Seepage to Ditches and Topographic Depressions in Saturated and Unsaturated Soils

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Abstract. An isobar generated by a line or point sink draining a confined semi-infinite aquifer is an analytic curve, to which a steady 2-D plane or axisymmetric Darcian flow converges. This sink may represent an excavation, ditch, or wadi on Earth, or a channel on Mars. The strength of the sink controls the form of the ditch depression: for 2-D flow, the shape of the isobar varies from a zero-depth channel to a semicircle; for axisymmetric flow, depressions as flat as a disk or as deep as a hemisphere are reconstructed. In the model of axisymmetric flow, a fictitious J.R. Philip's point sink is mirrored by an infinite array of sinks and sources placed along a vertical line perpendicular to a horizontal water table. A topographic depression is kept at constant capillary pressure (water content, Kirchhoff potential). None of these singularities belongs to the real flow domain, evaporating unsaturated Gardnerian soil. Saturated flow towards a triangular, empty or partially-filled ditch is tackled by conformal mappings and the solution of Riemann's problem in a reference plane. The obtained seepage flow rate is used as a right-hand side in an ODE of a Cauchy problem, the solution of which gives the draw-up curves, i.e., the rise of the water level in an initially empty trench. HYDRUS-2D computations for flows in saturated and unsaturated soils match well the analytical solutions. The modeling results are applied to assessments of real hydrological fluxes on Earth and paleo-reconstructions of Martian hydrology-geomorphology.

Key Words:

Analytic and HYDRUS solutions for Darcian 2-D and axisymmetric flows in saturated and unsaturated soils towards drainage ditches and topographic depressions;

Evaporation and seepage exfiltration from shallow groundwater;

Complex potential and conformal mappings;

Method of images with sinks and sources for the Laplace equation and ADE;

Boundary value problems involving seepage faces on Earth and Mars;

Isobars, isotachs, constant piezometric head, and Kirchhoff potential lines.

"I had to live in the desert before I could understand the full value of grass in a green ditch."

Ella Maillart

1. Introduction

Analytical models of 2-D seepage towards drainage ditches and trenches, constructed by civil, geotechnical, and agricultural engineers, used the machinery of complex variables (Anderson, 2013, Aravin and Numerov, 1953, Bear, 1972, Kirkham and Powers, 1972, Polubarinova-Kochina, 1962,1977, hereafter abbreviated as PK-62,77, Skaggs et al., 1999, Strack, 1989, Vedernikov, 1939), in particular, by tackling free boundaries of Darcian flows, the so-called phreatic surfaces.

We recall (see, e.g., Radcliffe and Šimůnek, 2010) that transient, 3-D, saturated-unsaturated flows in porous media (when both water and soil are incompressible) obey the Richards equation:

$$\frac{\partial \theta}{\partial t} = \nabla\left(K(p)\nabla h\right) \tag{0}$$

where $\theta(t,x,y,Z)$ is the volumetric moisture content, ∇ is the nubla operator (in Cartesian or cylindrical coordinates), $K(p,x,y,Z)$ is the hydraulic conductivity function, $h(t,x,y,Z)=p+Z$ is the total head, p is the pressure head, Z is a vertical coordinate, and $p(\theta)$ is a capillary pressure (water retention) function, fixed for each soil. Eq. (0) involves the Darsy law $\vec{V}=-K(p)\nabla h$, where $\vec{V}$ is the Darcian flux vector, and the principal of mass conservation.

Boundary value problems (hereafter abbreviated as BVPs) are solved for eq. (0) by specifying initial conditions, e.g., $\theta(0,x,y,Z)$, as well as imposing physically meaningful boundary conditions (e.g., Dirichlet's, Neuman's). Only numerical codes like HYDRUS-3D (Šimůnek et al., 2016) tackle such problems for arbitrary 3D transient flows. Eq. (0) is a highly nonlinear parabolic PDE. For steady flows, its LHS vanishes, and the equation becomes elliptic. If the flow is purely saturated and the porous medium is homogeneous, then $K(p)=K_s$, where K_s is saturated hydraulic conductivity, and eq. (0) is reduced to a linear Laplace one. For a special class of soils with an exponential $K(p)$ function (Gardner, 1958), the Kirchhoff transform makes possible a linearization such that eq. (0) is reduced to a linear advective dispersion equation (ADE) with respect to a Kirchhoff potential. The ADE can be analytically solved for solitary or systematically placed singularities (Kacimov, 2007, Philip, 1968, 1971, Pullan, 1990, Raats, 1971, 1972). More details on BVPs for the Laplace equation and the ADE are given in Sections 2-5 of the MS.

The total head for saturated seepage in undeformable porous media obeys the Laplace equation and, therefore, the mathematical commonality between flows of ideal fluids (Zhukovskii, 1948) and pore water motion has been widely explored (PK-62,77, Strack,1989). The method of images and the theory of BVPs (Henrici, 1993) have been engaged. In this paper, we combine these traditional analytical and new numerical (HYDRUS-2D, Šimůnek et al., 2016) methods to problems of groundwater and soil water movement from shallow confined and unconfined aquifers towards ditches and topographic depressions, common in desert landforms of arid-hyperarid climates of Oman and Mars.

A practical motivation of our work stems from a daunting problem of groundwater inundation caused by a rapid rise of the water table in perched unconfined and shallow confined aquifers, detected in many urban areas of the world (see, e.g., Attard et al., 2016, Barron et al., 2013, Chaudhary, 2012, Coda et al., 2019, Howard and Israfilov, 2012, Jha et al., 2012, Kazemi, 2011, Lerner, 2003, Lerner and Harris, 2008, Medovar et al., 2018, Naik et al., 2008, Porse et al., 2016, Preene and Fisher, 2015, Quan et al., 2010, Schirmer et al., 2013, Vázquez-Suñé et al., 2005, Vogwill, 2016). Surprisingly, the arid countries of Arabia – despite the hydrological mantra on water deficit and groundwater depletion – encounter the same problem of waterlogging of urban structures by shallow groundwater as the cities in humid zones (e.g., Abu-Rizaiza et al., 1989, Abu-Rizaiza, 2006, Al-Rawas and Qamaruddin, 1998, Al-Sefry and Şen, 2006, Al-Senafy, 2011, Al-Senafy et al., 2015, Bob et al., 2016, Kreibich and Thieken, 2008). It is noteworthy that urbanization in the Southwest of the USA caused similar negative hydrological impacts (amplification of flash flood intensities due to reduced evapotranspiration, see, e.g., Goudie, 2013).

In the metropolitan area of Muscat, at the campus of Sultan Qaboos University (SQU), the groundwater inundation has become evident even without any piezometry. In recent years, subsurface water seeped out in natural topographic depressions, man-made excavations, ditches, and drainage channels called wadis. Large wet spots or ponded areas at SQU are bio-marked by adjacent lush, wild vegetation. In the desert climate of Oman, this vegetation in some places pops up virtually as Maillart’s “grass in green ditches” (see the epigraph). The studied SQU field is located in a hyperarid zone of the Batinah Coast, Oman, Arabia (e.g., Alsharhan et al., 2001). The investigated wadi section was about 1 km long, with several excavated pits-trenches and natural depressions. For instance, one of the excavations has the following coordinates: Northing: 619721, Easting: 2608729, 47.5 m above the level in the Sea of Oman. Urbanization of the field started in the mid-1980s, before which the site was a desert with scattered shrubs and trees. Currently, a significant part of the campus area is paved, which reduced natural evapotranspiration. Waterlogging and associated lush vegetaiton is, geotechnically deleterious, putting footings of the SQU buildings, roads, subsurface cables, etc., in jeopardy, even when the salinity of the rising groundwater is small. Secondary salinization of the topsoil is also widespread in Oman when the draw-up of the water table is from an aquifer of poor groundwater quality. Fieldwork has been carried out in November-June 2020 at the SQU campus: auger holes, ditches, and excavations have been constructed, and groundwater monitored, to assess soil moisture oozing fluxes above shallow perched and confined aquifers detected in the study area.

The motivation of this work also stems from our intention to attract the attention of Planet Mars hydrologists, who model seepage of Martian groundwater into remotely scanned relief features of the Red Planet, to the legacy of experts in irrigation and drainage engineering, who – during the 20th century - assembled a good arsenal of analytical and numerical methods for investigations of the motion of pore water towards drains (Bear, 1972, Kirkham and Powers, 1972, Skaggs et al., 1999, 2012, Strack, 1989, Vedernikov, 1939, Zhukovskii, 1948).

Mars is a hyperarid planet where the remotely analyzed landforms (craters, dunes, endorheic lakes and playas, pits, alcoves, gullies, recurring slope lineae, fluvio-lacustrine basins, fans, shallow dust horizons, streaks, amphitheater-headed valleys, sapping valleys, etc.) and hydrological elements (springs, outseeps, deep fracture conduits, seepage spots), which convey liquid water, brine, and Martian gases are studied. The alleged paleo and current dynamics of Martian pore fluids are elaborated, for example, in Abotalib and Heggy (2019), Bhardwaj et al. (2019), Boatwright and Head (2019), Edwards and Piqueux (2016), Goldspiel and Squyres (2011), Grimm et al. (2014), Hobbs et al. (2014), Kereszturi et al. (2011), Kochel and Piper (1986), Luo et al. (2011), Malin and Carr (1999) Malin and Edgett (2000), Marra et al. (2014, 2015), Michalski et al. (2013), Mukherjee et al. (2020), and Salese et al. (2019).

For hypothetical Martian aquifers (both unconfined and artesian), K_s is assessed from proxy data (see, e.g., the posited "nominal" value in Goldspiel and Squyres (2011); Luo et al. (2011) evaluated the hydraulic conductivity from the solution of inverse problems of the hydrological cycle on Mars; Boatwright and Head (2019) assumed that the conductivity exponentially decreases with the Martian depth). We note that even the Darcy law is sometimes not correctly formulated by Martian hydrologists (see, e.g., an erroneous eq. (1) in Abotalib and Heggy, 2019). On Mars, purely unsaturated flows are also hypothesized (Edwards and Piqueux, 2016, Grimm et al., 2014), but – to the best of our knowledge - Martian hydrologists (e.g., Luo et al., 2011) so far have used only flow models for saturated soils.

Models of Darcian flows towards drainage ditches commonly conceptualize the ditches as rectangles in vertical cross-sections having vertical slopes (Afruzi et al., 2014, Barua and Alam, 2013, Chahar and Vadodaria, 2008, Gureghian and Youngs, 1975, Sarmah and Tiwari, 2018, Youngs, 1975, 1990, 1994). However, natural ephemeral river channels, both on Earth and Mars, as well as constructed drainage ditches and trenches often have non-rectangular shapes (Grotzinger et al., 2014, Kocurek et al., 2020). Excavations (pits) and natural depressionsare often axisymmetric and mild-sloped (see, e.g., Goldspiel and Squyres, 2011). Only in few studies have phreatic seepage and flows in the unsaturated zone to non-rectangular (trapezoidal and curvilinear) draining entities been analytically examined (PK-77 reported solutions by Bazanov, p. 150 of her book, and Vedernikov, pp.181-182, see also Ilyinsky and Kacimov, 1992a, Kacimov, 2005, 2006a, Kacimov and Obnosov, 2002). We note that although Luo et al. (2011) sketched trapezoidal draining channels in their models of groundwater discharge into these channels, they posited the Dupuit-Forchheimer (hereafter abbreviated as DF) model, which actually ignores the shape of channels. Marra et al. (2014, 2015) conceptualized, and studied experimentally in sand boxes, seepage into real trapezoidal channels, allegedly occurring on Mars, but unfortunately did not refer to a plethora of theoretical studies of this type of seepage to draining channels on our own planet (see, e.g., Aravin and Numerov, 1953, PK-62,77, Vedernikov, 1939).

This work is organized as follows. In Section 2, we develop a simple analytical solution for a 2-D (not DF!) confined flow to a curvilinear ditch draining a saturated, confined aquifer. Some natural wadi channels in our study area are arcuate, like ones on Mars (Malin and Carr, 1999), but as the first approximation, we ignore the 3-D effect. Our isobaric ditch contour is an isobar generated by a line sink. In Section 3, we obtain a similar solution for a point sink in an axisymmetric flow towards a crater-shaped isobar, the piezometric head being still a harmonic function. In Section 4, we assemble an infinite array of J.R. Philip's sinks-sources of the advective dispersion equation (ADE) for the Kirchhoff potential, which models an axisymmetric, steady water movement from a horizontal water table towards a topographic depression in a partially saturated so-called Gardner soil. In Section 5 and Appendix I, 2-D seepage towards a triangular empty or partially-filled ditch is studied by the machinery of holomorphic functions. All analytical solutions of Sections 2-5 are tested against HYDRUS-2D numerical simulations. In Section 6, we outline applications to arid zone hydrology, and in particular, to Martian hydrology.

