Dynamics of Multidimensional Wave Structures of the Soliton and Vortex Types in Complex Continuous Media Including Atmosphere, Hydrosphere and Space Plasma

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***Abstract*—The results of theoretical and numerical study of the structure and dynamics of 2D and 3D solitons and nonlinear waves described by Kadomtsev-Petviashvili, 3-DNLS classes of equations and also the vortex systems described by Euler-type equations are presented. The generalizations (relevant to various complex physical media), accounting for high-order dispersion corrections, dissipation, instabilities, and stochastic fluctuations of the wave fields are considered. Special attention is paid to the applications of the theory in different fields of modern physics including plasma physics, hydrodynamics and physics of the upper atmosphere.**

Keywords—solitons, vortices, generalized KP equation, DNLS equation, Euler equations

I. Introduction

The paper presents the main results on the theoretical and numerical study of the dynamics of 2D and 3D nonlinear wave structures of the soliton and vortex types, described by equations of the Kadomtsev-Petviashvili (KP) class, generalized by taking into account the dispersion effects of higher order and dissipation processes, equations of the 3-DNLS type, and a system of differential equations of Euler type. Such objects are interesting because their study plays an important role both in studying their general dynamics and in modeling nonlinear wave processes in the upper atmosphere (ionosphere) and hydrosphere, as well as in studying the propagation of wave structures in a magnetized plasma. The relevance of the topic is determined by the existing problems of the theory of multidimensional nonlinear waves and vortex formations in media with dispersion, the role that wave processes of hydrodynamic type in dispersing media can play, as well as the need to take into account the effects inherent in real media.

II. Classes of Nonlinear GKP and DNLS Models

As initial, considering the dispersion is negligible, we consider the following set of hydrodynamic equations with boundary conditions [1]:

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\,\mathbf{v} + (c^2/\rho)\nabla\rho = 0, \quad \partial_t\rho + \nabla(\rho\,\mathbf{v}) = 0;$$
$$\partial_t\Phi + \tfrac{1}{2}(\nabla\Phi)^2 + \frac{c^2(\rho-\rho_0)}{2\rho} + \frac{c^2 z}{\rho} = 0, \quad \Delta\Phi = 0, \qquad (1)$$
$$\partial_t\eta + \partial_x\eta\,\partial_x\Phi + \partial_y\eta\,\partial_y\Phi - \partial_z\Phi = 0,$$
$$\partial_t\Phi + \tfrac{1}{2}(\nabla\Phi)^2 + (c^2/\rho)\eta = 0,$$
$$z = \eta(x,y,t), \quad \partial_z\Phi\big|_{z=-\rho_0} = 0$$

where the concepts of generalized density ρ and velocity of "sound" $c(\rho)$ at neglecting of dispersion are introduced. Set (1), depending of the sense that we attach to the functions and variables included in it, describes

- Waves on water surface: $\rho \equiv H$ has the sense of the water depth, and velocity $c(\rho) = c_0 = \sqrt{gH}$ is the phase velocity of waves of small amplitude;
- Ion-acoustic waves in a plasma: ρ is the density of gas, $c(\rho) = c_0 = c_s = \sqrt{T_e/m}$ is the velocity of ionic "sound";
- Fast magnetosonic (FMS) waves in magnetized plasma: $\rho \equiv B$ has a sense of magnetic field of the wave, $c(\rho) = c(B) = v_A = B/\sqrt{4\pi nm}$ is the Alfvén velocity; *n* is the electron density, $m = m_e + m_i$ is the sum of the masses of the plasma component – the electron and the ion.

Other notations are standard. The first two equations are the equations of motion and continuity for the generalized velocity and density, respectively.

For waves on shallow water **v** is the particle velocity ("mass" velocity), for ion-acoustic waves it has a sense of the velocity of ion "sound", for magnetosonic waves $\mathbf{v} \equiv \mathbf{h} = \mathbf{H}_\sim / \mathbf{H}_0$ is the dimensionless magnetic field ($\mathbf{H}_\sim$ is a magnetic field of the wave). The next two equations are the equations for potential (gas flow is assumed to be potential, $\mathbf{v} = \mathrm{grad}\,\Phi$), the last four relations are boundary conditions where, e.g. for a fluid, the third and fourth relations can be interpreted respectively as the equation for the fluid surface and the boundary condition on at bottom, i.e. at $z = -H$.

Thus, we will use further implement a general approach, distracting from a specific type of medium, that is, we will work with generalized equations.

