# The Modified Method of Contour Dynamics for Modeling of Vortical Structures

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***Abstract*—One of the most effective methods of modeling of the vortical structures described by the 2-dimensional equation of carry of a vortex and by the Poisson equation for a flow function, namely, the contour dynamics (CD) method which is based on representation of a vortical stream by the finite area vortical regions is considered. The modification of the CD method minimizing the errors arising at its direct application to the description of vortical structures is offered.**

***Keywords—vortices, modeling, hydrodynamics, modified contour dynamics method, algorithm***

## I. Introduction

When solving any problem, inevitably a question arises about the optimal method for solving it, and the choice of method is determined by a number of factors, among which it is necessary to point out its temporal characteristics, simplicity (at condition of method adequacy to the solved problem ), reliability and universality. For problems studied in the framework of the theory of vortex motions, the choice of methods for numerical simulation is often limited by a set of well-known schemes based on a finite-difference representation of partial differential equations describing the motion of a fluid, gas, and plasma. However, this approach is not always acceptable due to the limitations that are imposed on difference grids.

Significant progress in the field of the study of vortex phenomena began in the late 60s of the last century, when powerful computers appeared and a new science – computational fluid dynamics was born [1], [2].

In this paper we consider one of the most effective methods based on the representation of vortex structures in the form of vortex regions of finite area (FAVRs) [3] when the description of their dynamics is simplified significantly. For numerical study, the use of the contour dynamics (CD) algorithm [4] is proposed and its modification, allowing to model not only individual vortex structures, but also to study the dynamics of *N*-vortex systems consisting of individual vortices, depending on their relative position, order of symmetry, value and sign of vorticity, is considered.

## II. Basic Equations and Methods for Modeling of Vortical Phenomena

In the simplest case of an incompressible viscous medium, the equations that describe the motion of a fluid are the Navier-Stokes equations:

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{v}, \qquad \nabla\mathbf{v} = 0 \tag{1}$$

where $\nu$ is a kinematic viscosity, $\rho$ is the density of medium, $p$ is a pressure. Since we are considering a vortex motion, excluding pressure and calculating the rotor of both sides, we transit from these equations to the transport equation for vorticity and the Poisson equation for the flow function. As a result, we obtain the non-stationary equation for the vorticity of medium: $\partial_t\zeta + (\mathbf{v}\nabla)\zeta = \nu\nabla^2\zeta$ where $\zeta$ is the vorticity, $\zeta = [\nabla, \mathbf{v}]$, and the Poisson equation

$$\Delta\psi = -\zeta \,. \tag{2}$$

The velocity in this case is defined as $\mathbf{v} = [\nabla, \psi]$, where $\psi$ is the flow function. Thus, we obtain a set of equations describing the motion of an incompressible medium. In 2-dimensional case this set has rather simple form:

$$\partial_t\zeta + (\mathbf{v}\nabla)\zeta = \nu\nabla^2\zeta, \quad \Delta\psi = -\zeta, \quad \mathbf{v} = [\nabla, \psi\mathbf{e}_z], \tag{3}$$

where $\mathbf{e}_z$ is the unit normal vector.

The processes and phenomena described by these equations are modeled using special methods, which include: finite-difference schemes, the method of "particles in a cell" (PIC model), the method of "water bag", the methods of discrete vortices, finite elements and a number others [5, 6]. However, all these methods in their classical formulation have a number of drawbacks that significantly reduce the effectiveness of modeling the dynamics and interaction of vortex structures [7], [8].

The CD method considered in this paper is a generalization of the “water bag” method, traditionally used in the study of plasma dynamics described by the Vlasov equation [7]. The main idea of the method is to represent the medium as a distribution function of a certain physical function that defines its configuration. For example, for an ideal fluid with a local vortex perturbation, this is the distribution of vorticity, which will be a contour within which the fluid rotates at a constant angular velocity. In this case, to simulate the evolution of a fluid, there is no need to consider all area occupied by the medium (which you have to do when using finite-difference grids) or the area inside the vortex (as in the discrete vortex method), it is enough to solve numerically the equations for boundary of the vortex region (or regions), that significantly saves computer time and memory. Another advantage of the CD method is the absence of artificial dispersion and dissipation that is characteristic with using of finite difference schemes.