2. A Line-Sink-Generated Seepage Face Ditch Draining a Confined Aquifer

Unlike Hobbs et al. (2014) who could not identify confining layers, in the form of bedrocks and caprocks, of hydrostratigraphic units in the Martian regolith massif, we - when making excavations on our SQU site - indeed crashed a low-permeable caliche layer at a depth of 1.5-2 m. Typical values of K_s for low-permeable (cemented) layers (also called calcrete, petrocalcic horizons, see, e.g., Duniway et al., 2010), formed by the precipitation of carbonates or gypsum due to intensive evaporation in semi-arid, arid and hyper-arid regions, are given in the literature: 10^{-4} to 10^{-6} cm/s (Al-Yaqout and Townsend, 2004), 0.0002 to 0.008 m/day (Heilweil and Watt, 2011), 2.8*10^{-7} m/s (Mujica and Bea, 2020), among others. In most reported cases, these strata hydrologically cap the vadose zone rather than a perched or confined aquifer in our situation.

We could not excavate deeper to the next low-permeable horizon, and therefore, in Fig. 1a, the aquifer is not confined from below. We note that genesis of the caliche in the SQU soil profile is similar to what Michalski et al. (2013) argued about the impact of the upwelling Martian groundwater on the sequence of cemented sediments of the Red Planet. Specifically, a periodic "upwelling" (rise) of groundwater in SQU shallow perched unconfined aquifers, and intensive evaporation from the water table has – allegedly – formed the caliche and "self-confined" this aquifer by a caprock (Fig. 1a). When we "punctured" the caliche by the JCB scoop, groundwater gushed into the excavation and filled it to the level of h_A^* (Fig. 1a) within 2 hours.

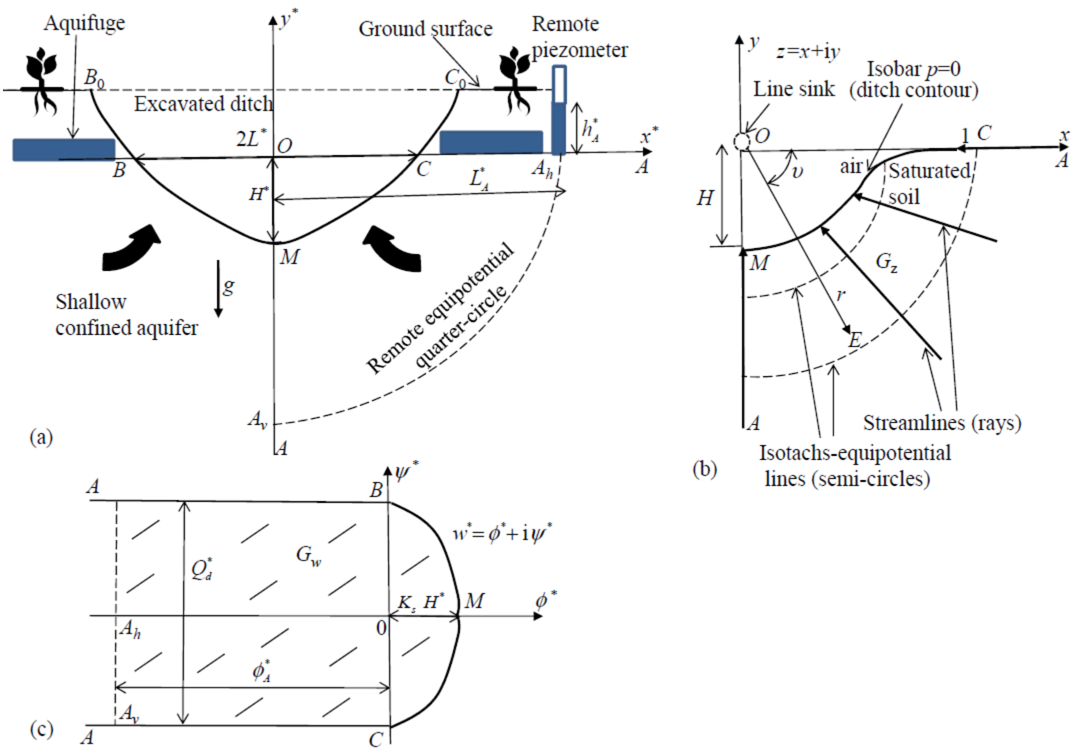


Fig. 1. A vertical cross-section of an empty ditch B_0MC_0 a); the right half of the empty ditch constructed such that the bottom of the ditch, *BMC*, is a seepage face of the flow in the domain G_z, generated by a fictive line sink at the origin O(x,y) b); the complex potential domain G_w for the whole ditch c).

Fig. 1a shows a vertical cross-section of an empty ditch B_0MC_0, which drains a dry soil layer above a horizontal aquifuge and a confined, homogeneous and isotropic aquifer (having at saturation a hydraulic conductivity K_s) beneath an originally impermeable layer (caprock) *ABCA*.

Point *A* in Fig. 1a is at infinity on the Riemann sphere. If the excavation depth H^*, counted from *ABCA* (point *M* is the deepest point of the ditch), is below the aquifuge, groundwater discharges into the ditch *BMC* (symmetric and having the width 2 *L*) with the total flow rate Q_d^* (m^2/s).

We assume that flow takes place only below the aquifuge. Goldspiel and Squyres (2011) call this layer "aquiclude", that is – in the vernacular of groundwater hydrology (Strack, 1989) – a "Martian misnomer," because Goldspiel and Squyres (2011) model impermeable rather than leaky confining layers. At a point A_h (Fig. 1a), located a certain distance $L_A^* ((L_A^* >> L)$ from O, a remote piezometer shows an elevation h_A^*. For simplicity, we ignore groundwater exfiltration upward through the caliche in Fig. 1a, i.e., leaky layer (aquitard) flow scenarios, as considered by Kacimov and Obnosov (2008, 2019).

We introduce Cartesian coordinates Ox^*y^* and the complex physical coordinate $z^* = x^* + \mathrm{i}\, y^*$ (Fig. 1b). In this section, we study the case of an empty ditch, i.e., the pressure head, p^*, along *BMC* is zero. The emptiness of the ditch can be ensured by either a high Manning slope in the direction perpendicular to the plane of Fig. 1a, so that all seeped water rapidly flows away as surface water (see, e.g., Al-Shukaili et al., 2020a), or due to intensive evaporation, provided H^* is relatively small and climate is arid enough. We note that surface water flow perpendicular to the plane of Fig. 1, i.e., along the ditch, can be easily tracked on Earth. In our fieldwork, we detected what Malin and Edgett (2000) called a Martian "seepage-fed surface runoff," slow Hortonian motion downslope a ditch, with velocities of few mm/s. On Mars, the juxtaposition of groundwater and surface water flows, reconstructed from the contemporary Martian landforms, is purely hypothetical (Malin and Carr, 1999).

As we have already pointed out in Section 1, a saturated 2-D flow from a homogeneous aquifer into a ditch (Fig. 1a) obeys Laplace':

$$\Delta h^*(x^*, y^*) = 0, \quad (1)$$

where $\vec{V}^*(u^*, v^*) = -K_s \nabla h^*$ is the Darcian velocity vector, u^* and v^* are its horizontal and vertical components, h^* is the total piezometric head, $h^* = p^* - y^*$, where the pore water pressure is $P^* = \rho_d g\, p^*$, g is gravity acceleration (9.8 m/s^2 on Earth and 3.7 m/s^2 on Mars), and ρ_d is pore fluid's (groundwater, brine, soil moisture) density. We emphasize that eq. (1) and all the results below do not assume a quasi-horizontal pore water motion, as it is postulated, for instance, in the DF model of Boatwright and Head (2019), and Luo et al. (2011).

We collected soil-sediment samples, measured K_s using standard techniques (see, e.g., Al-Shukaily et al., 2020b), and ascribed the obtained value to the textural class of HYDRUS-2D sands.

We introduce the complex potential $w^* = \phi^* + \mathrm{i}\psi^*$, $\phi^* = -K_s h^*$. The stream function ψ^* and a velocity potential ϕ^* are conjugate harmonic functions. Due to symmetry, we consider only the right half of the physical domain G_z (Fig. 1b). The following dimensionless quantities are introduced:

$$(H, L_A, x, y, z, r, h, h_A, p) = (H^*, L_A^*, x^*, y^*, z^*, r^*, h^*, h_A^*, p^*)/L, \quad (w, Q_d) = (w^*, Q_d^*)/(K_s L),$$

$$(V, u, v) = (V^*, u^*, v^*)/K_s$$

where r $(0 \le r < \infty)$ is the modulus of point $z = x + \mathrm{i}\, y$ in the aquifer (Fig. 1a)

The complex potential domain G_w is depicted in Fig. 1c. The piezometric head is assumed to be zero at points B and C. The image of the BMC isobar in G is a specific curve whose shape is unknown. Far away from the ditch, the piezometric head is constant (h_A) along a circle of a radius L_A in G_z (Fig.1a). The image of this circle is a segment A_hA_v in G_w (Fig. 1c).

Now we use the trick implemented by Zhukovskii (1948) in fluid mechanics of ideal fluids (see recent applications in Belotserkovsky and Lifanov, 1992), also employed in problems of subsurface mechanics of steady and transient Darcian flows (Fujii and Kacimov, 1998, Kacimov, 1992, 2000a, 2007, 2009, Kacimov et al., 2009, Kacimov and Obnosov, 2002, PK-62,77, Strack, 1989, 2020, Vedernikov, 1939). We place a line sink of intensity $Q=2Q_d$ at the origin of coordinates. The complex potential of this sink is:

$$\begin{gathered} w=-\mathrm{i}\frac{Q_d}{2}-\frac{Q_d}{\pi}\mathrm{Ln}\, z, \quad z=r\exp[\mathrm{i}\,\nu], \quad r=\sqrt{x^2+y^2}, \\ h=\frac{Q_d}{\pi}\mathrm{Ln}\, z, \quad \psi=-\frac{Q_d}{2}-\frac{Q_d}{\pi}\theta, \quad |\mathrm{V}|=\frac{Q_d}{\pi r}, \end{gathered} \tag{2}$$

where ν is the argument of z. It is evident from eq. (2) that the streamlines of this flow are rays converging to point O in Fig. 1b. The isotachs and equipotential lines are semicircles centered at the same point.

Now we restrict r in eq. (2) to only a positive-pressure part of the half-plane in Fig.1a, i.e., we use eq. (2) for the reconstruction of BMC by plotting the curve, which is determined by the following explicit formula:

$$Q_d \mathrm{Ln}\,\sqrt{x^2+y^2}=\pi y, \quad 0\le x\le 1, \quad -H\le y\le 0. \tag{3}$$

The depth H of the ditch, found from the solution of the equation

$$Q_d\,\mathrm{Log}\,|\,y\,|=\pi\, y\,, \tag{4}$$

is equal to

$$H=-Q_d W(-\pi/Q_d)/\pi\,, \tag{5}$$

where W stands for the Lambert W-function (implemented as **ProductLog** in Wolfram's (1991) *Mathematica*).

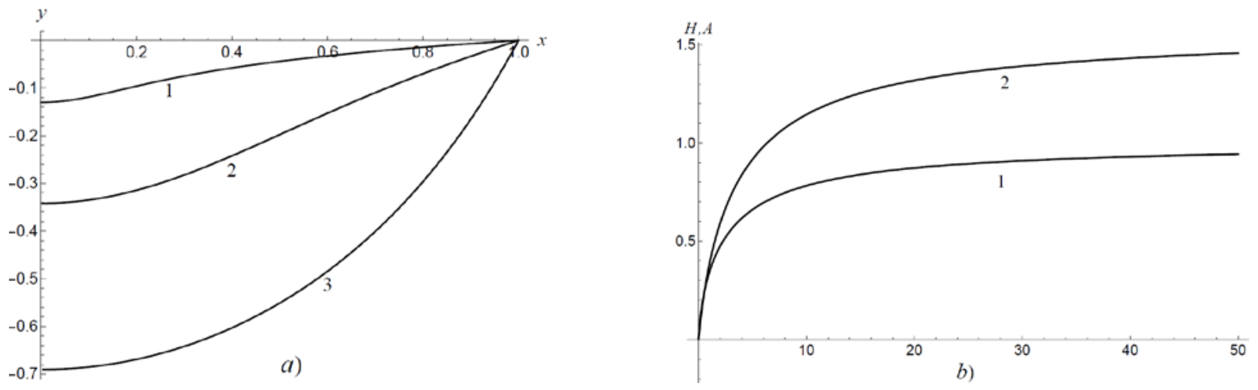


Fig. 2. a) Half-contours of sink-generated ditches for $Q_d = 0.2$, 1, and 5.86 (curves 1, 2, and 3, respectively). b) The depth H of a sink-generated ditch and its area A (curves 1 and 2, respectively) as functions of the seepage flow rate Q_d.

