Interaction of Multidimensional NLS Solitons in Nonuniform and Nonstationary Medium

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***Abstract*—On the basis of the analytical and numerical approaches the stability and dynamics of interaction of the multidimensional soliton-like solutions of the generalized nonlinear Schrödinger equation, which describes the waves in a plasma, fiber and planar optical waveguides, taking into account inhomogeneity and nonstationarity of propagation medium, is studied. The sufficient conditions of stability of the 2-dimensional and 3-dimensional solutions are obtained, and it is shown that even in the simplest 1-dimensional case the GNLS equation can have stable and quasi-stable solutions of the soliton and breather types and also unstable solutions which disperses with time. Obtained results can be useful in numerous applications in plasma physics, nonlinear optics and in many other fields of physics.**

Keywords—generalized nonlinear Schrödinger equation, envelop solitons, breathers, interaction, nonuniform and nonstationary media

I. Introduction

The generalized 3-dimensional (3D) nonlinear Schrödinger (3-GNLS) equation [1] describes the dynamics of the envelop of modulated nonlinear waves and pulses in weakly nonlinear media with dispersion, and has numerous important applications in plasma physics (e.g., it describes propagation of the Langmuir waves in hot plasmas and propagation of plane-diffracted wave beams in the focusing regions of the ionosphere), in nonlinear optics (propagation of light in nonlinear optical fibers and planar waveguides) and in other fields of physics (e.g., such as small-amplitude gravity waves on the surface of deep inviscid water; the Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime; propagation of the Davydov's alpha-helix solitons, which are responsible for energy transport along molecular chains), and many others (e.g., the NLS equation is a simplified 1D form of the Ginzburg-Landau equation introduced in 1950 in their work on superconductivity [2]). Note, that the multidimensional 3-GNLS equation is not completely integrable and its analytical solutions in unknown in general case (except for, perhaps, smooth solutions of type of solitary waves). But, using approaches developed in [3]-[6] for other equations of the Belashov-Karpman (BK) system we can study analytically stability of possible solutions of the 3-GNLS equation, and the dynamics of the soliton interaction can be investigated numerically. Such approach is realized in the present paper.

II. 3-GNLS Equation. Stability of Solutions

If in the BK system

$$\partial_t u + \hat{\mathrm{A}}(t,u)u = f, \quad f = \sigma\int_{-\infty}^{x}\Delta_\perp u\,\mathrm{d}x + f', \qquad \Delta_\perp = \partial_y^2 + \partial_z^2 \tag{1}$$

the differential operator has form

$$\hat{\mathrm{A}}(t,u) = i\,[\gamma|u|^2 - \beta\partial_x^2] + \alpha/2, \tag{2}$$

we have the 3-GNLS class of equations:

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = \sigma\int_{-\infty}^{x}\Delta_\perp u\,\mathrm{d}x + f' \tag{3}$$

where $\alpha, \beta, \gamma = \varphi(t,x,y,z)$, $f' = f'(t,x,y,z)$, and $(\alpha/2)u$ describes dissipative effects, and u is an envelope of the wave packet (the pulse). In the Hamiltonian form eq. (3) with $\alpha = 0$ (the 3-NLS equation) has form

$$\partial_t u = \partial_x(\delta\mathrm{H}/\delta u) \tag{4}$$

where $\mathrm{H} = \int_{-\infty}^{\infty}\left[\frac{\gamma}{2}|u|^4 + \beta u u^*\partial_x\varphi + \frac{1}{2}\sigma(\nabla_\perp\partial_x w)^2\right]\mathrm{d}\mathbf{r}$, $\partial_x^2 w = u$, $\varphi = \arg(u)$.

Using the method described in detail in [3], we study stability of the 2D and 3D solutions of (3). Thus, the problem for (4) is stated in the form of variational equation $\delta(\mathrm{H} + \upsilon P_x) = 0$, $P_x = \frac{1}{2}\int u^2\,\mathrm{d}\mathbf{r}$, which sense consists that all finite solutions of (4) are the stationary points of Hamiltonian H at the fixed value of the momentum projection P_x. In conformity with the Lyapunov's theorem, in dynamical system the points that correspond to minimum or maximum of H are absolutely stable. If the extremum is local, that we have locally stable solutions.