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### III. ALGORITHM OF THE CONTOUR DYNAMICS METHOD

Main states of the method are described in [4], however at its direct use there is a number of difficulties, therefore for correct application to simulation of the vortical structures the CD method requires some modification. Initial equations in the CD method are the equation of carry of a vortex (for ideal medium)

$$\partial_t \zeta + u\partial_x \zeta + v\partial_y \zeta = 0 \tag{4}$$

and the Poisson equation for flow function (2). Set of equations (2) and (4) should be also added with expressions for the components of velocity: $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$.

As it has been told, a general idea of CD method is that the interaction between the boundaries of the areas with constant $\zeta$ is considered, and due to this the dimension of the problem decreases on unit.

The areas with positive vorticity correspond to rotation of convective fluid elements counter-clockwise, therefore we use here the right-hand coordinate system where vector $\boldsymbol{\zeta}$ is directed along the axis $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$.

Analytical solution of the Poisson equation (2) for flow function $\psi$ has form

$$\psi(x,y) = -(2\pi)^{-1} \iint d\xi d\eta [\ln r] \zeta(\xi,\eta),$$
$$r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}, \tag{5}$$

where $\ln r$ is the Green's function of Eq. (2). A value of velocity in any point of stream can be obtained by differentiation of integral (5), namely: $\mathbf{u} = \nabla \times \mathbf{e}_z \psi = \mathbf{e}_x \partial_y \psi - \mathbf{e}_y \partial_x \psi$. Using changes $\partial_y \to -\partial_\eta$ and $\partial_x \to -\partial_\xi$, from the second formulae of (5) we obtain

$$\mathbf{u} = (2\pi)^{-1} \iint d\xi d\eta\, \zeta \left[\mathbf{e}_x \partial_\eta [\ln r] - \mathbf{e}_y \partial_\xi [\ln r]\right]. \tag{6}$$

Integrating (4) by parts we obtain [8]: $\mathbf{u} = (2\pi)^{-1} \mathbf{e}_z \times \iint d\xi d\eta\, [\ln r] \nabla_\xi \zeta$ where $\nabla_\xi = \mathbf{e}_x \partial_\xi + \mathbf{e}_y \partial_\eta$ and $\nabla_x = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y$. This integral consists of two parts which describe: 1) area inside of a contour where $\nabla_\xi \zeta = 0$ which does not influence on velocity, and 2) thin strip on contour boundary where $\nabla_\xi \zeta \neq 0$, which gives the unique final contribution to change of velocities of the vortices. Further, introducing on a contour the localized orthogonal coordinate system with axes $s$ and $q$ as it is shown in Fig. 1, after simple transformations (see [7], [8] for details) we obtain:

$$\mathbf{u} = (2\pi)^{-1} \sum_{j=1}^{N_c} [\zeta]_j \oint_j [\ln r] \left[\mathbf{e}_x d\xi_j + \mathbf{e}_y d\eta_j\right] \tag{7}$$

where $[\zeta]_j$ is the value of $[\zeta]$ connected with a contour $j$, and $[\zeta] = \zeta_0 - \zeta_1$ (square brackets mean jump of function; $\zeta_0$ is the vorticity outside of a contour, and $\zeta_1$ is the vorticity inside of a contour where on all its area it is a constant).