Fig. 2a shows the *MC* contours for sink-generated ditches (Fig. 1b) plotted for $Q_d = 0.2$, 1, and 5.86 (curves 1, 2, and 3, respectively). Curve 1 in Fig. 2b shows $H(Q_d)$. From eq. (3) it is obvious that for small Q_d, the ditch degenerates into a horizontal segment, while in another limit, $\lim_{Q_d \to \infty} H = 1$, *BMC* becomes a semicircle. The analytical solution (2)-(5) is similar to those obtained by Bazanov (see PK-62, 77) and Kacimov and Obnosov (2002) for empty ditches reconstructed by the specification of boundaries corresponding to *BMC* in the hodograph or other auxiliary planes (e.g., the plane of the Zhukovskii holomorphic function, which is defined as w–i z, see PK-62,77). We also recall the analytical and numerical solutions of Dagan (1964) and Gjerde and Tyvand (1992), who studied potential transient 2-D (not DF) flows towards horizontal drains (modeled by linear sinks and empty circles, correspondingly).

Curve 2 in Fig. 2b shows $A(Q_d)$. The cross-sectional area, A, of *BMC* (in Fig. 1a) is evaluated by the **NIntegrate** routine of *Mathematica* as:

$$A = -2\int_0^1 y(x)\,\mathrm{d}\,x\,, \tag{6}$$

Fig. 3 shows the results of modeling with HYDRUS-2D (Šimůnek et al., 2016). We selected $Q_d = 0.2/\pi$ in the analytical solution (3). With the help of eqs. (3) and (5) we generated 10 points on *MC*, imported them from *Mathematica* into HYDRUS, and made a spline curve in dimensional quantities used by HYDRUS. Also, HYDRUS uses *z* for the vertical coordinate. Fig. 3a shows the HYDRUS modeling domain bounded by an isobaric curve *MC*, for which we selected *L*=100 cm. Soil is loam from the HYDRUS soil catalog. The horizontal and vertical segments CA_h and MA_v in Fig. 3 are no-flow lines. We selected $L_A^* = 500$ cm, i.e., we assumed a quarter-circle A_hA_v with a radius of 500 cm to be a line with a constant piezometric head (we recall that HYDRUS, unlike PK-62,77 and Strack, 1989, uses *h* as a notation for the pressure head), along which we set $h_A^* = 32.2$ cm. This value was found at r^* =500 cm for a selected value of Q_d^* from the logarithmic variation of $h^*(x^*)$ along *CA*, counting from the fiducial point *C* where $h_C^* = 0$, see eq. (2). Fig. 3b shows the isotachs (also constant h^* and ϕ^* contours), and Fig. 3c shows the streamlines. The origin of the spatial coordinates is retained in Fig. 3 to indicate the position of the generating sink. HYDRUS results match very well the analytical solution.

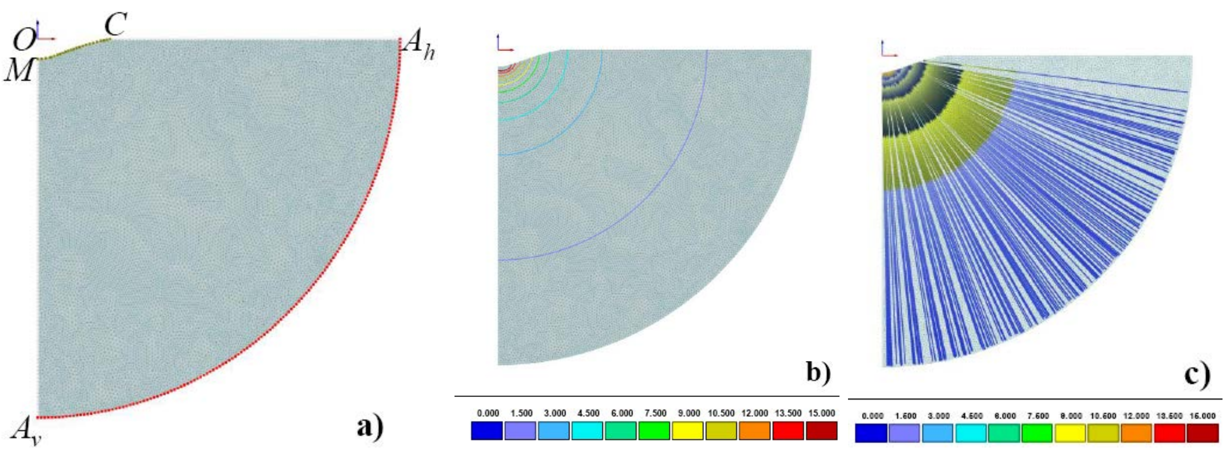


Fig. 3. The HYDRUS flow domain for a half-ditch generated by a line sink a), isotachs b), and streamlines c).

3. A Point-Sink-Generated Dimple Draining a Confined Aquifer

In this section, we use the same procedure as in Section 2 for the case of axisymmetric flow towards a depression *BMC* shown in Fig. 4 in an axial cross-section. We select a system of cylindrical coordinates $Oz^*\rho^*\kappa$, where z^* is a vertical coordinate (the same now as in HYDRUS-2D), $\rho^* = \sqrt{x^{*2} + y^{*2}}$, $r^* = \sqrt{\rho^{*2} + z^{*2}}$ and the angular coordinate κ vanishes from the analysis due to axial symmetry. At the origin *O*, we now place a point sink of intensity Q_d^* (m³/s).

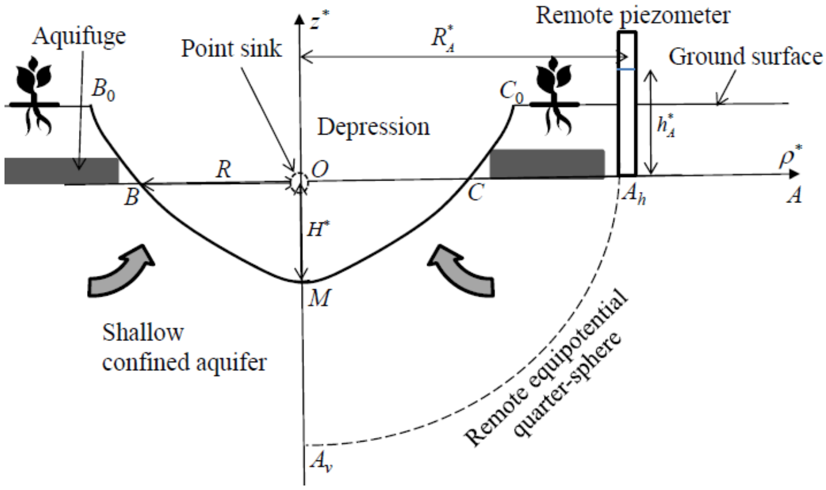


Fig. 4. An axial cross-section of an axisymmetric crater draining a confined aquifer.

The piezometric head caused by a point sink is (PK-62,77):

$$h^* = \frac{Q_d^*}{2\pi K_s}\left(\frac{1}{R} - \frac{1}{r^*}\right), \tag{7}$$

where *R* is the radius of the depression that coincides with the r^* (and ρ^*) coordinate of points *B* and *C* in Fig. 4. Eq. (7) sets up the total head equal to zero in these reference points. The depression of a depth H^* is empty and, therefore, the pressure head $p_{BMC}^* = 0$ along this curve in Fig. 4 (a surface in 3-D). At a remote point A_h, located at a distance R_A^* ($R_A^* >> R$) from point *O* (Fig. 4), we have:

$$h_A^* = \frac{Q_d^*}{2\pi K_s R}. \tag{8}$$

Point A_h is a surrogate infinity of the infinite point A on the Riemann sphere. The head of eq. (8) is illustrated in a remote piezometer sketched in Fig. 4. We recall (PK-62,77) that for flow to a 3-D sink, the piezometric head at infinity is finite, unlike for a 2-D sink (Section 2), for which the head at infinity (point A) in Fig. 1b logarithmically blows up to infinity.

We introduce dimensionless quantities: $(H, z, \rho, r, h, p, h_A, R_A) = (H^*, z^*, \rho^*, r^*, h^*, p^*, h_A^*, R_A^*)$, $Q_d = Q_d^* / (2\pi K_s R^2)$, $(V, u, v) = (V^*, u_\rho^*, v_Z^*) / K_s$, where (u,v) are now the radial and vertical velocity components.

From eq. (7), a zero-pressure isobar (seepage face) *BMC* is reconstructed, similarly to eq. (3), by an equation:

$$\rho = \sqrt{\frac{1}{(1 - z/Q_d)^2} - z^2}, \quad 0 \le \rho \le 1, \ -H \le z \le 0. \tag{9}$$

We note that some Martian hydrologists use the term “aquifer sapping face” and “crater wall” (Goldspiel and Squyres, 2011) for what is called a “seepage face” in terrestrial civil engineering (Strack, 1989).

Eq. (9) defines a quartic line. The depth H is found from eq. (9), which at $\rho = 0$ is reduced to a quadratic equation having the following physically meaningful solution:

$$H = \frac{\sqrt{Q_d^2 + 4Q_d} - Q_d}{2}. \tag{10}$$

We used eqs. (9-10) and calculated the volume of the depression *BMC* in Fig. 4 as a body of revolution:

$$V = \pi \int_{-H}^{0} \rho^2(\mathrm{z})\,\mathrm{d}\,z = \frac{\pi Q_d}{6}\left(Q_d{}^2 + 6Q_d + 6 - (4 + Q_d)\sqrt{Q_d(4 + Q_d)}\right), \tag{11}$$

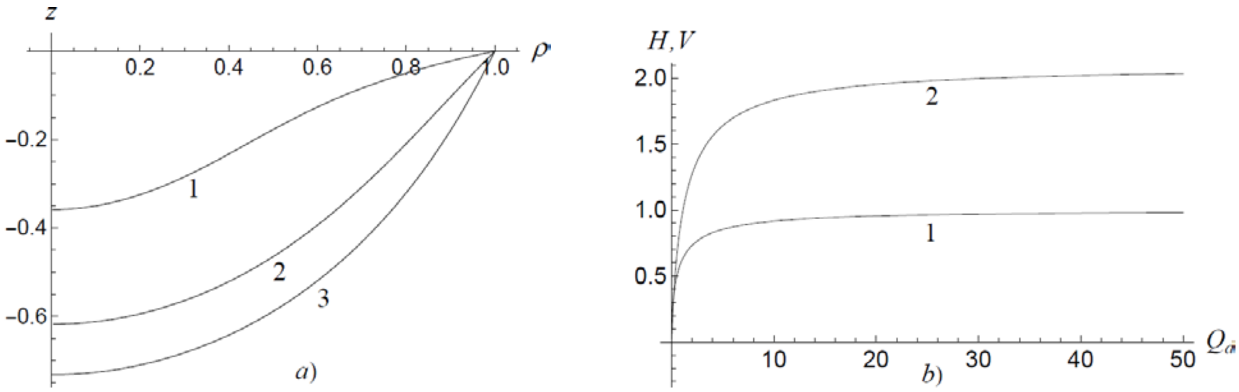


Fig. 5. a) Half-contours of a sink-generated axisymmetric topographic depression draining a confined aquifer for $Q_d = 0.2$, 1, and 2 (curves 1, 2, and 3, respectively). b) Depth and volume (curves 1 and 2, respectively) of an axisymmetric sink-generated depression in a confined aquifer.