Let's consider the deformations of H conserving momentum projection:

$$u(x, r_\perp) \to \zeta^{-1/2}\eta^{-1}u(x/\zeta,\ \mathbf{r}_\perp/\eta), \qquad \zeta,\eta \in \mathrm{C}. \tag{5}$$

In this case the Hamiltonian takes form $\mathrm{H}(\zeta,\eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} - c\zeta^2\eta^{(1-d)}$ with the integral coefficients

$$a = (\gamma/2)\int|u|^4\,\mathrm{d}\mathbf{r},\ b = \beta\int u u^*\partial_x\varphi\,\mathrm{d}\mathbf{r}, \\ c = (\sigma/2)\int(\nabla_\perp\partial_x w)^2\,\mathrm{d}\mathbf{r}. \tag{6}$$

From the necessary conditions of extremum, $\partial_\zeta\mathrm{H} = 0$, $\partial_\eta\mathrm{H} = 0$, we obtain at once its coordinates:

$$\zeta_0 = -ac^{-1}, \quad \eta_0 = \left[-ab^{-1}\left(1 + a^2c^{-2}\right)\right]^{1/2} \tag{7}$$

where $b < 0$ if $\eta \in \mathrm{R} \subset \mathrm{C}$ because $a > 0$, $c > 0$ by definition,

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and $b > 0$ if $\eta \in C$. Sufficient condition of minimum in point (ζ_i, η_j) are

$$\begin{vmatrix} \partial_\zeta^2 H(\zeta_i,\eta_j) & \partial_{\zeta\eta}^2 H(\zeta_i,\eta_j) \\ \partial_{\eta\zeta}^2 H(\zeta_i,\eta_j) & \partial_\eta^2 H(\zeta_i,\eta_j) \end{vmatrix} > 0, \ \partial_\zeta^2 H(\zeta_i,\eta_j) > 0. \quad (8)$$

Solving this set of inequalities we obtain the following results. For waves when $b < 0$ (positive nonlinearity) we have:

$$a/c < d = (2\sqrt{2})^{-1}\sqrt{13+\sqrt{185}}\,, \quad (9)$$

whence follows that $H > -3bd/(1+2d^2)$ i.e. the Hamiltonian is bounded from below.

For waves when $b > 0$ (negative nonlinearity) the change $b \rightarrow -b$ is equivalent changes $y \rightarrow -iy$, $z \rightarrow -iz$ and $H < -3bd/(1+2d^2)$, i.e. the Hamiltonian is not bounded from below in this case (it is bounded from above).

So, we have proved a possibility of existence of stable 3D solutions in the 3-NLS model and obtained the conditions of their stability, i.e. found the regions of values of the equation coefficients (characteristics of a medium) when 3D solitons are stable.

III. Numerical Simulation of Evolution and Interaction of the GNLS Solitons

Consider, at first, more simple 1D case ($\sigma = 0$) when (32) takes form

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = f' \quad (10)$$

where in general case (nonuniform and nonstationary medium), $\alpha, \beta, \gamma = \varphi(x,t)$, $f' = f'(x,t)$.

At modeling we used initial conditions in form of the soliton-like envelop modulation pulse $u_0 = u(x,0)$:

a) $u(x,0) = A\exp(-x^2/l)$;

b) $u(x,0) = A\exp[-(x-5)^2/l] + A\exp[-(x+5)^2/l]$;

c) $u(x,0) = A[\mathrm{sch}(x-s/2) + \mathrm{sch}(x+s/2)]$;

d) $u(x,0) = A[\mathrm{sch}(x) + \mathrm{sch}(x-s/2) + \mathrm{sch}(x+s/2)]$.