Using the expansion in powers of small parameters, as was done in [1], [2], one can obtain the following generalized equation:

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$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \Re, \quad (2)$$

where, e.g. for the waves on water surface

$$\alpha = 3c_0 / 2H, \quad c_0 = (gH)^{1/2},$$
$$\Re = -(c_0/2)\nabla_\perp w, \;\; \partial_x w = \nabla_\perp u, \;\; \beta = \frac{c_0}{6}\left(\frac{3\sigma}{\rho g} - H^2\right) \quad (3)$$

Note, that when $H \to (3\sigma/\rho g)^{1/2}$, the dispersion in the medium "disappears", and in order to take into account this "non-physical" effect, it is necessary to keep the next order term in the expansion of the full dispersion equation in *k*, and the dispersion correction $-\gamma\partial_x^5 u$ appears in the functional $\Re[u]$ with coefficient

$$\gamma = (c_0/6)\left[H^2\left(\tfrac{2}{5}H^2 - \sigma/\rho g\right) - \tfrac{1}{12}(3\sigma/\rho g - H^2)^2\right]. \quad (4)$$

For the FMS waves in magnetized plasma which are excited at $B_0^2 >> 8\pi nT$ in the frequency region $\omega << \omega_{Bi}$, we have $\Re = \kappa \nabla_\perp w, \;\; \partial_x w = \nabla_\perp u$ in (2) and the dispersive coefficient has form $\beta = v_A\,(c^2/2\omega_{0i}^2)(\cot^2\theta - m_e/m_i)$. It is clear that when the angle between the wave vector and magnetic field $\theta \to \arctan(m_i/m_e)^{1/2}$, we have the same situation and the functional $\Re[u]$ in (2) should be supplemented by term $-\gamma\partial_x^5 u$ with the dispersion coefficient of the next order:

$$\gamma = v_A\,(c^4/8\omega_{0i}^4)[3\,(m_e/m_i - \cot^2\theta)^2 - \\ - 4\cot^4\theta(1+\cot^2\theta)] \quad (5)$$

When considering dissipative effects in the medium, two cases can take place. If the Landau damping is small then the correction $-i\nu k_x^2$ appears in dispersion relation, and it takes form $\omega = c_0 k(1 - i\mu k - \beta k^2/c_0)$, and, therefore, the term appears in the right-hand side of (2): $\Re[u] = \nu\partial_x^2 u$ where, as it was shown in [1, 2], e.g. for ion-acoustic waves in a plasma without magnetic field the coefficient коэффициент $\nu = (\rho_0/2\rho)(c_\infty^2 - c_0^2)\tau\int_0^\infty \xi\varphi(\xi)d\xi$ has a sense of the coefficient of relaxation damping of "sound" oscillations, and c_∞ and c_0 are the velocities of the high-frequency and low-frequency "sound", respectively (the latter coincides with $c_S = (T_e/m_e)^{1/2}$); function $\varphi(t,\tau)$ defines relaxation process. If the Landau damping is significant, that dissipation can be taken into account by introducing of integral term

$$\Re[u] = -\hat{L}[u] = -\sigma\int_{-\infty}^{\infty}\frac{dk}{2\pi}|k|\int_{-\infty}^{\infty}u(x')e^{ik(x-x')}dx' \quad (6)$$

into the right-hand side of (2). Here $\sigma = c_0(\pi m_e/8m_i)^{1/2}$. The dispersion law takes form

$$\omega = c_0 k\,(1 - i\sigma|k|/c_0 - \beta k^2/c_0). \quad (7)$$

We confine ourselves to the hydrodynamic approximation when, e.g. for a plasma $\omega << \omega_{0e}$, i.e. the inverse time scale of ion oscillations is much less than the electron plasma frequency $\tau^{-1} << (4\pi n_0 e^2/m)^{1/2}$ (in this case for $T_e >> T_i$ Landau damping is negligible). With all the effects considered we can write the generalized KP equation (the GKP equation) in the following form:

$$\partial_x(\partial_t u + \alpha u\partial_x u - \nu\partial_x^2 u + \beta\partial_x^3 u + \gamma\partial_x^5 u) = \kappa\Delta_\perp u, \quad (8)$$

that corresponds in linear approach to the dispersion relation

$$\omega \approx c_0 k_x\left[1 + k_\perp^2/2k_x^2 - i\nu k_x/c_0 + (-\beta k_x^2 + \gamma k_x^4)/c_0\right]. \quad (9)$$