Fig. 5a shows the isobars *MC* for $Q_d = 0.2$, 1, 2 (curves 1-3, correspondingly). In Fig. 5b, the curves (10) and (11) are plotted. Obviously, for $\lim_{Q_d \to 0} H = 1$ and $\lim_{Q_D \to \infty} V = 2\pi/3$, *BMC* becomes a hemisphere in this limit, while for small Q_d, we get a disk.

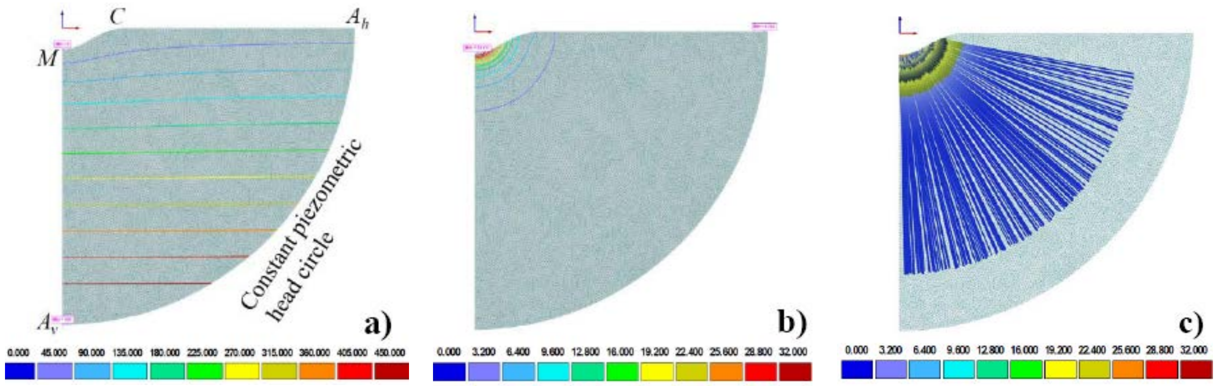


Fig. 6. HYDRUS isobars corresponding to $Q_d = 0.2$ a), isotachs b), and streamlines c).

The results of HYDRUS-2D simulations (in dimensional quantities) corresponding to the analytical case of $Q_d = 0.2$ are presented in Fig. 6. As in Section 2 (Fig. 3), we considered a loamy soil. We selected the radius *R* of the HYDRUS axisymmetric depression to be 100 cm. Using eqs.

(9) and (10), we generated ten discrete points on *MC* and made a HYDRUS spline (seepage face). The radius R_A^* of a constant piezometric head quarter-circle A_hA_v was 500 cm. According to eq. (8) the total head $h_A^* = p_{Ah}^* = 20$ cm at point A_h and, therefore, along A_hA_v, i.e., the HYDRUS pressure head is p_{AV}^* =520 cm at point A_v. Fig. 6a presents steady-state pressure head isolines. Fig. 6b illustrated isotachs and Fig. 6c plots the streamlines. Figs. 6bc perfectly fit the analytical solution.

In the next section, we focus on an opposite case of steady flow in a purely unsaturated Gardner soil, albeit the same trick will be used, *viz.* we place a fictitious sink “in the air” and use real isobars to construct topographic craters (dimples).

4. An Isobaric Depression Reconstructed From Philip’s Point-Sink Array and HYDRUS-2D Simulations

Analytical solutions of Sections 2 and 3 ignored flow in the unsaturated zone and the capillary fringe, which are important in the case of small-scale drainage entities (Abit et al., 2008, Silliman et al., 2002

In this section, we incorporate into an analytical quasilinear model all three physical factors, which control seepage: the gravitational force on Earth or Mars, Darcian resistance of the soil, and capillarity.

We use a constitutive relation for the phase conductivity:

$$k_{un} = K_s \exp[\alpha_p p^*], \tag{12}$$

where $k_{un}(p^*)$ for $p^* \leq 0$ is an unsaturated hydraulic conductivity, α_p (const > 0) is the sorptive number (1/m) (Gardner, 1958, see also Raats, 1973). J.R.Philip (1968) pioneered utilizing eqn. (12) for modeling 2D and 3D unsaturated flows (see, e.g., Pullan, 1990, Raats et al., 2002). The soil constant α_p is related to the VG-HYDRUS parameters α and n via well-known empiric relations (Acharya et al., 2012); we used eqn. (10) from Ghezzehei et al. (2007).

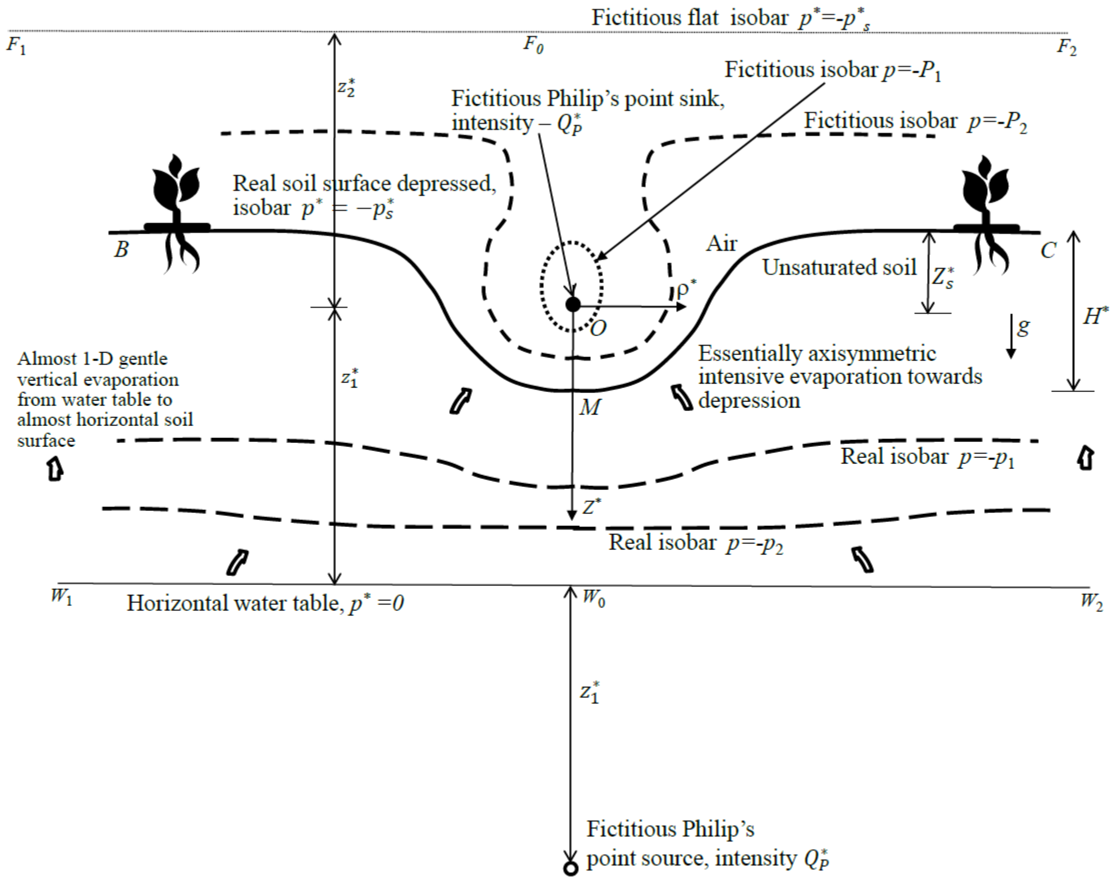


Fig. 7. An axial cross-section of evaporative flow in an unsaturated soil from a horizontal water table $W_1W_0W_2$ to a topographically depressed isobar *BMC*.

The evaporation arises from a steady upward flow from a horizontal (shallow) water table $W_1W_0W_2$ to a hot and dry soil surface *BMC* (Fig. 7) where the pressure head is constant ($p^* = -p_s^*$, $p_s^* > 0$). This surface has a topographic depression with the deepest point *M* at a depth $H^* > 0$ counted from the soil surface, which is horizontal and flat, far away from the topographic trough in Fig. 7, i.e., the water table is at a given depth b^* above a flat soil surface (near remote points *B* and *C*). *BMC* in Fig. 7 can also be related to the "sapping valleys" on Mars (see, e.g., Salese et al., 2019), albeit there is a difference in scales: the Martian depth of the water table $(Z_s^* - Z_1^*)$ is 4-5 km, while at SQU, the shallow groundwater table is only several tens of cm deep.

The depression in Fig. 7 is axisymmetric, like in Section 3. The axis of symmetry coincides with the OZ^* axis of a cylindrical system of coordinates $OZ^*\rho^*$ (introduced similarly as in Section 3 above). To be consistent with Philip, we orient OZ^* downward.

Thus, unsaturated soil in 3-D is sandwiched between the $W_1W_0W_2$ plane and the soil surface obtained by the revolution of *BMC* with respect to the axis OZ^* in Fig. 7. There are no physical singularities representing drains, subsurface emitters, or plant roots in this soil layer.

The contour *BMC* in Fig. 7 is obtained in the following manner (Philip, 1989, Obnosov and Kacimov, 2018). A fictitious point sink of intensity Q_p^* (m^3/s) is placed at the origin of coordinates, such that the water table is beneath this sink at a depth of $Z_1^* > 0$. A fictitious horizontal isobaric plane $F_1F_0F_2$ is placed at the level $Z^* = -Z_2^*$, $Z_2^* > 0$, as illustrated in Fig. 7. At this plane, the pressure head $p^* = -p_0^*$ is specified. Points *B* and *C* in Fig. 7 of an "intermediate isobar" between two sandwiching isobaric planes $F_1F_0F_2$ and $W_1W_0W_2$ are at the level of Z_s^*, which can be either positive or negative. Fictitious point sources and sinks of intensity Q_p^* mirror each other with respect to the planes $F_1F_0F_2$ and $W_1W_0W_2$. Only the first fictitious source under the water table, placed on the OZ^* axis at a depth of $2Z_1^*$, is shown in Fig. 7. The array of an infinite number of such sinks-sources generates a family of isobars, some of which are real (two dashed lines $p^* = -p_2^*$, $p^* = -p_1^*$, $p_2^* < p_1^* < p_s^*$ are sketched in Fig. 7), and others are fictitious (two dotted lines $p^* = -P_2^*$, $p^* = -P_1^*$, $p_s^* < P_2^* < P_1^*$ are sketched in Fig. 7). Mathematically, instead of *BMC* in Fig. 7, we can select another soil surface, for example, any of the two dashed isobars.

Practically, we proceed in the following manner. One goes to the field with a theta-probe and collects the moisture content, θ_s, from the topsoil of a real depression, having a certain geometrical sizes b^*, H^*. If θ_s (and hence p_s^*) is almost constant in the depression and on a flat surface away from the depression, then the position (Z_1^*, Z_2^*) of the origin of coordinates (sink's locus) and the strength Q_p^* of Zhukovskii's sinks-sources array should be mathematically adjusted in such a manner that the physical (field) depression becomes close to a mathematical isobar *BMC* in Fig. 7. In what follows, we illustrate this algorithm.

Routinely (see Philip, 1968, Raats, 1970, Obnosov and Kacimov, 2018), we introduce the Kirchhoff potential, Ω^*, which is often called the matric flux potential:

$$\Omega^*(\rho^*, Z^*) = \int_{-\infty}^{p^*} k_{un}(u)\,\mathrm{d}u = k_{un}[p^*(\rho*, Z*)]/\alpha_P, \quad \Omega^*_\infty = K_s/\alpha_P, \tag{13}$$

i.e., Ω^*_∞ is the Kirchhoff potential along $W_1W_0W_2$. Obviously, the lines of a constant Kirchhoff potential are also isobars (the values of corresponding pressure heads are found directly from eq. (13)) and isowetness curves (the corresponding water content is found from eq. (13) and the soil-water retention function).

We introduce dimensionless quantities: $(z, b, H, z_s, z_1, z_2, \rho, r, h, p)$ = $(Z^*, b^*, H^*, Z_s^*, Z_1^*, Z_2^*, \rho^*, r^*, h^*, p^*) \times \alpha/2$, $\Omega = \Omega^*/\Omega_\infty$, $Q_p = Q_p^*\alpha/\Omega_\infty$, $(V, u, v) = (V^*, u_p^*, v_Z^*)/K_s$. Then Richards' equation (0) is reduced to a linear ADE:

$$\Delta\Omega^* = \alpha_P \frac{\partial \Omega^*}{\partial Z^*}, \quad u_\rho^* = -\frac{\partial \Omega^*}{\partial \rho^*}, \quad v_Z^* = \alpha_P \Omega^* - \frac{\partial \Omega^*}{\partial Z^*}, \tag{14}$$

where Δ is the Laplacian operator in (ρ, Z), u_ρ^* and v_Z^* are the radial and vertical velocity components.