Figs. 1 and 2 show the results obtained for the initial conditions (a) and (b) in the simplest case of the NLS equation with $\beta, \gamma =$ const (stationary medium); $\alpha, f' = 0$ for

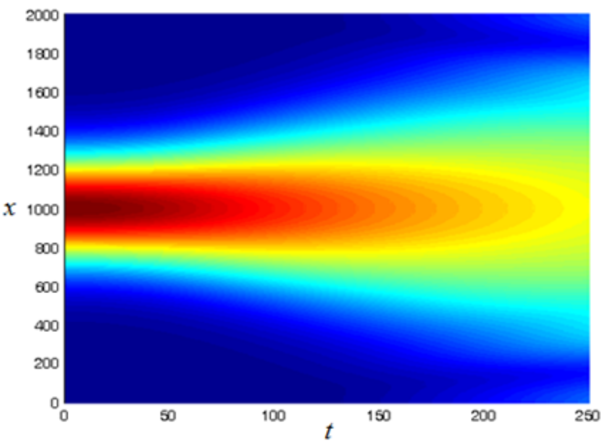

Fig. 1. Evolution of a Gaussian soliton-like envelop pulse (a) for $A = 2$, $l = 2$; $\beta=0.5$, $\gamma=0$.

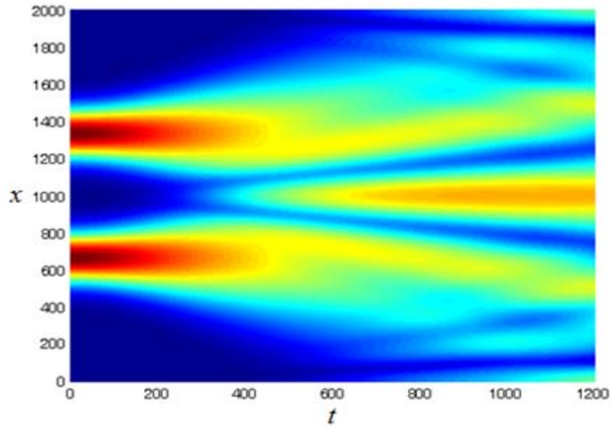

Fig. 2. Evolution of a Gaussian soliton-like envelop pulse (b) for $A=1$, $l=4$; $\beta=0.5$, $\gamma=0$.

negative nonlinearity, $\beta > 0$. In this case $b > 0$ in the Hamiltonian and $H > -3bd/(1+2d^2)$, and it means that the stability condition, $H < -3bd/(1+2d^2)$, is not satisfied, and as we can see from Figs. 1 and 2, we observe dispersion of the envelop pulses with time.

Other cases when coefficient $\gamma \neq 0$ are shown in Figs. 3 and 4 for the cases when stability condition is not satisfied and it is satisfied, respectively.

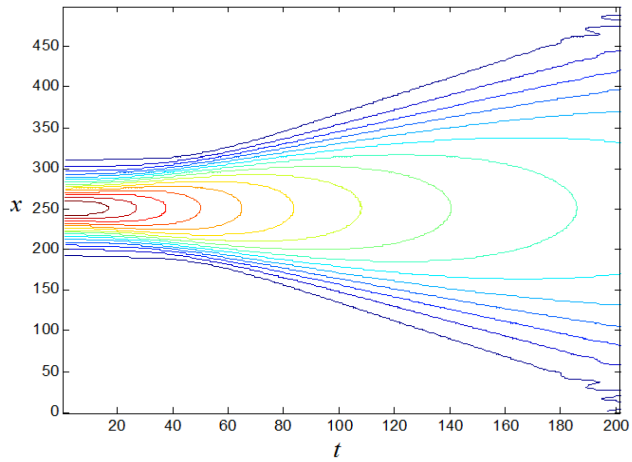

Fig. 3. Evolution of a Gaussian soliton-like envelop pulse (a) for $A=1$, $l=4$; $\beta<0$, $\gamma>0$ (positive nonlinearity).