Let us now generalize (8) to another class of models describing another branch of oscillations – Alfvén waves. Following [3] we write the modified GKP+DNLS model in form

$$\partial_t u + \hat{A}(t,u)u = f, \;\; f = \kappa\int_{-\infty}^{x}\Delta_\perp u dx, \;\; \Delta_\perp = \partial_y^2 + \partial_z^2. \quad (10)$$

This model describes two classes of systems:

- The equations of the GKP class if differential operator is $\hat{A}(t,u) = \alpha u\partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3)$, then it takes the form analogous (8): $\partial_\eta\left(\partial_t u + \alpha u\partial_\eta u - \nu\partial_\eta^2 u + \beta\partial_\eta^3 u + \gamma\partial_\eta^5 u\right) = \kappa\Delta_\perp u,$ $\Delta_\perp = \partial_{\zeta_1}^2 + \partial_{\zeta_2}^2$;
- The equations of the DNLS class if $\hat{A}(t,u) = 3s|p|^2 u^2\partial_x - \partial_x^2(i\lambda + \nu)$ where

$$u = h = (B_y + iB_z)/2B_0\,|1-\beta|^{1/2}, \quad (11)$$
$$\mathbf{h} = \mathbf{B}_\perp/B_0, \quad p = (1+ie).$$

In this case the model takes form of the 3D generalized DNLS equation (3-DNLS equation):

$$\partial_t h + s\partial_x\left(|h|^2 h\right) - i\lambda\partial_x^2 h - \nu\partial_x^2 h = \sigma\int_{-\infty}^{x}\Delta_\perp h dx. \quad (12)$$

Both models are not completely integrable in the mathematical sense, analytically we can only perform stability analysis of multidimensional solutions using the method of studying the transformation properties of the Hamiltonian of the corresponding system [2], and also qualitative and asymptotic analysis of multidimensional solutions and, as a result, to construct a classification of solutions in $(n-1)\times d$–phase space and on the character of the asymptotics [4].

III. Analysis of Stability of Multidimensional Solutions

Write the GKP equation in Hamiltonian form $\partial_t u = \partial_x(\delta\mathcal{H}/\delta u)$ with the Hamiltonian

$$\mathcal{H} = \int[-(\varepsilon/2)(\partial_x u)^2 + (\lambda/2)(\partial_x^2 u)^2 + (\nabla_\perp\partial_x v)^2/2 - u^3]\,d\mathbf{r} \quad (13)$$

having the sense of energy of the system. Consider the variational problem: $\delta(\mathcal{H} + \upsilon P_x) = 0, \; P_x = \frac{1}{2}\int u^2 d\mathbf{r}$. These equations mean that all finite solutions are stationary points of Hamiltonian $\mathcal{H}$ at fixed momentum projection P_x. The problem of stability is that, in accordance with the Lyapunov theorem, in dynamical system the points that correspond to minimum or maximum of $\mathcal{H}$ are absolutely stable. Consider the deformations of $\mathcal{H}$ conserving momentum projection P_x:

$$u(x, \mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{(1-d)/2}u(x/\zeta, \mathbf{r}_\perp/\eta). \quad (14)$$

The Hamiltonian of the GKP equation as a function of deformation variables takes form

$$\mathcal{H}(\zeta,\eta) = a\zeta^{-2} + b\zeta^2\eta^{-2} - c\zeta^{-1/2}\eta^{(1-d)/2} + e\zeta^{-4} \quad (15)$$

where $a = -(\varepsilon/2)\int(\partial_x u)^2 \mathrm{d}\mathbf{r}$, $b=(1/2)\int(\nabla_\perp \partial_x v)^2 \mathrm{d}\mathbf{r}$, $c=\int u^3 \mathrm{d}\mathbf{r}$, $e=(\lambda/2)\int(\partial_x^2 u)^2 \mathrm{d}\mathbf{r}$. The necessary condition of extremum is

$$\partial_\zeta \mathcal{H} = 0, \quad \partial_\eta \mathcal{H} = 0. \quad (16)$$

The sufficient condition for the minimum of the Hamiltonian is the following:

$$\begin{vmatrix} \partial_\zeta^2 \mathcal{H}(\zeta_i,\eta_j) & \partial_{\zeta\eta}^2 \mathcal{H}(\zeta_i,\eta_j) \\ \partial_{\eta\zeta}^2 \mathcal{H}(\zeta_i,\eta_j) & \partial_\eta^2 \mathcal{H}(\zeta_i,\eta_j) \end{vmatrix} > 0, \quad \partial_\zeta^2 \mathcal{H}(\zeta_i,\eta_j) > 0. \quad (17)$$

Solution of (16) and (17) enables to obtain the results presented in Figs. 1 and 2. Thus, we have proved the possibility of the existence of absolutely and locally stable solutions in the GKP model, the stability conditions for 2D and 3D soliton solutions are presented in [3].