We adapt Philip's (1989) and Obnosov and Kacimov (2018) solutions to the case of flow in Fig. 7 and obtain a series:

$$\Omega = \frac{\Omega_0 \left[\exp[2(z_1+z_2)] - \exp[2(z+z_2)]\right] + \exp[2(z+z_2)] - 1}{\exp[2(z_1+z_2)] - 1} -$$
$$\frac{Q_P \exp z}{8\pi}\left[\frac{\exp(-r)}{r} - \frac{\exp(-r_1)}{r_1} - \frac{\exp(-r_2)}{r_2} + \frac{\exp(-r_3)}{r_3} + \frac{\exp(-r_4)}{r_4}\right] - \dots, \qquad (15)$$
$$r = \sqrt{\rho^2 + z^2}, \quad r_1 = \sqrt{\rho^2 + (z - 2z_1)^2}, \quad r_2 = \sqrt{\rho^2 + (z + 2z_2)^2},$$
$$r_3 = \sqrt{\rho^2 + (z + 2z_1 + 2z_2)^2}, \quad r_4 = \sqrt{\rho^2 + (z - 2z_1 - 2z_2)^2}, \dots$$

where the constant $\Omega_0 = \exp[-\alpha_p z_2]$, $0 \le \Omega_0 \le 1$ is the Kirchhoff potential along $F_1F_0F_2$. At points *B* and *C*, which are located on a physical isobar, whose remote «wings» are above the sink in Fig. 7, we have $\Omega = \Omega_s = \exp[-\alpha_p z_2]$ for $\Omega_0 \le \Omega_s \le 1$.

The first term on the RHS of eq. (15) determines 1-D unsaturated evaporative flow from $W_1W_0W_2$ to $F_1F_0F_2$ in the layer $z_1 < z < -z_2$. The terms in the square brackets of eq. (15) are responsible for sinks and sources, and consequently, for mirrored (imaged) with respect to the plains $z = z_1$ and $z = -z_2$. We emphasize that Philip (1989) and Obnosov and Kacimov (2018) studied flow for one real source representing a subsurface irrigation emitter, while all in Fig. 7 and eq. (15) are fictitious. A trick similar to ours was used by Zhukovskii (1948) in his famous formula for aerodynamics of ideal fluids. He combined mathematical singularities of a characteristic holomorphic function (two dipoles and vortex), the latter placed inside an airfoil (e.g., an impermeable cylinder). Zhukovskii also ignored a fictitious flow inside an airfoil (in the vicinity of the vortex) and only considered an exterior of a mathematical separatrix-streamline (cylinder). We confine real Darcian flow not by streamlines (for the sake of brevity, we have not even introduced the Stokes stream function, see Obnosov and Kacimov, 2018), but by two isobars (*BMC* and $W_1W_0W_2$ in Fig. 7).

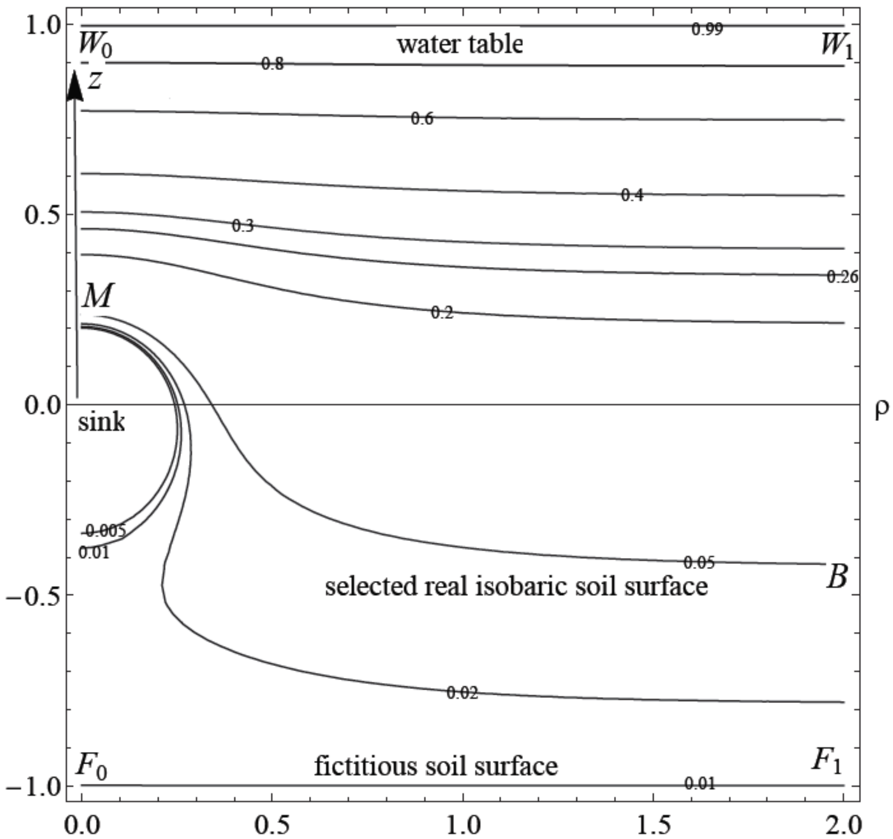


Fig. 8. Analytically computed isolines of Kirchhoff's potential for (z_1, z_2, Ω_0, Q_P) =(1, 1, 0.2, 1).

The results of computations based on eq. (15), where the series was truncated to 5 terms, are plotted by the **ContourPlot** routine of *Mathematica* in Fig. 8 for (z_1, z_2, Ω_0, Q_P) =(1, 1, 0.2, 1). As an example, in Fig. 8, we selected an isobar *BM* as the real depressed isobaric soil surface maintained at the Kirchhoff potential $\Omega_s = 0.26$. For this isobar, we used the **FindRoot** routine of *Mathematica* and evaluated $z_M = 0.389$, $z_B = z_s = -0.177$, i.e., the dimensionless depth of the depression in Fig. 8 is H=0.566.

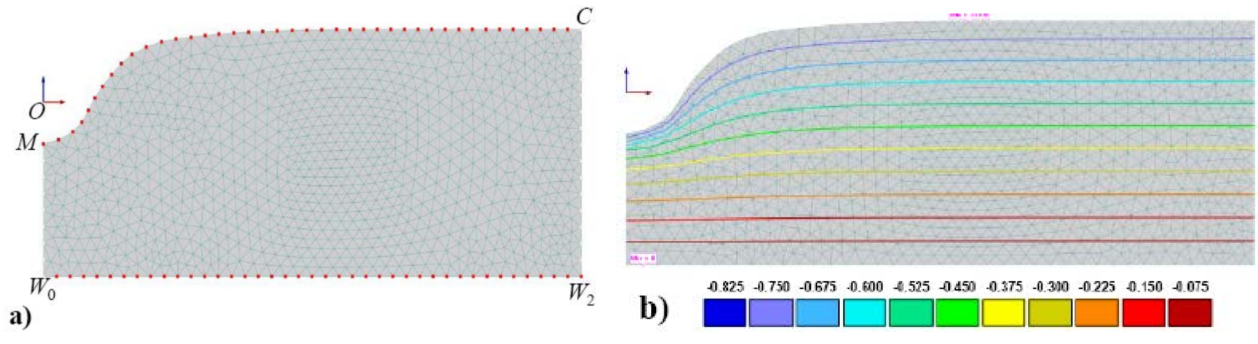
HYDRUS-2D simulations are presented in Fig. 9. We used silt loam with the VG-HYDRUS triad of soil hydraulic parameters (α, n, K_s)=(2 1/m, 1.41, 1.08 m/day). We followed Ghezzehei et al. (2007) and made Gardners's soil having α_P=1.3$\alpha\, n$, i.e., α_P=3.67 1/m. In Fig. 9a, a HYDRUS axial section is shown for the case of the pressure head $p^*_{MC} = -0.82\,\text{m}$. The flow domain, which corresponds to the analytical depression, has the HYDRUS coordinates $z_M = -0.13\,\mathrm{m}$, $z_C = 0.23\,\mathrm{m}$, and $z_{W2} = -0.55\,\mathrm{m}$. The analytical isobar *MC* is converted to the HYDRUS spline curve similarly as in Sections 2 and 3. Figs. 9b and 9c show the HYDRUS isobars and streamlines. As is evident from Fig. 9c, the ditch funnels up the flux of moisture from the water table that also evaporates at the almost flat part of the soil surface (near point *C*). In other words, a simple flow tube is realized in Fig. 9c.

In Fig. 9d, we assumed a wetter soil surface having $p^*_{MC} = -0.63\,\mathrm{m}$. The origin of coordinates O is again, like in Figs. 8 and 9a, at the locus of the analytic imaginary sink. The geometry of the flow domain has changed as compared to Fig. 9a. The corresponding analytical depression MC has now the HYDRUS coordinates $z_M = -0.1\,\mathrm{m}$, $z_C = 0.176\,\mathrm{m}$, and $z_{W2} = -0.55\,\mathrm{m}$. The isobars are shown in Fig. 9d. They seem uninteresting, like ones in Fig. 9b. Fig. 9e presents HYDRUS streamlines, which are more interesting. Indeed, the flow topology is different from one in Fig. 9c. Specifically, like in Kacimov and Youngs (2005), who studied a seemingly trivial flow domain having a nontrivial two-sheet Riemann surface in the hodograph plane (see also Anderson, 2013). In Fig. 9e, we see a separatrix $S_sS_wS_d$, which divides the unsaturated domain into three subdomains. In the first one (counted from the left in Fig. 9e), moisture ascends from segment W_0S_w of the water table to the closest (deepest) section MS_d of the ditch surface (like in Fig. 9c). At the rest of the water table, segment S_wW_2, moisture descends (infiltrates) from S_sC on the soil surface (third subdomain). Point S_w is a stagnation point. Stagnation points and separatrix streamlines also appeared in Raats (1977) for steady flows to an array of the parallel line sinks in unsaturated Gardner soils. In HYDRUS, the numerical value of the velocity vector at this point (x=0.35 m) attains a sharp minimum (1.3×10^{-5} m/day).

The second, and most interesting, subdomain in Fig. 9e is bounded by the separatrix (streamline) and a curve segment $S_dH_pS_s$. Along this isobar (we recall that $p^*_{MC} = -0.63\,\mathrm{m}$, moisture infiltrates into the soil through H_pS_s and exfiltrates back into the atmosphere through S_dH_p, making a U-turn. Therefore, the whole flow domain in Fig. 9e represents a complex flow tube similar to one in the potential flow shown in Fig. 62 of PK-77. It is noteworthy that in potential flows governed by the Laplace equation, this type of separatrix (complex flow tubes) emerges in situations when the boundary of the flow domain has three or more equipotential components. Figs. 9c and 9e illustrate that in ADE-governed unsaturated flow, a complex topology can appear in a domain bounded by two lines (surfaces) of a constant Kirchhoff potential.

A hinge point H_p in Fig. 9e is similar to ones in the classical Tothian (see, e.g., Kirkham, 1947, Toth, 2009) topology of purely saturated flows controlled by gravity, Darcian resistance, and topography of the isobaric ground surfaces (postulated by Toth), along which groundwater infiltrates-exfiltrates (but without taking into account unsaturated flow in the vadose zone). At the point H_p, the Darcian velocity vector is tangential to the soil slope. We rephrase: if the velocity vector is decoupled into the components normal and tangential to the local soil surface, then at H_p, the normal component is zero. Along S_dH_p, this normal component is oriented from the soil to the air, whereas along H_pS_s, the normal component points into the soil.

Summarizing, in the analysis of flow in an unsaturated soil represented in Fig. 9d and e, we amended the Tothian (2009) physical factors by capillarity of the soil. We then got a topological pattern shown (upside-down) at the forefront of the Freeze and Cherry (1979) and in a normal way in Figs. 2.25 – 2.26 of Radcliffe and Šimůnek (2010).



