}^{1}\alpha+D_{02}^{2}\frac{1}{\alpha}+L_{02}^{2}\ \alpha\ln\alpha\right].$$

For $m=1$ we get

$$\theta_{11}(\alpha)=A_{11}^{1}\alpha+A_{11}^{2}\alpha^{-1},\ \ \theta_{12}(\alpha)=A_{12}^{1}\alpha+A_{12}^{2}\alpha^{-1},$$

$$w_{11}=\frac{2A}{\varepsilon}\left\{C_{11}^{1}\alpha+C_{11}^{2}\frac{1}{\alpha}+\frac{1}{8}L_{12}^{1}\alpha^{3}+\frac{1}{2}L_{12}^{2}\alpha\ln\alpha\right\},$$

$$w_{12}=-\frac{2A}{\varepsilon}\left(C_{12}^{1}\alpha+C_{12}^{2}\frac{1}{\alpha}\right),$$

$$u_{11}=\frac{A}{2}\left[\left(D_{11}^{1}+L_{11}^{1}\right)\alpha^{2}+D_{11}^{2}\alpha^{-2}+D_{11}^{4}\ln\alpha+D_{11}^{3}+L_{11}^{2}\right],$$

$$v_{11}=\frac{A}{2}\left[\left(D_{11}^{1}-L_{11}^{1}\right)\alpha^{2}+D_{11}^{2}\alpha^{-2}-D_{11}^{4}\ln\alpha-D_{11}^{3}+L_{11}^{2}\right],$$

$$u_{12}=\frac{A}{2}\left[\left(D_{12}^{1}+L_{12}^{1}\right)\alpha^{2}+D_{12}^{2}\alpha^{-2}+D_{12}^{4}\ln\alpha+D_{12}^{3}+L_{12}^{2}\right],$$

$$v_{12}=\frac{A}{2}\left[\left(D_{12}^{1}-L_{12}^{1}\right)\alpha^{2}+D_{12}^{2}\alpha^{-2}-D_{12}^{4}\ln\alpha-D_{12}^{3}+L_{12}^{2}\right].$$

For $m>1$ we have

$$\theta_{m1}(\alpha)=A_{m1}^{1}\alpha^{m}+A_{m1}^{2}\alpha^{-m},\ \ \theta_{m2}(\alpha)=A_{m2}^{1}\alpha^{m}+A_{m2}^{2}\ \alpha^{-m},$$

$$w_{m1}=\frac{A}{2\varepsilon}\left[C_{m1}^{1}\alpha^{m}+C_{m1}^{2}\alpha^{-m}+\frac{L_{m2}^{1}}{(m+1)}\alpha^{m+2}-\frac{L_{m2}^{2}}{(m-1)}\alpha^{-m+2}\right],$$

$$w_{m2}=-\frac{R}{(1-2\upsilon)\,\varepsilon}\left(C_{m2}^{1}\alpha^{m}+C_{m2}^{2}\alpha^{-m}\right),$$

$$u_{m1}=\frac{A}{2}\left[\left(D_{m1}^{1}+L_{m1}^{1}\right)\alpha^{(m+1)}+D_{m1}^{2}\alpha^{-(m+1)}+D_{m1}^{3}\alpha^{(m-1)}+\left(D_{m1}^{4}+L_{m1}^{2}\right)\alpha^{-(m-1)}\right],$$

$$\upsilon_{m1}=\frac{A}{2}\left[\left(D_{m1}^{1}-L_{m1}^{1}\right)\alpha^{(m+1)}+D_{m1}^{2}\alpha^{-(m+1)}-D_{m1}^{3}\alpha^{(m-1)}-\left(D_{m1}^{4}-L_{m1}^{2}\right)\alpha^{-(m-1)}\right],$$

$$u_{m2}=\frac{A}{2}\left[\left(D_{m2}^{1}+L_{m2}^{1}\right)\alpha^{(m+1)}+D_{m2}^{2}\alpha^{-(m+1)}+D_{m2}^{3}\alpha^{(m-1)}+\left(D_{m2}^{4}+L_{m2}^{2}\right)\alpha^{-(m-1)}\right],$$

$$\upsilon_{m2}=\frac{A}{2}\left[\left(D_{m2}^{1}-L_{m2}^{1}\right)\alpha^{(m+1)}+D_{m2}^{2}\alpha^{-(m+1)}-D_{m2}^{3}\alpha^{(m-1)}-\left(D_{m2}^{4}-L_{m2}^{2}\right)\alpha^{-(m-1)}\right].$$

The correctness of the obtained solutions can be easily established by their direct substitution into equations either manually or by using any computer system.