To study stability of solutions of the 3-DNLS equation we also write it in the Hamiltonian form [2]: $\partial_t h = \partial_x(\delta\mathcal{H}/\delta h)$ with Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{\infty}\left[\tfrac{1}{2}|h|^4 + \lambda s h h^* \partial_x \varphi + \tfrac{1}{2}\sigma(\nabla_\perp \partial_x w)^2\right] d\mathbf{r}, \quad \partial_x^2 w = h, \quad \varphi = \arg(h). \quad (18)$$

Variational problem is formulated as follows: $\delta(\mathcal{H} + \upsilon P_x) = 0$, $P_x = \frac{1}{2}\int |h|^2 \, d\mathbf{r}$. As in the previous case, all finite solutions are stationary points of the Hamiltonian at fixed momentum projection P_x. Solving the problem of stability we consider deformations of $\mathcal{H}$ conserving the momentum projection P_x:

$$h(x, \mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{-1}h(x/\zeta, \mathbf{r}_\perp/\eta), \quad \zeta,\eta \in \mathrm{C}. \quad (19)$$

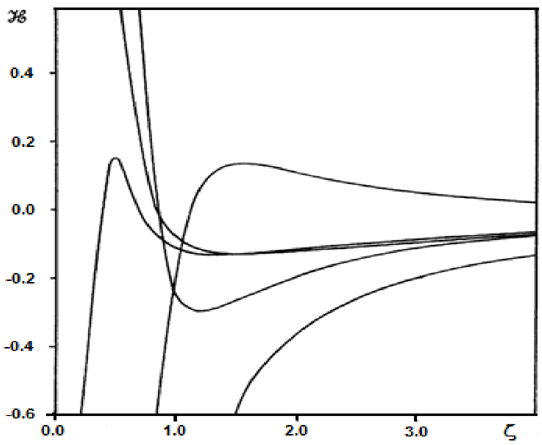

Fig. 1. Change of $\mathcal{H}(\zeta,\eta)$ at d=2 for different values of the coefficients along lines $\eta = [(4b/c)^2\zeta^5]^{1/3}$.

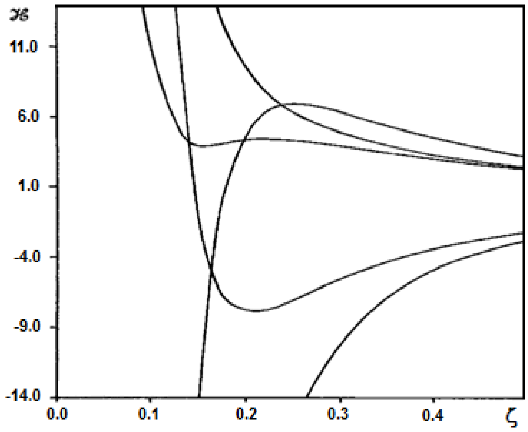

Fig. 2. Change of $\mathcal{H}(\zeta,\eta)$ at d = 3 for different values of the coefficients along lines $\eta = (2b/c)\zeta^{5/2}$.

The Hamiltonian of the 3-DNLS equation takes form

$$\mathcal{H}(\zeta,\eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} + c\zeta^2\eta^{-2} \quad (20)$$

where

$$a = (1/2)\int |h|^4 \,\mathrm{d}\mathbf{r}, \quad b = \lambda s\int h h^* \partial_x \varphi \,\mathrm{d}\mathbf{r}, \quad c = (\sigma/2)\int(\nabla_\perp \partial_x w)^2 \mathrm{d}\mathbf{r}. \quad (21)$$

The analysis of the boundness of the Hamiltonian $\mathcal{H}$ was performed similarly to the case of GKP. The possibility of the existence of absolutely and locally stable 3D solutions in the 3-DNLS model is proved, and the conditions for their stability are obtained (that is, the range of the coefficients of the equation) [3].