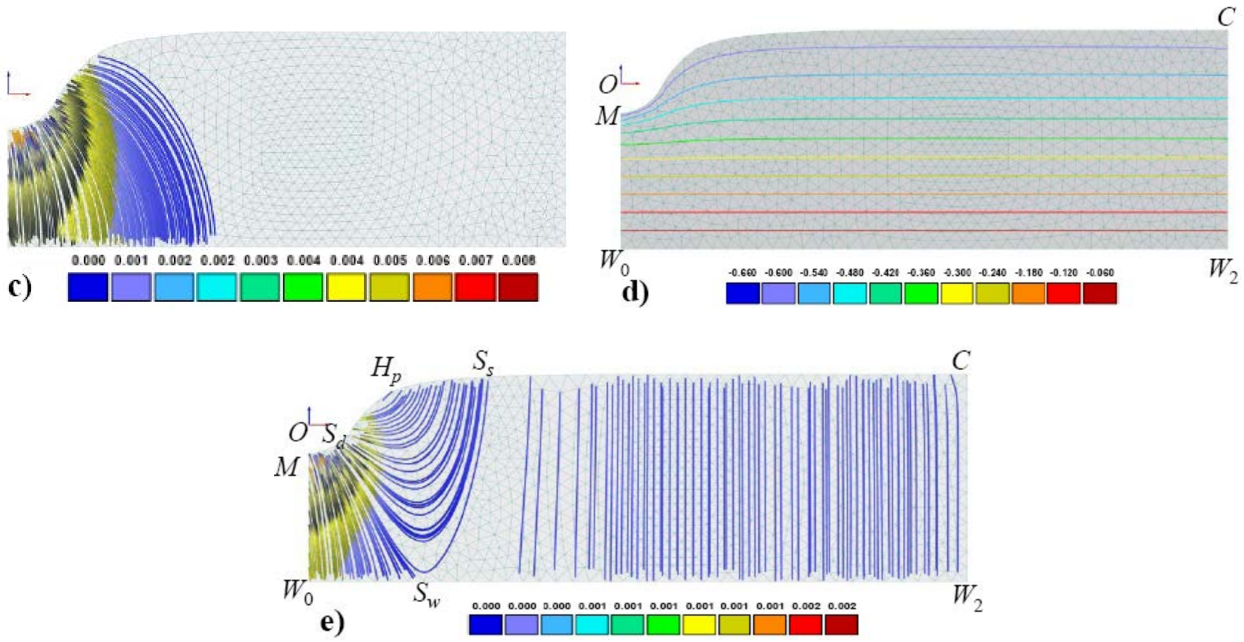


Fig. 9. HYDRUS simulations of purely unsaturated flow between a water table and an analytically found isobaric depression: a) finite element discretization of the flow domain and the boundary conditions, b) and d) pressure head contours, c) and e) streamlines. b) and c) flow to a relatively dry soil surface, d) and e) evaporation-infiltration at a relatively wet soil surface.

5. A Triangular Ditch in a Confined Aquifer

In this section, we return to steady, and capillarity-free flow in saturated soil. We consider 2-D potential flow in a vertical plane (Fig. 10a) and use the notations of Section 2, i.e., the complex physical coordinate is again $z^*=x^*+\mathrm{i}\,y^*$.

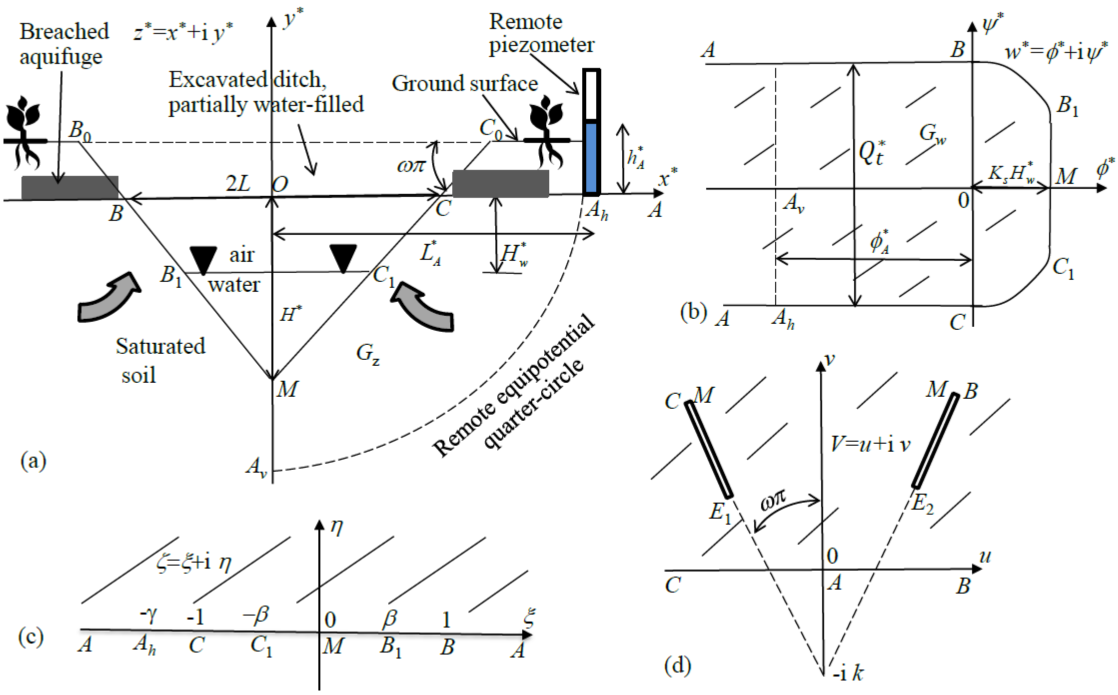


Fig. 10. A vertical cross-section of seepage flow towards a triangular ditch in a confined aquifer a), a complex potential domain b), a reference plane c), and a mirror-image in the hodograph plane d).

A caprock of a confined aquifer is now breached by a triangular isosceles ditch B_0MC_0, having a bank slope $\pi\omega$, $0 \le \omega \le 1/2$. The flow domain, G_z, is now the half-plane $y^* < 0$ with a triangular indent *BMC*. Ditch's depth is $H^* = L^* \tan \pi\omega$.

Unlike in Section 2, we allow water to accumulate in the ditch up the horizon B_1C_1 at the level H_w^*. The piezometric head at infinity (point *A* in Fig. 10ab) is infinite but, as in Section 2, we replace this infinity by a "surrogate infinity," *viz.* an equipotential line A_vA_h (shown as a dashed semi-circle and a segment in Figs. 10a and 10b, respectively). Practically, in a remote piezometer located distance L_A^* from the ditch axis, the total head, h_A^*, counted from points *B* and *C*, is measured. The overall flow rate into the ditch is Q_t^*. The domain G_w of the complex potential *w* is sketched in Fig. 10b. The boundary conditions for *w* are

$$\begin{cases} \phi^* = -K\, y^* \quad \text{along } BB_1 \text{ and } CC_1, \\ \phi^* = K\, H_w^* \quad \text{along } B_1M \text{ and } MC_1, \\ \psi^* = Q_t^*/2, \quad \text{along } AB, \\ \psi^* = -Q_t^*/2, \quad \text{along } AC. \end{cases} \qquad (16)$$

To the dimensionless quantities introduced in Section 2, we add

$$(H_w, L_A, h_A) = (H_w^*, L_A^*, h_A^*)/L, \quad Q_t = Q_t^*/(K_s L).$$

We map the tetragon G_z onto a half-plane $\eta \geq 0$ of a reference plane $\zeta = \xi + \mathrm{i}\,\eta$ (Fig. 10c) with the correspondence of points $M \to 0$, $A \to \infty$, $B \to 1$, $C \to -1$, $B_1 \to \beta$, $C_1 \to -\beta$. Appendix I presents the details of our solution. An alternative method for the analytical solution (see the taxonomy of these methods in Aravin and Numerov, 1953, see also Bear, 1972) involves a conformal mapping of G_z onto a mirror-image of the pentagon in the hodograph plane. Fig. 10d sketches such a half-plane with two semi-infinite cuts for the case of an empty ditch.

Fig. 11 shows the function $Q_t(H)$ (curve 1) computed by eq. (A9) for $h_A = 3$ and $L_A = 5$. Similarly to Al-Shukaili et al. (2020b) and Kacimov (1985), we introduce the Morel-Seytoux dimensional seepage flow rate $\mu = Q_t^*/(K_s\sqrt{A^*})$, $\quad A^* = H^* L^*$.

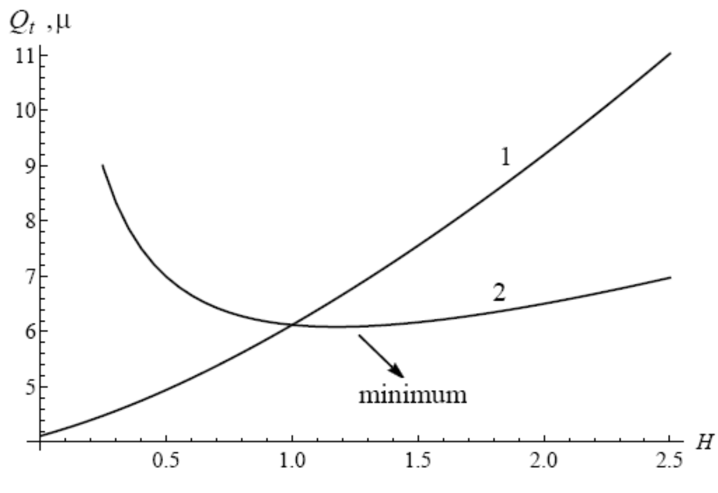
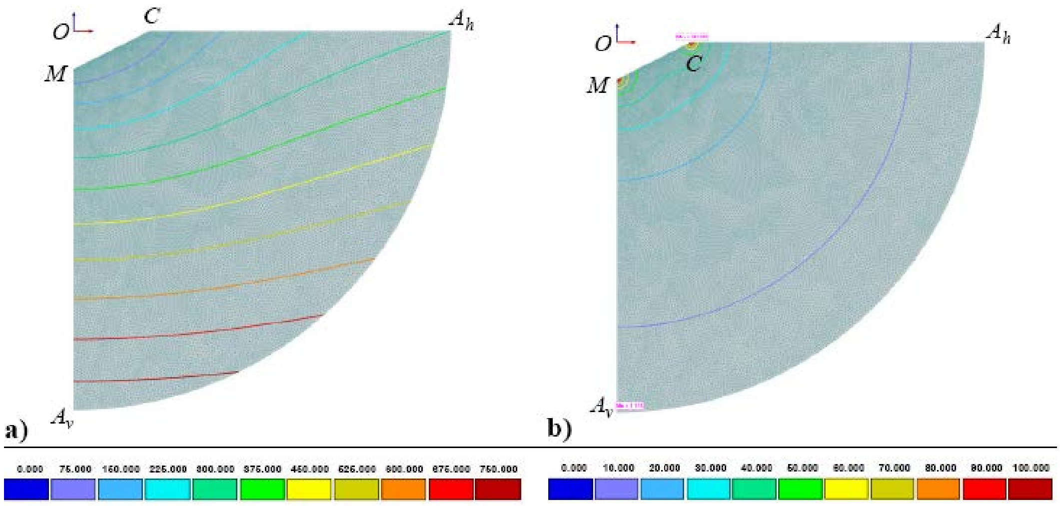


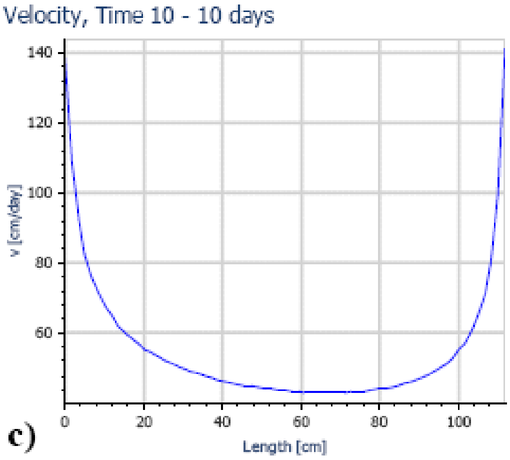
Fig. 11. Dimensionless seepage flow rates Q_t (curve 1) and μ (curve 2) as functions of H for an empty ditch, $h_A = 3$, and $L_A = 5$.