It remains to fulfill the additional condition (5). From this condition, there follows the pair of identities for each m:

$$\theta_{m1}\equiv\frac{1}{R}\left(\frac{1}{\alpha}\frac{d\left(\alpha u_{m1}\right)}{d\alpha}+\frac{m}{\alpha}\upsilon_{m1}+\frac{1}{\varepsilon}w_{m2}\right),\ \ \theta_{m2}\equiv\frac{1}{R}\left(\frac{1}{\alpha}\frac{d\left(\alpha u_{m2}\right)}{d\alpha}+\frac{m}{\alpha}\upsilon_{m2}\right),$$

which allow one to express the values of A_{mn}^{k} by the remainder constants:

$$A_{01}^{1}=\frac{1}{2\,(1-\upsilon)\,(1-2\upsilon)}\left\{(1+\upsilon)\,(1-2\upsilon)\,\alpha_{T}B_{01}^{2}+\frac{C_{02}^{2}-4\,(1-\upsilon)\,C_{02}^{1}}{2\varepsilon^{2}}-2\,(1-\upsilon)\,D_{01}^{1}\right\},$$

$$A_{01}^{2}=\frac{(1+\upsilon)}{(1-\upsilon)}\alpha_{T}B_{01}^{2}-\frac{1}{2\,(1-\upsilon)\,\varepsilon^{2}}C_{02}^{2},\ \ A_{02}^{1}=\frac{(1+\upsilon)}{2\,(1-\upsilon)}\alpha_{T}B_{02}^{2}-\frac{1}{(1-2\upsilon)}D_{02}^{1},$$

$$A_{02}^{2}=\frac{(1+\upsilon)}{(1-\upsilon)}\alpha_{T}B_{02}^{2},$$

$$A_{11}^{1}=\frac{2}{(3-4\upsilon)}\left[(1+\upsilon)\,\alpha_{T}B_{11}^{1}-\frac{1}{\varepsilon^{2}}C_{12}^{1}-D_{11}^{1}\right],$$

$$A_{11}^{2}=\frac{1}{2\,(3-4\upsilon)}\left[4\,(1+\upsilon)\,\alpha_{T}B_{11}^{2}-\frac{4}{\varepsilon^{2}}C_{12}^{2}-D_{11}^{4}\right],$$

$$A_{12}^{1}=\frac{2}{(3-4\upsilon)}\left[(1+\upsilon)\,\alpha_{T}B_{12}^{1}-D_{12}^{1}\right],\ \ A_{12}^{2}=\frac{1}{2\,(3-4\upsilon)}\left[4\,(1+\upsilon)\,\alpha_{T}B_{12}^{2}-D_{12}^{4}\right],$$

$$A_{m1}^{1}=\frac{1}{(3-4\upsilon)}\left[2\,(1+\upsilon)\,\alpha_{T}B_{m1}^{1}-\frac{2}{\varepsilon^{2}}C_{m2}^{1}-(m+1)\,D_{m1}^{1}\right],$$

$$A_{m1}^{2}=\frac{1}{(3-4\upsilon)}\left[2\,(1+\upsilon)\,\alpha_{T}B_{m1}^{2}-\frac{2}{\varepsilon^{2}}C_{m2}^{2}+(m-1)\,D_{m1}^{4}\right].$$

The remaining integration constants are found as follows: B_{mn}^{k} are defined from the boundary conditions of the temperature problem, C_{mn}^{k}, D_{mn}^{k} are determined from the conditions of the elastic problem.

Let us consider the case of an axisymmetric deformation ($m = 0$) of a tank filled with a fluid. Suppose that both its ends and the inner lateral surface are thermally insulated from the surrounding environment, the outer lateral surface has a constant temperature Θ. In this case, the higher is the thermal conductivity of the material, the faster the temperature of the tank walls takes the value $T = T_{01} = \Theta$, $T_{02} = 0$. Obviously, $B_{01}^1 = \Theta$, $B_{01}^2 = B_{02}^1 = B_{02}^2 = 0$.

Then

$$A_{01}^1 = \frac{1}{2(1-\upsilon)(1-2\upsilon)}\left[\frac{C_{02}^2 - 4(1-\upsilon)C_{02}^1}{2\varepsilon^2} - 2(1-\upsilon)D_{01}^1\right],$$

$$A_{01}^2 = -\frac{1}{2(1-\upsilon)\varepsilon^2}C_{02}^2, \;\; A_{02}^1 = -\frac{1}{(1-2\upsilon)}D_{02}^1, \;\; A_{02}^2 = 0,$$

$$L_{01}^1 = \frac{1}{2(1-\upsilon)(1-2\upsilon)}\left[\frac{C_{02}^2 - 4(1-\upsilon)C_{02}^1}{2\varepsilon^2} - 2(1-\upsilon)D_{01}^1\right] - 2(1+\upsilon)\alpha_T B_{01}^1,$$

$$L_{01}^2 = -\frac{1}{2(1-\upsilon)\varepsilon^2}C_{02}^2, \;\; L_{02}^1 = -\frac{1}{(1-2\upsilon)}D_{02}^1, \;\; L_{02}^2 = 0.$$