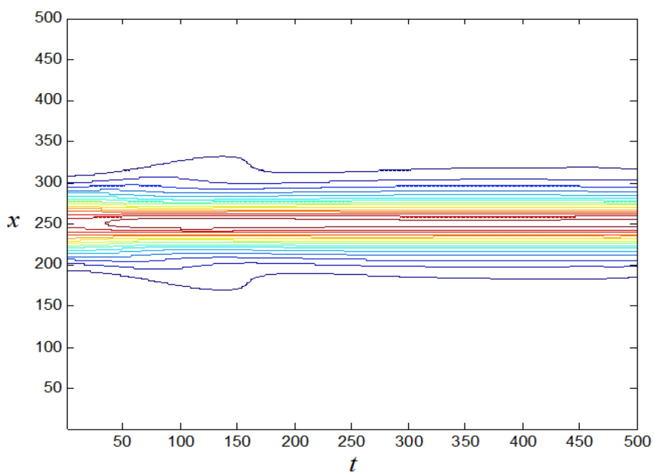

Fig. 4. Evolution of a Gaussian soliton-like envelop pulse (a) for $A=1$, $l=4$; $\beta<0$, $\gamma=-0.5<0$ (negative nonlinearity).

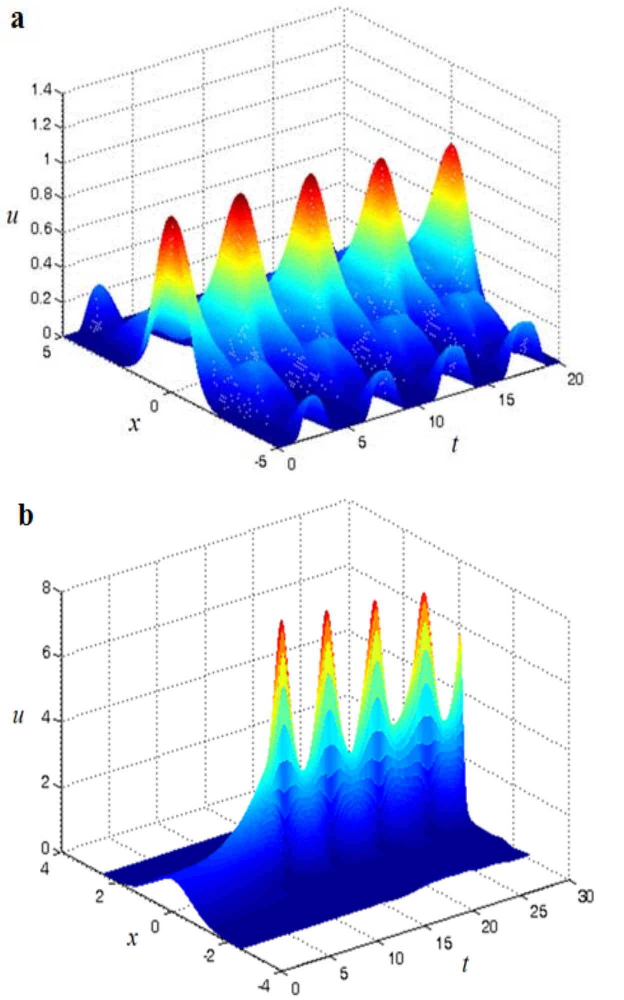

Fig. 5. Evolution of a Gaussian soliton-like envelop pulse in nonstationary medium at $\alpha, f' = 0$: a) $\beta = 0.5$, $\gamma = -1+0.01\sin 2\pi t$; b) $\gamma = -1$, $\beta(t) = -0.5$ for $t \leq 5$ and $\beta(t) = 0.5(1+0.2\sin 2\pi t)$ for $t>5$; the cases of negative nonlinearity.

In the first case (Fig. 3) we observe dispersion of envelop pulse with time, and in the second case (Fig. 4) one can see stabilization of solution, and soliton formation from initial pulse.

Fig. 5 shows two examples of the results of evolution of the Gaussian pulse (a) in nonstationary medium with negative nonlinearity when condition of solution stability $H<-3bd/(1+2d^2)$ is satisfied.

As a result of evolution, we observe the dynamics of arising of strong stable pulsations of breather type from initial solitary pulse in this case.