Curve 2 in Fig. 11 plots the function $\mu(H)$. The minimum of the function is $\mu \approx 6.085$ and $H \approx 1.177$. This minimum can be interpreted as a solution to the following optimal shape design problem, OSDP (similar to ones solved in Kacimov, 2005, 2006b):

OSDP 1. Determine an empty ditch, into which a minimal quantity Q_t^* of water seeps, provided the ditch area A^* and remote piezometric data (h_A^* and L_A^*) are fixed.

Curve 2 in Fig. 11 provides evidence that there is a unique and global solution to OSDP1. The minimum is "mild" (similar to ones in Kacimov, 1985), i.e., triangular ditches having slopes close to the best one will not deviate much from the best (minimal) seepage exfiltration rates. Problems similar to OSDP1 are common in civil engineering when a pit (e.g., for a building foundation) is excavated, and minimum exfiltration is desired, or pit's drainage measures are planned.





c)

Fig. 12. HYDRUS simulations for a triangular empty ditch having H^*=50 cm, L^*=100 cm, L_A^* = 500 cm, and h_A^*=300 cm: isobars a), isotachs b), and Darcian velocities along the ditch side MC c).

For comparisons, in Fig. 12, we present the results of HYDRUS simulations for the tetrad of parameters (H^*, L^*, L_A^*, h_A^*) =(50 cm, 100 cm, 500 cm, 300 cm). Fig. 12a shows the isobars. Fig. 12b illustrates the isotachs. In the analytical solution, the Darcian velocity approaches infinity at points *M* and *C*. At the same time, it attains a minimal value, which corresponds to the tip of the cut in the hodograph domain (Fig. 10d). HYDRUS shows the same. In Fig. 12c, we plot the distribution of the velocities along the ditch side *MC*, where a minimum $|V|$=43.1 cm/day is attained at s_d =66.7 cm, where s_d is the arc coordinate counted from point *M* in Fig. 10a. Curves like the one in Fig. 12c are useful for the assessment of erosional stability of slopes of ditches, since heaving and suffusion are determined by exit values and directions of hydraulic gradient vectors $\vec{i}$. For example, PK-62,77 considers the condition $|\vec{i}|>1$ as a criterion of instability, according to which Fig. 12c illustrates that the ditch side *MC* in Fig. 12 is absolutely unstable at any s_d.

The HYDRUS-computed Morel-Seytoux factor μ is equal to 6.97 for the trench in Fig. 12, while the analytical solution gives μ=7.08. We computed other triangular ditches in HYDRUS-2D, by using a discrete variation of the slope angle. We evaluated Q_t^* from the steady-state limit of the cumulative flux into the ditch from a "feeding semicircle" (Fig. 10a, dashed curve). Then we converted this dimensional Q_t^* into the dimensionless μ, which fits well the analytical curve 2 in Fig. 11.

Collation of solutions for seepage into a triangular ditch and sink-generated ditch in Section 2 is done in the following manner. For the case presented in Fig.11, we use eq. (2) and evaluate $Q_d = \pi h_A / \text{Ln } L_A = 3\pi / \text{Ln } 5 = 5.86$. The corresponding isobar is shown as curve 3 in Fig.2. Using eq.(6) we evaluated the area and the Morel-Seytoux factor $\mu = 5.94$. We surmise that this is a global optimum in the class of arbitrary ditch shapes.

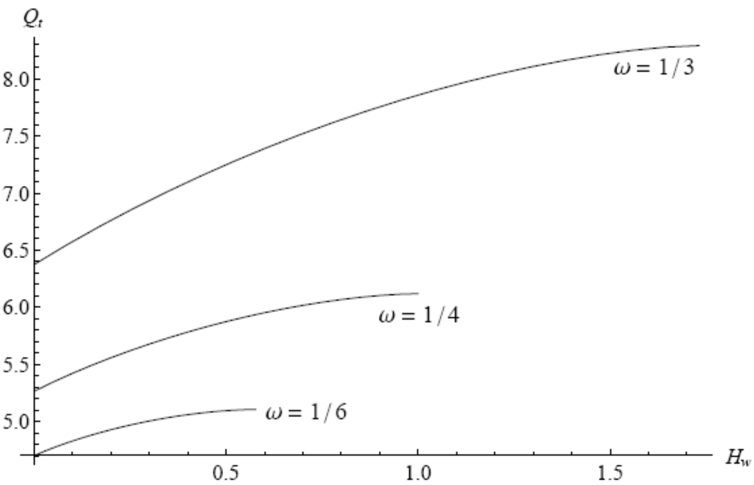


Fig. 13. The dimensionless seepage rate Q_t into a triangular partially filled ditch as a function of H_w for three slopes (ω=1/6, 1/4, and 1/3), h_A=3, and L_A=5.

Next, we considered a general case of a partially-filled ditch (Fig. 10a). In Fig. 13, curves 1-3 show the graphs of the functions $Q_t(H_w)$ for three ditch slopes: π/6, π/4, and π/3, and the same $h_A = 3$ and $L_A = 5$ as in Fig. 11. In the regimes of partial filling, we can solve OSDPs similar to

OSDP1. Regional groundwater flow can be easily taken into account, similarly to Ilyinsky and Kacimov (1992b). The analytical solution to the steady problem can be readily extended to a transient regime of exfiltration into a gradually-filling ditch. We assume that the value h_A^* in Fig. 10a is high enough such that when we make a ditch, seepage into it does not reduce much h_A^*, and the soil in Fig. 10a remains saturated (an aquifer remains confined).

Let us assume, similarly as in HYDRUS, that during the transient seepage phase, the porous skeleton and water are incompressible. If evaporation from the ditch is ignored, then from the principle of mass conservation, the following ODE follows:

$$\frac{\mathrm{d}A_s^*}{\mathrm{d}t^*} = Q_t^*[H^*(t)], \qquad (17)$$

where t^* is dimensional time and $A_s^*(t)$ is the area of the swelling triangle B_1C_1M in Fig. 10a (for comparisons see Al-Shukaili et al., 2020b, where emptying of triangular trenches has been studied). Obviously, the initial condition in eq. (17) is $A_s^*(0) = 0$, i.e., seepage-filling commences from an empty ditch stage. We put $A_s^*(t) = \left(H^* - H_w^*(\mathrm{t})\right)^2 \operatorname{cotan} \pi\omega$ into eq. (17) and get the following Cauchy problem:

$$2\operatorname{cotan}\pi\omega\left(H_w^*(t) - H^*\right)\frac{\mathrm{d}H_w^*(t)}{\mathrm{d}t^*} = Q_t^*[H_w^*(t)], \quad H_w^*(0) = 0, \quad 0 < t < \infty. \qquad (18)$$

We used the **NDSolve** routine of *Mathematica* to solve the nonlinear 1-st order ODE (18) numerically, with the RHS taken from eq. (A9). For this purpose, we interpolated 51 point values of $Q_t^*[H_w^*]$ (see curve 1) in the steady-state problem. The results of computations are shown in Fig. 13 in dimensionless quantities, to which we added a dimensionless time $t = t^* K_s / L$. In Fig.14, we plotted the functions $H_w(t)$ obtained from the solution of eq. (18) for $h_A = 3$ and $L_A = 5$, and the same ditch slopes ω = 1/6, 1/4, and 1/3, (curves 1-3, correspondingly). As is evident from these draw-up curves, the ditches are filled out in finite time intervals, t_f =(0.115, 0.175, 0.24), respectively. At $t > t_f$, seepage into a ditch continues due to an artesian pressure in an aquifer, and exfiltrating water has to be removed (e.g., pumped out) if we want to keep points C and B in Fig. 10a at zero pressure heads (i.e., not ponded). That is equivalent to the condition that the horizon BC remains static (see in Kacimov, 1991, a similar flow regime in a horizontal plane). The type of curves in Fig. 14 can be used for the determination of K_s (similarly to Al-Shukaili et al., 2020b and Kacimov, 2000b) that is also useful for deep structure constructors when pumping tests in deep confined aquifers are not feasible. If an aquifer is unconfined, then for a suddenly emptied ditch (or an auger hole, Kacimov 2000,b), $t_f = \infty$

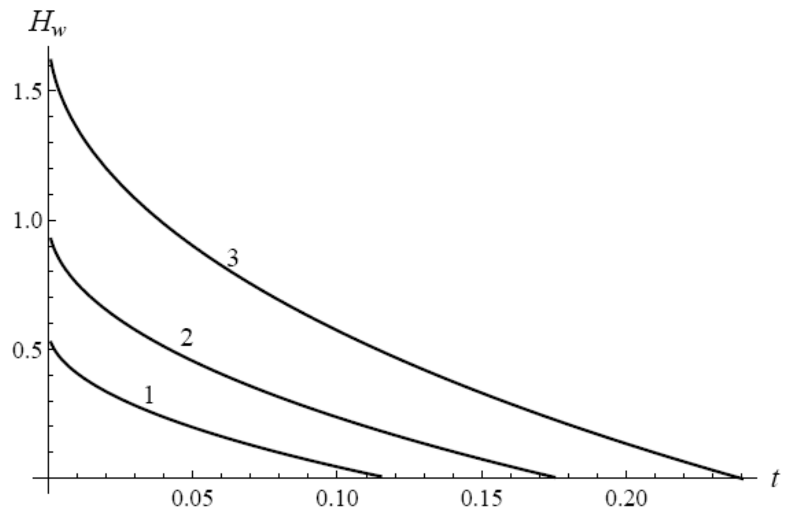


Fig. 14. Dimensionless draw-up curves H_w (1, 2, and 3) as functions of dimensionless time *t* for different values of ω (1/6, 1/4, and 1.3,respectively).

A transient analytical solution can be compared with a numerical one, which involves a new reservoir boundary condition of HYDRUS (Šimůnek et al., 2018). This new HYDRUS option allows for the consideration of variable storage of a furrow channel, or well bore (e.g., Bristow et al., 2020, Sasidharan et al. 2018, 2019, 2020), the contours of which are subject to the condition of a constant piezometric head on the submerged part and a seepage face on the empty part of the draining entity. HYDRUS then considers flow into or out of the reservoir through its interface with the subsurface transport domain depending on the prevailing conditions in the transport domain, such as the position of the groundwater table or piezometric heads, and external fluxes, such as pumping, or injection, or evaporation, or precipitation. The furrow reservoir can be adapted for our triangular ditch. The problem considered in Fig. 12 was rerun under transient conditions, starting with an empty ditch with dimensional sizes of H^* =50 cm and L^*=100 cm, corresponding to the case illustrated by curve 1 in Fig. 14. The HYDRUS draw-up curve $H_w^*(t^*)$ is shown in Fig. 15.

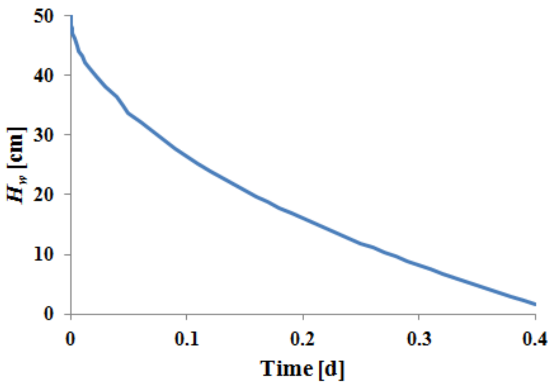


Fig. 15. A draw-up curve computed by HYDRUS for ω =1/6, H^*=50 cm, and L^*=100 cm.

There is a perfect match between the results in Fig. 15 and curve 1 in Fig. 14 from the analytical solution in dimensionless variables. We also compared the transient fluxes into the half-ditch, i.e., the analytical Q_t/2 from Fig. 11 (curve 1) and one computed by HYDRUS. For example, at t^*=0.1 day, the HYDRUS flux is about 6075 cm^2/day, whereas the analytical flux $Q_t^*/2$ is 6063 cm^2/day.

6. Comparison of Terrestrial and Martian Evaporating Spots

As an example, let us consider depression 1 (Q_d=0.2) in Fig. 5. Assume R^* =100 cm, H^*=35 cm, and a loamy soil having K_s=25 cm/day. Let the water evaporate from the disk πR^{*2} with a constant intensity e. In order to keep the crater empty, this value has to be $e = 0.2 \times 25/\pi \approx 1.6$

cm/day, which is - according to the Penman-Monteith equation and field measurements of evaporation - a too high value even in the Empty Quarters of Oman.