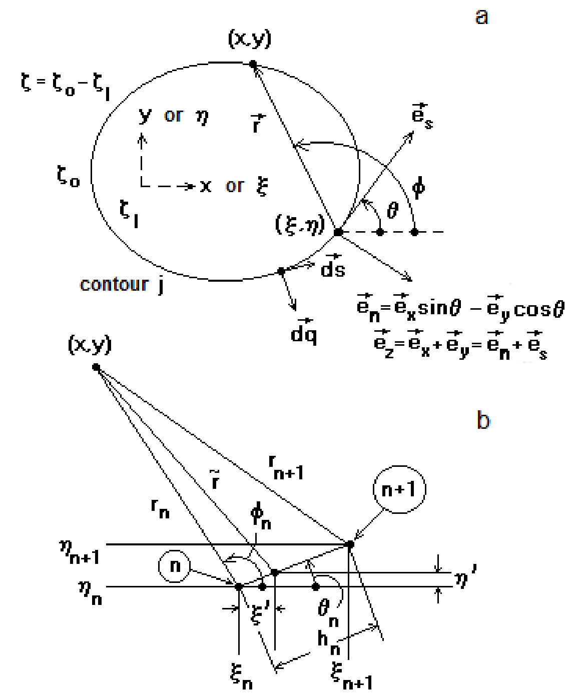


Fig. 1. Illustration to the algorithm of the contour dynamics method.

Thus, areas of constant vorticity are replaced with distribution of radiants with logarithmic forces along the $j$ contours surrounded by finite number of $N_C$ vortex areas in field of vorticity. Spatial and time discretization of the equations written above is considered in detail in [7], [8].

Let's make the important remark. In order for the CD method described above gives stable solutions, it must be modified, because, firstly, an overstepping scheme gives a tangible error at sufficiently long time intervals, and secondly, as experiments show, the evolution of FAVR leads to the "scattering" of the contour nodes, and the main condition that the contour must be piecewise continuous, i.e. must fit the continuous line as accurately as possible, is violated. The first problem can be solved using the technique proposed in [7]. To eliminate errors in the overstepping method, it is necessary to synchronize the values of the variables on two temporal layers. In this case, it is assumed that at some time layer, the coordinates and velocities of all contour nodes are known. Then the asynchronous component can be filtered using averaged variables for the time moment $p+1/2$, and $p$ must be even. These variables are obtained trivially:

$$\mathbf{x}^{p+1/2} = (\mathbf{x}^{p+1} + \mathbf{x}^p)/2, \quad \mathbf{u}^{p+1/2} = (\mathbf{u}^{p+1} + \mathbf{u}^p)/2, \tag{8}$$

and, specifying the function increment over time $\tau/2$ as

$$\mathbf{x}^{p+1} = \mathbf{x}^{p+1/2} + (\tau/2)\mathbf{u}^{p+1/2},$$
$$\mathbf{x}^{p} = \mathbf{x}^{p+1/2} - (\tau/2)\mathbf{u}^{p+1/2}, \tag{9}$$

we calculate the new position of system of the contours.

The second difficulty can be overcome by introducing additional nodes at the points of "discontinuity" of the contour, when $h_n \geq 2h_{\max}$ : to conserve the accuracy of calculations in the "discontinuity" area the point $C$ with coordinates $x_C = (x_A + x_B)/2$, $y_C = (y_A + y_B)/2$ is entered

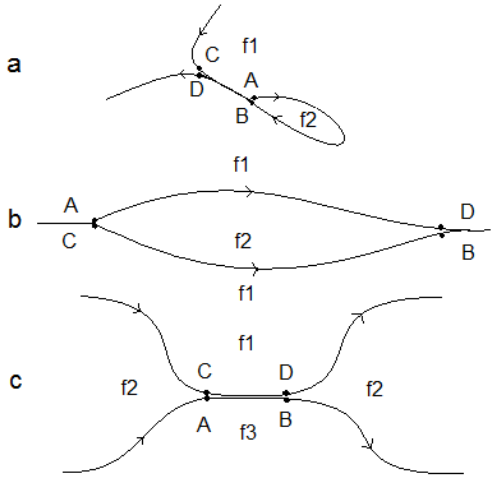


Fig. 2. Three cases of double curves formation.

between the already existing points $A$ and $B$. In this case, if an overstepping scheme is used, additional nodes must also be added on the two previous time layers. Here it should be noted that the opposite situation can also take place – with a close approach of two nodes, one of them can be excluded from the calculations, in this case the accuracy of the calculations will not change.

Another optimization method is the "correcting" of contours, described in [7]. With time, the lines of the same contour or lines belonging to different contours may become so close to each other that they become one curve, which has to be passed twice. Excluding this site, we do not change the state of the system [7], [8].

Fig. 2 shows possible cases of formation of "double" curves. Let us consider a case shown in Fig. 2a.