As a result

$$\theta_{01}(\alpha) = \frac{1}{2(1-\upsilon)\varepsilon^2}\left[\frac{1}{2(1-2\upsilon)} - \ln\alpha\right]C_{02}^2 - \frac{1}{(1-2\upsilon)}\left(\frac{1}{\varepsilon^2}C_{02}^1 + D_{01}^1\right), \quad (8)$$

$$\theta_{02}(\alpha) = -\frac{1}{(1-2\upsilon)}D_{02}^1, \;\; w_{01}(\alpha) = -\frac{R}{(1-2\upsilon)\varepsilon}\left[C_{01}^1 + C_{01}^2\ln\alpha - \frac{1}{4(1-2\upsilon)}D_{02}^1\alpha^2\right],$$

$$w_{02}(\alpha) = -\frac{R}{(1-2\upsilon)\varepsilon}\left(C_{02}^1 + C_{02}^2\ln\alpha\right), \;\; u_{02} = -\frac{R}{2(1-2\upsilon)}\left[D_{02}^1\alpha + D_{02}^2\,\frac{1}{\alpha}\right],$$

$$u_{01} = -\frac{R}{2(1-2\upsilon)}\left[D_{01}^1\alpha + D_{01}^2\frac{1}{\alpha} - \frac{1}{2(1-\upsilon)\varepsilon^2}C_{02}^2\,\alpha\ln\alpha\right].$$

Let us define the stresses necessary for the fulfillment of the boundary conditions. To do so, we use the Duhamel–Neumann relations [3]:

$$\sigma_{\alpha\alpha} = \frac{E}{2(1+\upsilon)(1-2\upsilon)}\left\{\frac{2(1-2\upsilon)}{R}\frac{\partial u}{\partial\alpha} + 2\upsilon\theta - 2(1+\upsilon)\alpha_T T\right\},$$

$$\sigma_{\gamma\gamma} = \frac{E}{2(1+\upsilon)(1-2\upsilon)}\left\{\frac{2(1-2\upsilon)}{R\varepsilon}\frac{\partial w}{\partial\gamma} + 2\upsilon\theta - 2(1+\upsilon)\alpha_T T\right\},$$

$$\sigma_{\alpha\gamma} = \frac{E}{2(1+\upsilon)R}\left(\frac{\partial w}{\partial\alpha} + \frac{1}{\varepsilon}\frac{\partial u}{\partial\gamma}\right).$$

If we assume that

$$\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha1} + \sigma_{\alpha\alpha2}\gamma, \;\; \sigma_{\gamma\gamma} = \sigma_{\gamma\gamma1} + \sigma_{\gamma\gamma2}\gamma, \;\; \sigma_{\alpha\gamma} = \sigma_{\alpha\gamma1} + \sigma_{\alpha\gamma2}\gamma,$$

then differentiating the relations (8), we determine

$$\sigma_{\alpha\alpha1} = \frac{E}{2(1+\upsilon)(1-2\upsilon)}\left\{-\frac{1}{(1-2\upsilon)}D_{01}^1 + D_{01}^2\frac{1}{\alpha^2} - \frac{2\upsilon}{(1-2\upsilon)\varepsilon^2}C_{02}^1 - 2(1+\upsilon)\alpha_T B_{01}^1 \right.$$
$$\left. + \frac{1}{2(1-\upsilon)\varepsilon^2}\left[\frac{(1-\upsilon)}{(1-2\upsilon)} + (1-2\upsilon)\;\ln\alpha\right]C_{02}^2\right\},$$

$$\sigma_{\alpha\alpha 2} = \frac{E}{2(1+\upsilon)(1-2\upsilon)} \left\{ -\frac{1}{(1-2\upsilon)} D_{02}^1 + D_{02}^2 \frac{1}{\alpha^2} \right\},$$

$$\sigma_{\gamma\gamma 1} = \frac{E}{2(1+\upsilon)(1-2\upsilon)} \left\{ -\frac{2(1-\upsilon)}{(1-2\upsilon)\varepsilon^2} C_{02}^1 - 2(1+\upsilon)\alpha_T B_{01}^1 \right.$$
$$\left. + \frac{1}{(1-\upsilon)\varepsilon^2} \left[\frac{\upsilon}{2(1-2\upsilon)} - (2-\upsilon)\ln\alpha \right] C_{02}^2 - \frac{2\upsilon}{(1-2\upsilon)} D_{01}^1 \right\},$$

$$\sigma_{\gamma\gamma 2} = -\frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)^2} D_{02}^1, \quad \sigma_{\alpha\gamma 2} = -\frac{E}{2(1+\upsilon)(1-2\upsilon)\varepsilon} C_{02}^2 \frac{1}{\alpha},$$

$$\sigma_{\alpha\gamma 1} = -\frac{E}{4(1+\upsilon)(1-2\upsilon)\varepsilon} \left\{ \left(2C_{01}^2 + D_{02}^2\right) \frac{1}{\alpha} - \frac{2\upsilon}{(1-2\upsilon)} D_{02}^1 \alpha \right\}.$$