IV. Asymptotics and Structure of the 2D Solutions of the GKP Class of Equations

The asymptotics of the solutions of the GKP class of equations were studied in detail in [4] for function $w = u(\eta,\zeta,t)/V$. We have obtained that

- For the cases $V > 0$, $\gamma = -1$ and $V < 0$, $\gamma = -1$:
  $$w = A_1 \exp\left\{(2\gamma)^{-1/2}\left[C^2 + \sqrt{C^4 \pm 4\gamma}\right]^{1/2}\chi\right\}, \quad (22)$$
  that is, solutions exponentially decay on $\pm\infty$;
- For case $V < 0$, $\gamma = 1$:
  $$w = A_2 \exp\left\{\left(2C^{-1}\gamma^{-1/2}\right)^{-1}\left(2C^{-2}\gamma^{1/2} - 1\right)^{1/2}\chi\right\} \times \cos\left\{\left(2C^{-1}\gamma^{-1/2}\right)^{-1}\left(2C^{-2}\gamma^{1/2} + 1\right)\chi + \Theta\right\} \quad (23)$$
  where $A1$, $A2$ and Θ are arbitrary constants, $C = |V|^{-1/4}$, $\chi = [\eta \pm \zeta + (\kappa - V)t]$, that is, the asymptotics are oscillating damped.

Thus, depending on the signs of V and β, the GKP equation can have 2D soliton solutions with monotonous and oscillating asymptotics.

V. Interaction of the 2D Solitons of the GKP Equation and Influence of Dissipation

When studying the interaction of 2D solitons of the GKP equation we used specially developed numerical integration

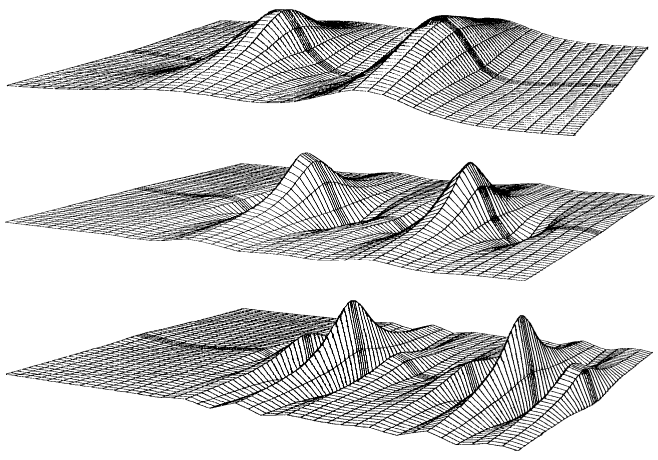

Fig. 3. Formation of 2D bi-soliton at $u_1(0)$=1.35, $u_2(0)$=1.3, $\Delta x(0)$=6, top down: t=0; t=0.6; t=1.3.

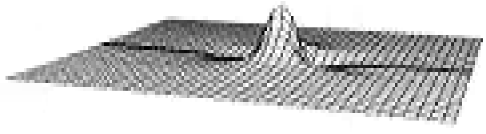

Fig. 4. Evolution of 2D soliton: ν =1, β, γ >0: t = 0.1.

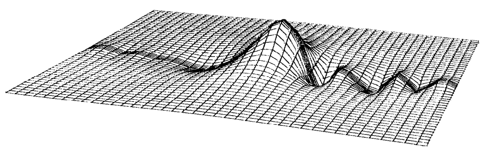

Fig. 5. Evolution of 2D soliton: ν =1, β<0, γ>0; t = 0.2.

methods based on finite difference and spectral approaches [2]. In numerous series of computer experiments we have found that for some values of the dispersion coefficients, trivial cases of interaction, similar to the cases of interaction of 2D solitons of the KP equation, is observed; but for some values of the parameters of the equation (and, consequently, the propagation medium) a completely non-trivial (and impossible in the "classical" KP model) case of the formation of stable soliton pairs (bound states), so-called bi-solitons, can be realized (Fig. 3).

When studying the effect of dissipation in the medium on the evolution and structure of 2D solitons, the GKP equation was written as

$$\partial_\eta\,(\partial_t u + \alpha u\partial_\eta u - \nu\partial_\eta^2 u + \beta\partial_\eta^3 u + \gamma\partial_\eta^5 u) = \kappa\Delta_\perp u \quad (24)$$

and integrated numerically. The examples of some results are shown in Figs. 4 and 5. One can see that dissipation in the system, along with the general damping of the amplitude of the wave field, directly affects the structure of the 2D solitons. In all cases, the effect of elongation of the soliton "tail", a decrease in the frequency of oscillations and damping of oscillations behind the main maximum, as well as asymmetrical changes of integrals P and $\mathcal{H}$ in the frontal and rear "cavities" (where u<0) were observed.