If the curve forms a loop we change it as follows: we are connecting points C and D among themselves, thus the new closed curve beginning in point A and terminating in point B is formed. Further, depending on a problem considered, the points between C – A and B – D are replaced by one curve or completely excluded from calculations. The situation represented in Fig. 2b differs from previous one that it is necessary to consider two sites, and we are connecting the points C and A, and B and D, respectively. Fig. 2c shows the case when the double curve divides two areas with different values of function $f$. Here it is possible to connect points C and A, and also B and D and to define curves of new type which connect points A and B of two other contours for which $\Delta f = f_3 - f_1$.

In view of the made remarks the scheme of calculation for modified CD method contains two additional stages (in comparison with the scheme for a standard CD method [4]) and takes following form:

1. For each contour the quantity of knots, the initial shape and a vorticity are set.
2. The contribution of all contours to change of velocity of each knot on a current contour is calculated.
3. Correction on knots is carried out, namely: additional knots are entered and/or superfluous knots are excluded.
4. Synchronization of values of velocity on three time layers for elimination of an error of the overstepping scheme is spent.
5. Steps 2-4 are repeated up to leave on stationary regime or up to the value of time, set in experiment.

So, setting initial configuration of the FAVRs we can investigate their time evolution. The detailed scheme of calculation according to algorithm of the modified CD method is presented in [7]. The diagnostics of the modified CD algorithm was considered in detail in [7], [8] where it was shown that this method can successfully be used for study of evolution of vortical structures.

Let us represent here some results of using of the method for the simple cases of evolution of vortical structures. Fig. 3 shows the results of simulation of interaction of two vortices with the same signs of vorticities for different values $r$ of initial distance between vortices. One can see that at distance greater critical, $3d/4$ [8] (where $d$ is the diameter of the vortices), the vortices rotate around of the common center (Fig. 3a) and velocity of this rotation depends on distance $r$. Thus we observe an appreciable change of the shape of vortices – from circle to the shape of the ellipse, but a return to the initial shape is observed with time (the so-called the "quasi-recurrence" phenomenon). At decreasing the distance between vortices a qualitative change in their behaviour is observed: they start to interact, and the common vortical area is formed with time, which consists of the more small scales vorticities (Fig. 3b).

Fig. 4 shows the interaction of two vortices with opposite signs of vorticities. One can see that their interaction differs essentially from the previous case. Vortices approach and move in a direction perpendicular an axis connecting their centers. The direction and velocity of a motion of vortices depend on signs of vorticities and a relative positioning.

Fig. 5 shows the result for a case of four vortex interaction when initially the distance $r$ between inner vortices is $d/2$, and the distance between each inner and

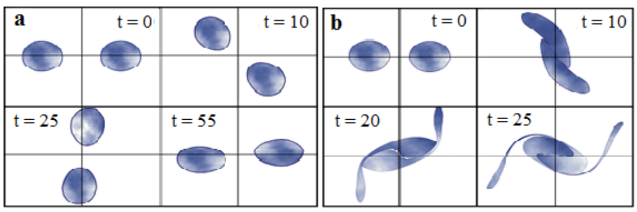


Fig. 3. Interaction of two circular vortices with the same signs of vorticities $\zeta_1 = \zeta_2 = -1$ at initial distance between them: a) $r = 2d$; b) $r = d/2$.

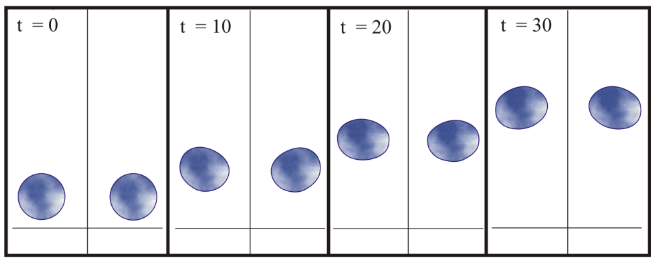


Fig. 4. Interaction of two circular vortexes with different signs of vorticities ($\zeta = -1$ and $\zeta = 1$).