The example of interaction of the soliton-like initial pulses (d) and (c) at negative nonlinearity in the GNLS model is shown in Figs. 6 and 7, respectively. In the first case the stability condition is not satisfied, and we observe arising of a strong pulse from 3-pulse initial disturbance and its disintergration onto two small pulses with time. In the second case the stability condition is satisfied and we observe stable evolution of the double-pulse initial perturbation.

In our numerical simulations we have also found that at weak negative nonlinearity in the stationary medium when stability condition is satisfied, the transit from stable state to regime of stable pulsations of breather type occurs with decreasing of distance between initial pulses s in initial condition (c) (see Fig. 8). So, the result of interaction strongly depends on initial distance between maxima of initial disturbance.

IV. Conclusion

We have discussed the problem of stability and dynamics of multidimensional solutions of the GNLS equation (as partial case of BK system), namely: stability of the 3D solutions of the 3-GNLS equation and dynamics of stable and unstable solutions of the NLS equation in stationary and nonstationary media.

We have obtained analytically the conditions dividing the classes of stable and unstable soliton-like solutions of the GNLS equation, so we have found the sufficient conditions of stability for multidimensional solutions. We have also studied numerically the cases of stable and unstable (with formation of breathers) evolution of pulses of the various form, and also the interaction of 2- and 3-pulse structures leading formation of stable and unstable solutions.

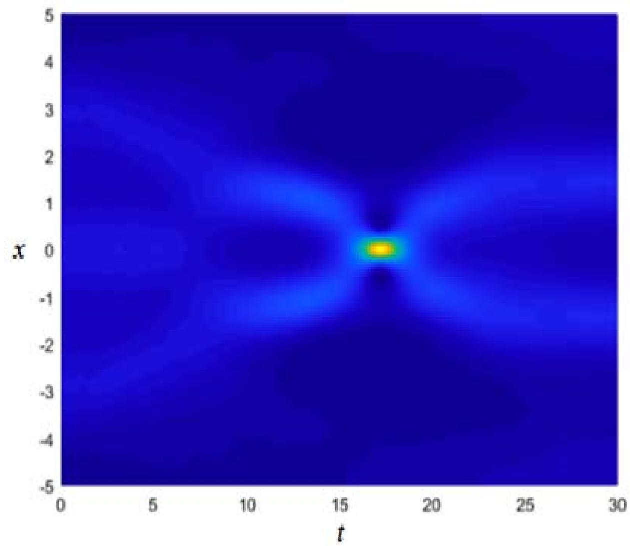

Fig. 6. Interaction of three GNLS pulses (stationary medium) at $\gamma = -1$, $\beta = 0.25$; the case of weak negative nonlinearity.

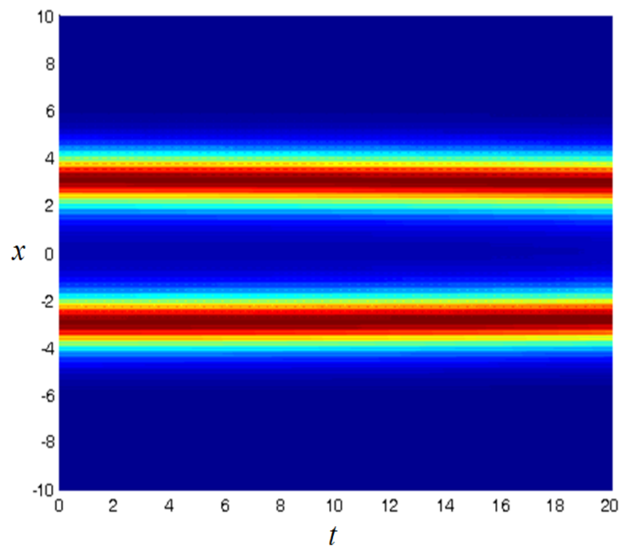

Fig. 7. Absence of interaction of the GNLS pulses (stationary medium) at $\gamma = -1$, $\beta = 0.05$; the case of negative nonlinearity.