VI. Some Applications of the GKP Model

In [1]-[4] numerous applications of the model of the GKP class equations in physics of real media with dispersion were investigated, in particular:

- Dynamics if the ion-acoustic of fast magnetosonic (FMS) waves in a plasma (Earth's ionosphere and magnetosphere, astrophysics including the relativistic limit);
- Soliton dynamics on the surface of a "shallow" fluid (gravity and gravity-capillary waves, tsunami waves);
- Solitary wave disturbances in the atmosphere and ionosphere induced by the pulse sources (seismic processes, the fronts of the solar eclipse and solar terminator, powerful artificial explosions – the possibility of identifying and bearing of the sources);
- Evolution in media with variable dispersion (waves in a fluid, waves in a plasma).

Let us consider here one of important applications when in a magnetized plasma with $\beta \equiv 4\pi nT/B^2 << 1$ in frequency range $\omega < \omega_B = eB/m_i c$ the FMS waves are excited, and in case $k\lambda_D << 1$, $k_x^2 >> k_\perp^2$, $v_x << c_A = B^2/4\pi n m_i$ the dispersion relation $\omega \approx c_A k_x (1 + k_\perp^2/k_x^2 + \chi(\theta)\lambda_D^2 k_x^2)$ is valid (see [4] and numerous references in it). In this case, at rather high temperature of ions ($\beta > m_e/m_i$) the dispersion "length" is $\chi(\theta)\lambda_D^2 = (c^2/2\omega_{0i}^2)\cot^2\theta - \frac{1}{2}\rho^2(3 - \frac{11}{4}\sin^2\theta)$, where $\rho = v_{Ti}/\omega_B$ is the ion Larmor radius. If $\beta = 4\pi nT/B^2 < m_e/m_i$ the structure of the FMS waves depends on sign of dispersion coefficient $\gamma_1 = -c_A\chi(\theta)\lambda_D^2 = c_A(c^2/2\omega_{0i}^2)(m_e/m_i - \cot^2\theta)$. In this case, near the cone of angles where the dispersion changes its sign, i.e. $|\pi/2 - \theta| \le (\beta/4)^{1/2}$, we have $\gamma_1 \to 0$, that leads to appearance in the dispersion equation of a term proportional to the fifth degree of k, i.e. $\gamma_2 k_x^5$, with the coefficient

$$\gamma_2 = c_A(c^4/8\omega_{0i}^4)\left[3(m_e/m_i - \cot^2\theta)^2 - 4\cot^4\theta(1+\cot^2\theta)\right]$$

and as a result we obtain the GKP equation in form

$$\partial_x(\partial_t h + \alpha h\partial_x h + \gamma_1\partial_x^3 h + \gamma_2\partial_x^5 h) = -(c_A/2)\Delta_\perp h\,. \quad (25)$$

In this case the character of the dispersion is determined by the ratio of the signs of the dispersion coefficiens γ_1 and γ_2. The following cases take place (Fig. 6):

- $\gamma_1 > 0,\ \gamma_2 < 0$ (region B) – a case of negative dispersion;
- $\gamma_1 > 0,\ \gamma_2 > 0$ (region A) and $\gamma_1 < 0,\ \gamma_2 < 0$ (region C) – the cases of "mixed" dispersion.

Let us formulate the following problem. Let there is a 3D stationary beam of FMS waves propagating in plasma at the angle to the magnetic field near the cone $\theta = \arctan\,(m_i/m_e)^{1/2}$. Performing scale transformations in (25), we make the transition from the initial problem to boundary one:

$$\partial_t(\partial_x h + 6h\partial_t h - \varepsilon\partial_t^3 h - \lambda\partial_t^5 h) = \Delta_\perp h \quad (26)$$

where $\Delta_\perp = \partial_\rho^2 + (1/\rho)\partial_\rho$ (axially-symmetric geometry).

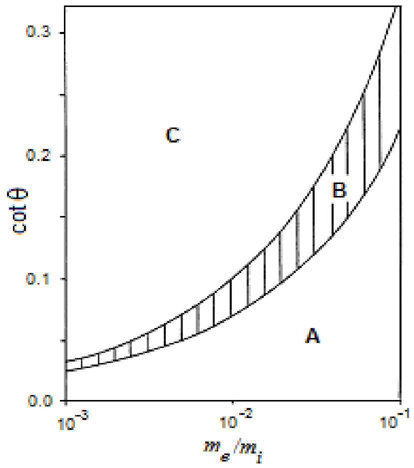

Fig. 6. Character of dispersion for the FMS waves depending on angle θ and ratio m_e/m_i.