If one wants to reconstruct desertification of Mars or intercept ascending moisture fluxes from the deep water table, for example in projects of Mars colonization, then the topography like the one in Fig. 9 should be taken into account similarly to what we have on Earth. Hummocky terrains or even earth dams (Kacimov and Brown, 2015) generate complex 2D and 3D Toth-type Darcian fluxes. Specifically, flat landforms with a homogeneous vadose zone and a shallow water table are easy to examine for water storage and fluxes: it suffices to measure the depth of the water table and the pressure head (moisture content) of the topsoil to infer what is the value and direction of the flux. If the vadose zone is heterogeneous, the flux magnitudes can attain nontrivial extrema, even of a simplest type of a two-layered soil (Kacimov et al., 2019). Fig. 9e illustrates that routine measurements of the boundary condition on soil surfaces with depressions are not sufficient for the determination of the directions and magnitudes of the motion of water governed by the Richards equation.

7. Concluding Remarks

The mainstream groundwater hydrology focuses on MAR (Managed Aquifer Recharge), a method of replenishing aquifers, especially in arid environments. We are putting forward the opposite concept of MAD (Managed Aquifer Discharge) of shallow aquifers, which have to be intelligently drained to mitigate the damage inflicted by groundwater inundation. The analysis in this paper will be useful in planned drainage of water-logged built-up areas and further studies in dryland ecohydrology (Camporeale et al., 2019). Maillart’s desert ditches of Central Asia (similarly as ones in humid Holland), can be viewed as a technique of lowering a shallow water table or piezometric surface. The shape of these ditches can be mathematically approximated by isobars made by hydrodynamic sinks, comparable with Zhukovskii’s method of a smart composition of singularities in aerodynamics. The Darcian flows governed by the Laplace equation in saturated media or ADE for steady flows in unsaturated media can be viewed as created by intelligently located singularities. The resulting isobars model the outlets in seepage to empty curvilinear ditches and depressions. Seepage to triangular ditches, both empty and partially filled, is modeled by conformal mappings and the BVP method. The transient stage of filling, i.e., the variable storage in ditches, is examined by solving a Cauchy problem for a nonlinear ODE with type curves showing how rapidly the ditch is filled up. HYDRUS-2D simulations well agree with analytical modeling. Analytical solutions and HYDRUS reconstruct jointly isobars, isohumes, isotachs, and streamlines - the theoretical arsenal of soil physicists and civil engineers working with Earth and Mars soil and water systems. The proper design of ditches, pits, and other artificial excavations to be operated in shallow water table environments, as well as understanding of subsurface hydrology of natural wadis and depressions, can incorporate the findings of this paper.

Appendix

The Schwarz-Christoffel integral maps conformally the reference half-plane on G_z:

$$z(\zeta)+\mathrm{i}H=-\mathrm{c}_1(1-\mathrm{i}H)\int_0^{\zeta}(1-\tau^2)^{-\omega}\tau^{2\omega}\,\mathrm{d}\tau=-(1-\mathrm{i}H)\frac{\mathrm{B}_{\zeta^2}(1/2+\omega,1-\omega)}{\mathrm{B}(1/2+\omega,1-\omega)}, \qquad \text{(A1)}$$

where B and B_{ζ^2} are the complete and incomplete beta-functions, respectively (Abramowitz and Stegun, 1969, formula 6.6.1). The positive constant $c_1=2/\mathrm{B}(1/2+\omega,1-\omega)$ in (A1) is determined from the condition $z(1)=-1$. Note that $z(\zeta)$ obeys the symmetry condition $z(-\bar{\zeta})=-\overline{z(\zeta)}$ as it should be.

From eq. (A1), along two seepage faces (BB_1 and CC_1), we have:

$$y(\xi)=H\left(\frac{\mathrm{B}_{\xi^2}(1/2+\omega,1-\omega)}{\mathrm{B}(1/2+\omega,1-\omega)}-1\right),\quad \beta\le|\xi|\le 1, \qquad \text{(A2)}$$

where $-\beta$ is the affix of point C_1. We determine β from the relation $y(-\beta) = -H_w$, that gives:

$$\frac{\mathrm{B}_{\beta^2}(1/2+\omega, 1-\omega)}{\mathrm{B}(1/2+\omega, 1-\omega)} = \frac{H - H_w}{H}. \qquad \text{(A3)}$$

Next, we introduce a holomorphic function:

$$W(\zeta) = w(\zeta) - \frac{\mathrm{i}Q_t}{\pi}\arcsin\zeta. \qquad \text{(A4)}$$

The function (A4), in accordance with (16) and properties of the function arcsin, satisfies the conditions:

$$\begin{cases} \mathrm{Re}\,W(\xi) = -y(\xi), & \xi \in (-1,-\beta)\cup(\beta,1), \\ \mathrm{Re}\,W(\xi) = H_w, & \xi \in (-\beta,\beta), \\ \mathrm{Im}\,W(\xi) = 0, & \xi \in (-\infty,-1)\cup(1,\infty). \end{cases}$$

Here $y(\xi)$ is determined in eq. (A2). Then the function $W_0(\zeta) = W(\zeta)/\sqrt{1-\zeta^2}$ obeys the following conditions along the ξ-axis:

$$\begin{cases} \mathrm{Re}\,W_0(\xi) = -y(\xi)/\sqrt{1-\xi^2}, & \xi \in (-1,-\beta)\cup(\beta,1), \\ \mathrm{Re}\,W_0(\xi) = H_w/\sqrt{1-\xi^2}, & \xi \in (-\beta,\beta), \\ \mathrm{Re}\,W_0(\xi) = 0, & \xi \in (-\infty,-1)\cup(1,\infty). \end{cases} \qquad \text{(A5)}$$

Eqs. (A5) comprise a Riemann (Schwartz) BVP (Henrici, 1993, PK-62,77). The function $W(\zeta)$ is finite at two transition points $\xi = \pm 1$, continuous at points $\pm\beta$, and has a logarithmic singularity at infinity. Hence, the function $W_0(\zeta)$ has integrable singularities at points $\xi = \pm 1$ and vanishes at infinity. Therefore, the unique solution of the problem (A5) gives the Schwartz operator for the upper half-plane (Henrici, 1993, PK-62,77). Thus, after simple algebra, we get

$$W(\zeta) = \frac{2\zeta\sqrt{1-\zeta^2}}{\pi \mathrm{i}}\left(H_w \int_0^\beta \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}(\tau^2-\zeta^2)} - \int_\beta^1 \frac{y(\tau)\,\mathrm{d}\tau}{\sqrt{1-\tau^2}(\tau^2-\zeta^2)} \right), \qquad \text{(A6)}$$

where, for the sake of definiteness, the branch of the radical $\sqrt{1-\zeta^2}$ is fixed positive for $\zeta = \xi \in (-1,1)$ in the upper half-plane. This branch and, therefore, the function (A6) satisfy the symmetry condition $W(-\bar{\zeta}) = \overline{W(\zeta)}$. The integrals in (A6) are evaluated by the Sokhotski-Plemely formula: the first for $\zeta \to \xi \in (0,\beta)$, and the second for $\zeta \to \xi \in (\beta,1)$ (Henrici, 1993).

We illustrate the computations for the case of an empty ditch (H_w=H). We use the piezometric data at point A_h (Fig.10a), i.e., $\phi_A = -h_A$ First, from eq. (A1) along the ray *CA* we get:

$$x(\xi) = 1 + 2\sqrt{1+H^2}\int_1^{-\xi}(\tau^2-1)^{-\omega}\tau^{2\omega}\,\mathrm{d}\tau \,/\, \mathrm{B}(1/2+\omega, 1-\omega), \quad -\infty \le \xi < -1. \qquad \text{(A7)}$$

From eq. (A5) we determine the affix $-\gamma$ of point A_h in the reference plane (Fig. 10c):

$$L_A = x(-\gamma). \qquad \text{(A8)}$$

We use the **FindRoot** routine of *Mathematica* to solve eq. (A8) for γ. We put the found root of eq. (A8) into eq. (A6), and at $\phi_A = \phi(-\gamma) = -Q_t \log(\gamma + \sqrt{\gamma^2-1})/\pi + \mathrm{Re}W(-\gamma)$ arrive at the equality:

$$Q_t = \frac{2\gamma\sqrt{\gamma^2-1}}{\log(\gamma+\sqrt{\gamma^2-1})}\left(H_w\int_0^\beta \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}(\gamma^2-\tau^2)} - \int_\beta^1 \frac{y(\tau)\,\mathrm{d}\tau}{\sqrt{1-\tau^2}(\gamma^2-\tau^2)}\right) - \frac{\pi\phi_A}{\log(\gamma+\sqrt{\gamma^2-1})}. \qquad \text{(A9)}$$

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Figure Captions

Fig. 1. A vertical cross-section of an empty ditch B_0MC_0 a); the right half of the empty ditch constructed such that the bottom of the ditch, *BMC*, is a seepage face of the flow in the domain G_z, generated by a fictive line sink at the origin O(x,y) b); the complex potential domain G_w for the whole ditch c).

Fig. 2. a) Half-contours of sink-generated ditches for $Q_d = 0.2$, 1, and 5.86 (curves 1, 2, and 3, respectively). b) The depth H of a sink-generated ditch and its area A (curves 1 and 2) as functions of seepage flow rate Q_d.

Fig. 3. The HYDRUS flow domain for a half-ditch generated by a line sink a), isotachs b), and streamlines c).

Fig. 4. An axial cross-section of an axisymmetric crater draining a confined aquifer.

Fig. 5. a) Half-contours of a sink-generated axisymmetric topographic depression draining a confined aquifer for $Q_d = 0.2$, 1, and 2 (curves 1, 2, and 3, respectively). b) Depth and volume(curves 1 and 2, respectively) of an axisymmetric sink-generated depression in a confined aquifer.

Fig. 6. HYDRUS isobars corresponding to $Q_d = 0.2$ a), isotachs b), and streamlines c).

Fig. 7. An axial cross-section of evaporative flow in an unsaturated soil from a horizontal water table $W_1W_0W_2$ to a topographically depressed isobar *BMC*.

Fig. 8. Analytically computed isolines of Kirchhoff’s potential for (z_1, z_2, Ω_0, Q_P) =(1, 1, 0.2, 1).

Fig. 9. HYDRUS simulations of purely unsaturated flow between a water table and an analytically found isobaric depression: a) finite element discretization of the transport domain and the boundary conditions, b) and d) pressure head contours, c) and e) streamlines. b) and c) pure evaporation to a relatively dry soil surface, d) and e) evaporation-infiltration to-from a relatively wet soil surface.

Fig. 10. A vertical cross-section of seepage flow towards a triangular ditch in a confined aquifer a), a complex potential domain b), a reference plane c), and a mirror-image in the hodograph plane d).

Fig. 11. Dimensionless seepage flow rates Q_t (curve 1) and μ (curve 2) as functions of H for an empty ditch, h_A=3, and L_A=5.

Fig. 12. HYDRUS simulations for a triangular empty ditch having H^*=50 cm, L^*=100 cm, L_A^* = 500 cm, h_A^*=300 cm: isobars a), isotachs b), and Darcian velocities along the ditch side MC c).

Fig. 13. The dimensionless seepage rate Q_t into a triangular partially filled ditch as a function of H_w for three slopes (ω=1/6, 1/4, and 1/3), h_A=3, and L_A=5.

Fig. 14. Dimensionless draw-up curves H_w (1, 2, and 3) as functions of dimensionless time t for different values of ω (1/6, 1/4, and 1.3, respectively).

Fig.15. A draw-up curve computed by HYDRUS for ω=1/6, H^* =50 cm, and L^*=100 cm.

Abbreviations:

1) ADE = advective dispersion equation

2) BVP = boundary value problem

3) DF = Dupuit-Forchheimer

4) LHS, RHS=left hand side, right hand side

5) ODE = Ordinary Differential Equation

6) OSDP = Optimal Shape Design Problem

7) PK-62,77 = Polubarinova-Kochina, P.Ya., 1962. Theory of Ground Water Movement. Princeton University Press, Princeton. The second edition of the book (in Russian) was published in 1977, Nauka, Moscow.

8) VG = van Genuchten

HIGHLIGHTS

- Steady 2-D flows in a vertical plane and axisymmetric flows from a shallow horizontal water table to isobaric ditches (topographic depressions) are studied
- Method of images applied to the Laplace and advective dispersion equations, conformal mappings and HYDRUS-2D are utilized
- Saturated and unsaturated moisture motion in homogeneous soils is illustrated by isobars, isohumes, isotachs and streamlines
- Applications to arid zone hydrology on Earth and Mars are proposed