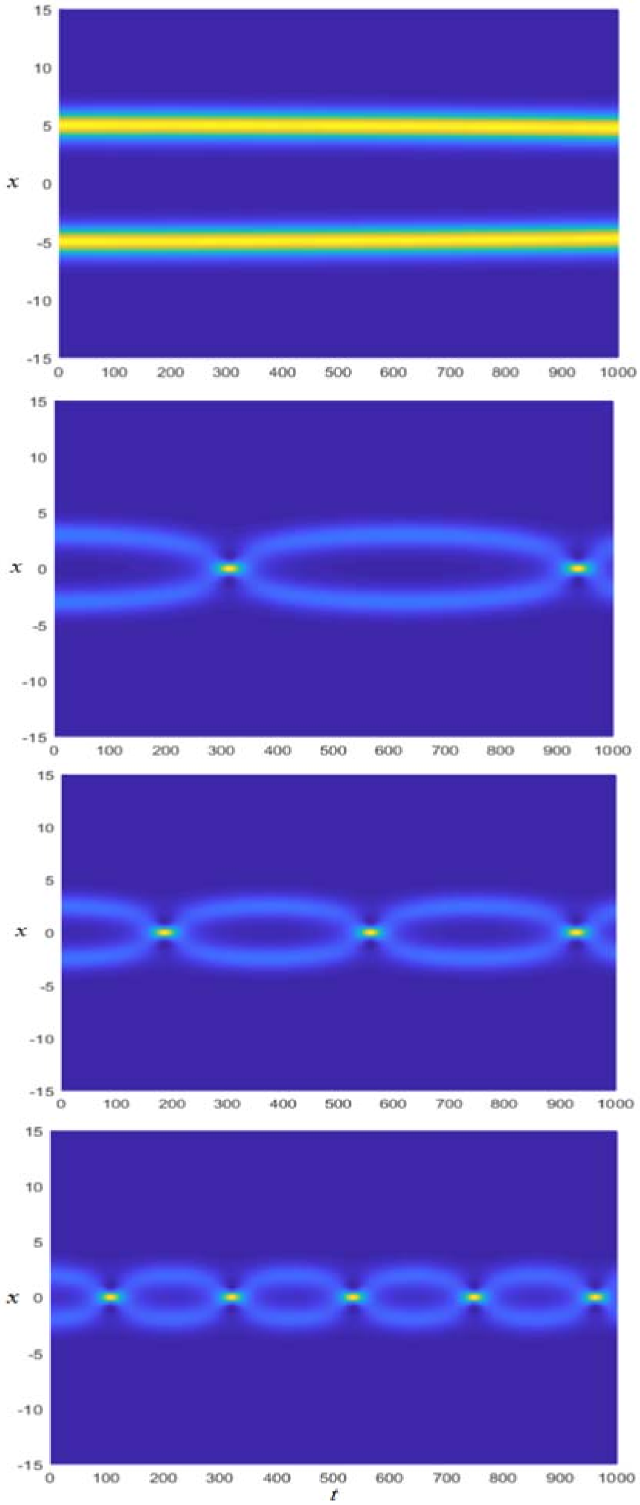

Fig. 6. Interaction of two GNLS pulses in stationary medium (weak negative nonlinearity) at $\gamma = -1$, $\beta = 0.05$ for $s = 10, 6, 5, 4$ (from top to down).

Let us note that the results of our numerical experiments well confirm our analytical study of solutions' stability presented in the part II of the paper. Obtained results can be useful in numerous important 3D applications of the theory, such as:

- Plasma physics (e.g. propagation of the Langmuir waves in hot plasmas and the plane-diffracted wave beams in the focusing regions of the ionosphere and the magnetosphere of the Earth);
- Nonlinear optics (e.g. propagation of light and the light beams in nonlinear optical fibers and planar waveguides);
- Other fields of physics (e.g. small-amplitude gravity waves on the surface of deep inviscid water; the Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime; the Davydson's alpha-helix solitons which can transport of energy along molecular chains), and
- Many others fields of physics (e.g. in the theory of superconductivity [2]).

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