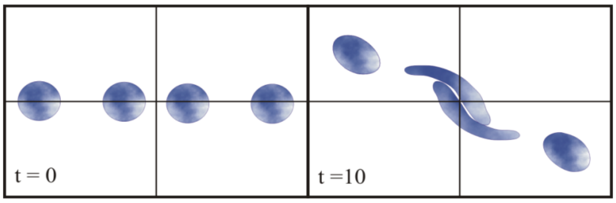


Fig. 5. Interaction of four linearly disposed at initial time vortices with the same signs of vorticities ($\zeta = -1$) at initial distance between them $r = d/2$ and $r = d$.

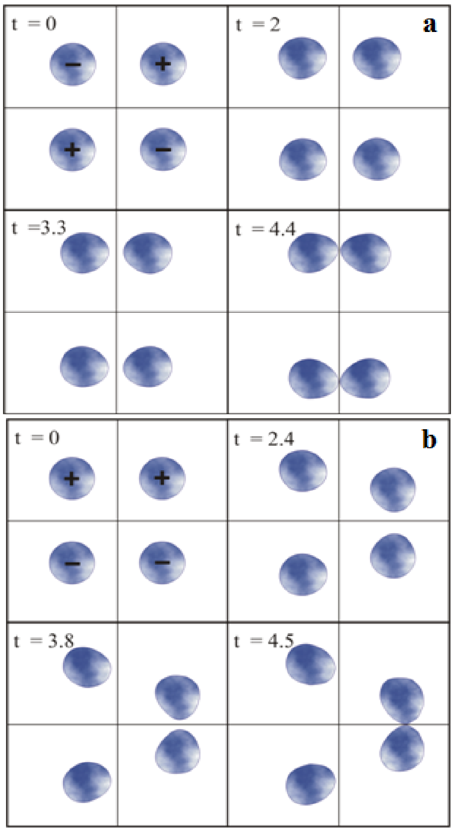


Fig. 6. Interaction of four vortices at initial distances between them $r_i = d$: a) $\zeta_1 = \zeta_3 > 0, \zeta_2 = \zeta_4 < 0$ ; b) $\zeta_1 = \zeta_2 > 0, \zeta_3 = \zeta_4 < 0$ .

corresponding outer vortices $r=d$. One can see that at evolution of such system the interaction of inner vortices takes place whereas two outer vortices do not interact, rotating around of the central vortical structure.

Fig. 6 shows the results of 4-vortex interaction for vortices with different signs of vorticities depending on their relative positioning. For case represented in Fig. 6a, evolution is similar to behaviour of pair of vortices with opposite vorticity signs – they pairly scatter to opposite sides. But, the interaction of vortices with opposite signs of vorticities is more intensive than in case of the same vorticities. In Fig. 6b we can see that the pair of vortices with opposite signs of $\zeta$ move to a right side, and two other vortices try "to catch up" with them.

The results of simulation for more complex vortical structures including the $N$-vortex systems of different configurations were presented in [8], [9].

## IV. Conclusion

So, we have shown that when studying vortex phenomena, the CD method, which has no artificial diffusion and dissipation, is most preferable. Representing the medium as system of FAVRs excludes the need to consider all space, as well as calculate the values of velocities and vorticities inside the vortex regions, which significantly speeds up the modeling process.

To eliminate the errors associated with the "discontinuity" of the contours and the error of the "over-stepping" method, we have proposed a special modification of the standard algorithm of the CD method. It allows to study effectively the evolution and dynamics of the interaction of $N$-vortex systems of various spatial configurations, consisting of FAVRs, depending on their spatial location, symmetry order, value and sign of the vorticity of the FAVRs.

The class of vortex structures that can be investigated by the CD method can also include vortex sheets [10], [11] which arise, for example, when studying a steady stratified medium with a velocity shift described by the Boussinesq equations.

The results obtained by numerical simulation using the modified CD method, along with their obvious significance for interpreting the effects of turbulence in gases and fluids (in particular, vortex motions in the Earth's atmosphere with regard to Coriolis forces), can be useful in describing turbulent processes in a plasma (for example, when describing a plasma by the model of Coulomb interacting quasi-particles and charged "filaments", as well as in studying the dynamics of the Alfvén vortices in a space plasma [9], [11]-[13]).

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