As examples, we define the displacements for three kinds of boundary conditions. The formulas for displacements are the same for all three problems:

$$w(\alpha,\gamma) = -\frac{R}{(1-2\upsilon)\varepsilon} \left\{ C_{01}^1 + C_{02}^1\gamma + \left(C_{01}^2 + C_{02}^2\gamma\right)\ln\alpha - \frac{1}{4(1-2\upsilon)} D_{02}^1 \alpha^2 \right\},$$

$$u(\alpha,\gamma) = -\frac{R}{2(1-2\upsilon)} \left\{ \left(D_{01}^1 + D_{02}^1\gamma\right)\alpha + \left(D_{01}^2 + D_{02}^2\gamma\right)\frac{1}{\alpha} - \frac{1}{2(1-\upsilon)\varepsilon^2} C_{02}^2 \alpha \ln\alpha \right\},$$

which follows from (8). It remains to determine the values of the integration constants.

1) For the boundary conditions

$$\begin{gathered} \sigma_{\alpha\alpha}(t,\gamma) = \rho H(1-\gamma), \quad \sigma_{\alpha\gamma}(t,\gamma) = 0, \\ w(1,0) = u(1,0) = 0, \quad w(1,1) = 0, \quad u(1,1) = 0 \end{gathered} \tag{9}$$

we obtain

$$\begin{gathered} C_{01}^1 = \frac{t^2 f}{4(t^2+1-2\upsilon)}, \quad C_{01}^2 = \frac{\left(2\upsilon t^2 + 1 - 2\upsilon\right)t^2 f}{2(t^2+1-2\upsilon)}, \quad C_{02}^1 = 0, \quad C_{02}^2 = 0, \\ D_{01}^1 = -D_{01}^2 = -\frac{(1-2\upsilon)\left[f + 2(1+\upsilon)\alpha_T B_{01}^1\right]t^2}{(t^2+1-2\upsilon)}, \quad D_{02}^1 = -D_{02}^2 = \frac{(1-2\upsilon)t^2 f}{(t^2+1-2\upsilon)}. \end{gathered} \tag{10}$$

Here ρ is the density of the fluid, $f = \frac{2(1+\upsilon)(1-2\upsilon)\rho H}{E}$.

2) If the boundary conditions take the form

$$\sigma_{\alpha\alpha}(t,\gamma) = \rho H(1-\gamma), \quad \sigma_{\alpha\gamma}(t,\gamma) = 0,$$

$$w(1,0) = u(1,0) = 0, \quad \sigma_{\gamma\gamma}(1,1) = 0, \quad u(1,1) = 0,$$

i.e., they differ from (9) by one condition, then for the constants C_{01}^1, C_{01}^2, C_{02}^2, D_{02}^1, D_{02}^2, formulas (10) hold,

$$\frac{1}{\varepsilon^2} C_{02}^1 = -\frac{(1+\upsilon)(1-2\upsilon)\left(t^2+1\right)}{\left[t^2(1+\upsilon)+1-\upsilon\right]} \alpha_T B_{01}^1,$$

$$D_{01}^1 = -D_{01}^2 = -\frac{2(1+\upsilon)(1-2\upsilon)t^2}{\left[t^2(1+\upsilon)+1-\upsilon\right]} \alpha_T B_{01}^1 - \frac{(1-2\upsilon)t^2 f}{(t^2+1-2\upsilon)}.$$

3) For the boundary conditions

$$\sigma_{\alpha\alpha}(t,\gamma) = \rho H(1-\gamma),$$

$$w\,(1,0) = u\,(1,0) = 0,\;\; w\,(1,1) = 0,\;\; u\,(1,1) = 0,\;\; w\,(t,0) = 0,\;\; w\,(t,1) = 0$$

the constants C_{01}^1, C_{01}^2, C_{02}^2, D_{02}^1, D_{02}^2 are determined by formulas (10),

$$C_{01}^2 = \frac{\left(t^2-1\right)t^2 f}{4\left(t^2+1-2\upsilon\right)\ln t},\;\; D_{01}^1 = -D_{01}^2 = -\frac{\left(1-2\upsilon\right)\left[f+2\left(1+\upsilon\right)\alpha_T B_{01}^1\right]t^2}{\left(t^2+1-2\upsilon\right)}.$$

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