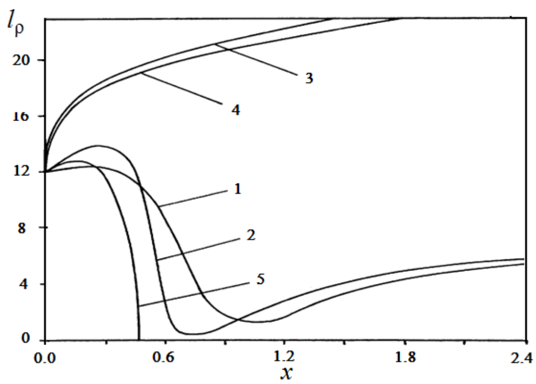

Fig. 7. Change of cross-section of the FMS wave beam at its propagation along the *x* axis.

We solved the problem (26) numerically [1] with condition $h_0 = h\,(t, 0, \rho) = \cos\,(mt)\exp\,(-\rho^2)$ on boundary x = 0: i.e. a harmonic in time and limited in the transverse direction beam of the FMS waves was set. Basic numerical results are shown in Fig. 7. One can see that in the regions of the angles B and C (curves 3 and 4) scattering of "magnetic sound" is observed at propagation, in region A (curves 1 and 2) the beam is first focused, then, due to nonlinear saturation, some defocusing is observed, and then evolution leads to the formation of a stable stationary beam of FMS waves, i.e. the 3D FMS soliton. Note that such a phenomenon in the standard KP model (curve 5) is not observed and only account of more thin dispersive effects allowed observing it for the first time.

Fig. 8 shows the solution of the problem of evolution of the FMS wave beam which corresponds to the stage of maximum of its amplitude. One can observe a gradual lag of the beam "wings" from its central part in the process of evolution from its main maximum (a) and then their "flapping" with the formation of a ring structure of the FMS behind the main maximum (b).

VII. The Euler Equations and Dynamics of Vortex Structures

As initial let us consider the Euler equations

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{v}, \quad \nabla\mathbf{v} = 0, \tag{27}$$

where ν is a kinematic viscosity.

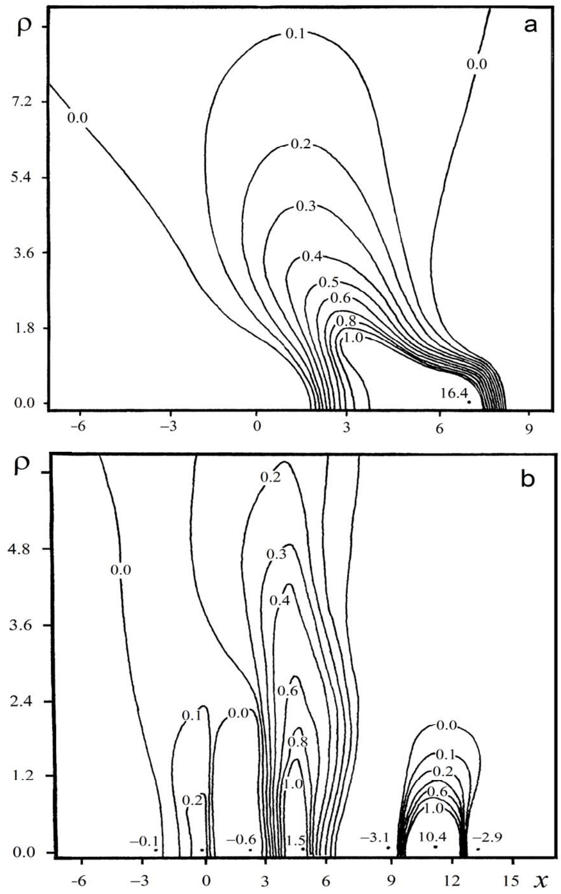

Fig. 8. Solution in plane (x, ρ) corresponding to the stage of maximum of the beam amplitude.

To study the vortex motion, excluding pressure p and calculating rot, we turn to the transport equation for density ρ and to the Poisson equation to the flow function ψ:

$$\partial_t \rho + (\mathbf{v}\nabla)\rho = \nu\nabla^2\rho, \quad \Delta\psi - f = -\rho,$$
$$\mathbf{v} = B^{-1}[\nabla, \psi\mathbf{e}_z], \ \ \mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y. \tag{28}$$

Equations (28) describe continuous medium (inviscid incompressible fluid) or quasi-particles (charged filaments aligned along a uniform magnetic field **B**) with Coulomb interaction. A sense of the variables in (28) depends on type of medium. For simulation we used the modified contour dynamics method developed in [5].

Fig. 9 shows an example of simulation of evolution of the synoptic vortex in comparison with the satellite photo of such vortex. One can see that the result of our modeling is qualitatively coincided with the real system which is simulated and well reflects the basic properties of evolution including formation of a vortex sheet of the cyclonic vortex.

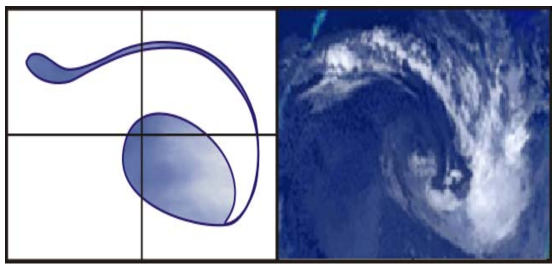

Fig. 9. Simulation of evolution of the cyclonic type synoptic vortex (numerical result and satellite photo).

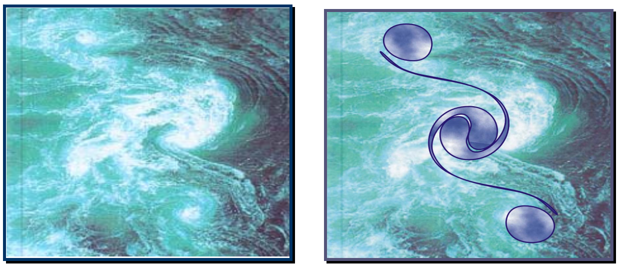

Fig. 10. Simulation of the 4-vortical interaction in channel Naruto, Japan (numerical result and air photo).

Fig. 10 shows the example of simulation of interaction of vortical structures in a fluid in comparison with the real situation displayed on the air photography frame. It is clearly seen that the result of the 4-vortex interaction is the formation of a complex structure in which the core of the system consists of two internal vortices. The formed vortex sheets that connect the vortices of the system are also clearly visible.

Fig. 11 shows the results of simulation of tornado evolution with use of quasi-2D approach with many-layer approximation of the 3D vortical structure by the vortex system (system of FAVRs) [5].

One can see that our simulation reflects the basic features of evolution of a tornado. In particular, we investigated an influence of the perturbation imposed on the tornado axis on its dynamics. We established as a result, that small cross-section indignation leads to inappreciable fluctuations of an axis and, as a whole, does not influence on structure and stability of a vortex. So, using approach proposed in [5] we can forecast tornado evolution and simulate interaction of such type of vortices.

As other important applications, the following can be considered, e.g.

- Hydro- and aerodynamics (formation of vorticities and vortical chains at flowing of solid bodies by streams of gas and a fluid);
- Modeling of formation and evolution of vortical structures in astrophysics (such as spiral structure of Galaxies and solar flare activity associated with the dynamics of magnetic loops and magnetic tubes in the solar corona);
- Problem of magnetic confinement of plasma and controlled fusion;

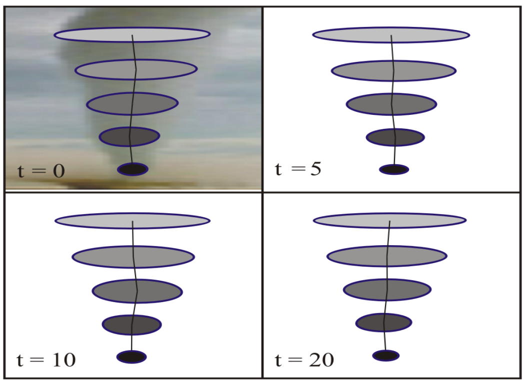

Fig. 11. Simulation of a tornado vortex evolution.

- Spiral and vortex motions in a plasma concerning plasma technologies.

VIII. Conclusion

In conclusion, we have presented here the results on the theoretical and numerical study of the dynamics of 2D and 3D nonlinear wave structures of the soliton and vortex types, described by equations of the generalized Kadomtsev-Petviashvili (GKP) class, equations of the 3-DNLS type, and a system of differential equations of Euler type. These models play an important role both in studying their general dynamics and in modeling nonlinear wave processes in the upper atmosphere (ionosphere) and hydrosphere, as well as in studying the propagation of wave structures in a magnetized plasma. The results obtained can be useful in investigations in fields such as plasma physics, hydrodynamics and physics of upper atmosphere.

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